A GENERAL SOLUTION OF THE GRAVITATIONAL EQUATIONS WITH A SIMULTANEOUS FICTITIOUS SINGULARITY

V. A. BELINSKIĬ and I. M. KHALATNIKOV

Institute of Physics Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor April 29, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 1000-1008 (September, 1965)

A general solution of the gravitational equations in vacuum is derived in a synchronous coordinate system, which possesses a simultaneous fictitious singularity reached by all points in space at the same time t = 0.

1. INTRODUCTION

In recent papers, E. M. Lifshitz and I. M. Khalatnikov[1,2] have considered the question of the existence of a time singularity 1) in the general solution of the gravitational equations. The authors reached the conclusion that the general solution may contain only a fictitious singularity caused by the geometric specification of the chosen coordinate system, which vanishes as one goes over to another system. A solution which contains a physical singularity leading to a real gravitational collapse can only be some particular solution which does not allow the specification of a complete set of arbitrary initial conditions. The investigation was carried out in a synchronous coordinate system in which the presence of a singularity is inescapable, as is well known.[1]

A general solution with a fictitious nonsimultaneous singularity on the hypersurface $t = \varphi(x^1, x^2, x^3)$ was constructed in [2]. This singularity is reached by different space points at different times. The authors of [2] also pointed out the possibility of constructing a general solution with a simultaneous singularity which is reached by all points in space at the same instant t = const. The present paper is concerned with an analytic construction of such a solution.

As it turned out, the general solution of the vacuum gravitational equations in a synchronous system containing a simultaneous singularity has an expansion in t whose coefficients cannot contain a single arbitrary function of the three space coordinates. The physical three-dimensional arbitrariness appears in this case in the coefficients of the expansion in one of the space coordinates. The reasons for this circumstance are contained in the geometrical considerations of [1]; it was pointed out there that the singularity in a synchronous system arises from the unavoidable 2) intersection of the time coordinate lines, which are a family of geodesics, on certain hypersurfaces 3) —the analogs of the caustics of geometrical optics. When the caustic is the hypersurface $t = \varphi(x^1, x^2, x^3)$ we obtain a metric with a nonsimultaneous singularity.

If in the case of a simultaneous singularity the caustic also were a hypersurface, viz., the "hyperplane" t = const, then there would be no difference in principle between the problems, and the desired solution could be found from the solution with a nonsimultaneous singularity in which one should set φ = const. However, it was shown in [1] that in this case the caustic degenerates into a two-dimensional surface since the hypersurface would manifestly contain time-like intervals (as it is tangent to the time lines), which excludes the simultaneity of the singularities. It is precisely this circumstance which gives rise to the characteristic of the solution mentioned earlier.

If the solution is expanded in a series in powers of the time near the point $t = t_0$, the physical arbitrariness in the coefficients of the expansion is connected with the character of the manifold $t = t_0$ on which the initial conditions are imposed. Expanding the desired solution near the singularity, we do not, of course, obtain three-dimensional arbitrary functions, since the manifold $t = t_0$ has in this case only two dimensions. In this form, the

1)The singularity may be in a point, along a line, on a two-dimensional surface, and in general, on a hypersurface. Here the determinant of the matrix of the metric tensor vanishes.

2)This is a consequence of the inequality $T_0^0 - (\frac{1}{2})T \leq 0$, which always holds in a real world.

3)The singular hypersurface is the envelope of the family of time lines and touches these in their mutual points of intersection.

solution can contain only two-dimensional arbitrary functions. The dependence on the one remaining variable will have a special form which depends essentially on the choice of the coordinate system. It does not follow from this, however, that such a solution will not be general, since it may admit an expansion in another variable x near the hypersurface x = const which manifestly contains three-dimensional arbitrary functions. This question is discussed in the Conclusion.

We note also that here we confirm the assertions made in [1] that one of the principal values of the metric tensor and the determinant vanish quadratically in the time at the singularity and that at the singularity the interval reduces to a quadratic form of only two differentials (not counting dt). These assertions can also be proved analytically in a rigorous fashion.

2. ANALYTIC CONSTRUCTION OF THE SOLUTION

The gravitational equations in a synchronous system can be written in the form

$$R_{\alpha\beta} = P_{\alpha\beta} + {}^1/_4\varkappa\varkappa_{\alpha\beta} + {}^1/_2\dot{\varkappa}_{\alpha\beta} - {}^1/_2\varkappa_{\beta\gamma}\varkappa_\alpha{}^\gamma = 0, \qquad (1)$$

$$R_\alpha{}^0 = {}^1/_2\varkappa_{;\alpha} - {}^1/_2\varkappa_{\alpha;\,\beta}^{\beta} = 0, \qquad (2)$$

$$R_0{}^0 = {}^1/_2\dot{\varkappa} + {}^1/_4\varkappa_\alpha{}^\beta\varkappa_\beta{}^\alpha = 0. \qquad (3)$$

The Greek indices run through 1, 2, 3. The Latin indices a, b, c, d, which we shall encounter later on, take the values 1 and 2. The semicolon denotes, as usual, the covariant derivative, the dot the simple time derivative, and the comma the simple derivative with respect to the space coordinates. All tensor operations are carried out in a three-dimensional space with the metric $g_{\alpha\beta}$. $\kappa_{\alpha\beta}$ denotes the three-dimensional tensor $\partial g_{\alpha\beta}/\partial t$, and $P_{\alpha\beta}$ denotes the three-dimensional Ricci tensor constructed from $g_{\alpha\beta}$ in analogy to the four-dimensional one.

We shall assume that the singularity occurs at the instant t = 0, and we shall seek the solution in the form of a regular expansion in t near t = 0. Recalling what has been said in the introduction on the structure of the solution, we shall seek it in the form

$$g_{ab} = \overset{(0)}{g}_{ab} + \overset{(1)}{g}_{ab} t + \overset{(2)}{g}_{ab} t^2 + \ldots,$$

$$g_{a3} = \overset{(1)}{g}_{a3} t + \overset{(2)}{g}_{a3} t^2 + \overset{(3)}{g}_{a3} t^3 + \ldots,$$

$$g_{33} = \overset{(2)}{g}_{33} t^2 + \overset{(3)}{g}_{33} t^3 + \overset{(4)}{g}_{33} t^4 + \ldots. \qquad (4)$$

where $\overset{(k)}{g}_{\alpha\beta}$ are functions of the space coordinates.

Let us now show that one can specialize the system of coordinates such, using the remaining arbitrariness in its definition, that, with the equations of motion we obtain the metric in the form

$$g_{ab} = q\delta_{ab} + \overset{(1)}{g}_{ab} t + \overset{(2)}{g}_{ab} t^2 + \ldots,$$

$$g_{a3} = \overset{(3)}{g}_{a3} t^3 + \overset{(4)}{g}_{a3} t^4 + \ldots,$$

$$g_{33} = t^2 + \overset{(4)}{g}_{33} t^4 + \ldots, \qquad (5)$$

where $q = q(x^1, x^2)$ is a two-dimensional function and δ_{ab} is the two-dimensional Kronecker symbol. In other words, we can arrange to satisfy the conditions

$$\overset{(0)}{g}_{ab} = q\delta_{ab}, \qquad (6)$$

$$\overset{(1)}{g}_{a3} = 0, \qquad (7)$$

$$\overset{(2)}{g}_{a3} = 0, \qquad (8)$$

$$\overset{(2)}{g}_{33} = 1, \qquad (9)$$

$$\overset{(3)}{g}_{33} = 0. \qquad (10)$$

Substituting (4) in the equations $R_a^0 = 0$ and $R_0^0 = 0$, we obtain in the leading order in the time

$$\overset{(1)}{g}_{a3} = \overset{(2)}{g}_{33}\Phi_a(x^1, x^2)/\sqrt{g^{(2)}}, \qquad S^{ab}_{(0)}\Phi_a\Phi_b = 0, \qquad (11)$$

where $S^{ab}_{(0)}$ is the free term in the expansion in the time of the matrix S^{ab}, the inverse of the matrix g_{ab}, and $\Phi_a(x^1, x^2)$ are arbitrary two-dimensional functions; $g^{(2)}$ is the coefficient of t^2 in the expansion of the determinant of g. It is seen from (11) that, if one of the Φ_a is zero, the other also vanishes since $S^{11}_{(0)} \neq 0$ and $S^{22}_{(0)} \neq 0$. But we can make one of the Φ_a vanish by the transformation $x^a = f^a(\bar{x}^b)$ which is allowed by the metric (4). Thus we obtain condition (7).

Now we are still free to make the three-dimensional transformation $x^3 = f^3(\bar{x}^\alpha)$, by which we can achieve the fulfilment of condition (9). Then the equations $R^0_\alpha = 0$ and $R^0_0 = 0$ yield in leading order

$$\frac{\partial}{\partial x^3}\overset{(2)}{g}_{a3} = 0, \qquad (12)$$

$$\frac{\partial}{\partial x^3}|\overset{(0)}{g}_{ab}| = 0, \qquad (13)$$

$$\overset{(3)}{g}_{33} = 0. \qquad (14)$$

Relation (14) agrees with (10), and (12) implies that $\overset{(2)}{g}_{a3}$ is two-dimensional. This being the case, it is easy to see that they can be made to vanish by the transformation

$$x^a = f^a(\bar{x}^b), \qquad x^3 = \bar{x}^3 + f^3(\bar{x}^b),$$

which is admissible. Hence we have achieved the fulfilment of (8).

After this we have still left at our disposal the transformations $x^a = f^a(\bar{x}^b)$, which can be used, as we shall see presently, to fulfil condition (6). We introduce the notation

$$g_{ab}^{(0)} = a_{ab}, \qquad g_{ab,3}^{(0)} = a_{ab,3} = \alpha_{ab}.$$

Then the equations $R_{ab} = 0$ and $R_{33} = 0$ in their leading orders and relation (13) give the following system:

$$\alpha_{a,3}^{b} = 0, \quad \alpha_a^b \alpha_b^a = 0, \quad \alpha_a^a = 0. \tag{15}$$

The raising of the indices is here achieved with the help of a_{ab}. The most general solution of this system has the form

$$\begin{aligned} a_{11} &= A_{11} + (A_{11}\Phi_1{}^1 + A_{12}\Phi_1{}^2)Z, \\ a_{12} &= A_{12} + (A_{11}\Phi_2{}^1 + A_{12}\Phi_2{}^2)Z, \\ a_{22} &= A_{22} + (A_{12}\Phi_2{}^1 + A_{22}\Phi_2{}^2)Z. \end{aligned} \tag{16}$$

Here and in the following, z stands for the coordinate x^3. The functions A_{ab} and Φ_a^b are two-dimensional and satisfy the relations

$$\begin{gathered} A_{11}\Phi_2{}^1 + A_{12}\Phi_2{}^2 = A_{22}\Phi_1{}^2 + A_{12}\Phi_1{}^1, \\ \Phi_a{}^a = 0, \quad \Phi_a{}^b\Phi_b{}^a = 0. \end{gathered} \tag{17}$$

It is seen from this that $A_{ab} = q\delta_{ab}$ implies $\Phi_a^b = 0$ and hence $a_{ab} = q\delta_{ab}$. But with the transformation $x^a = f^a(\bar{x}^b)$ we can always reduce the form

$$dl^2 = A_{ab}(x^1, x^2)\, dx^a dx^b$$

to the form $dl^2 = q(dx_1^2 + dx_2^2)$, i.e., obtain $A_{ab} = q\delta_{ab}$. Thus we see that we can also satisfy condition (6).

Thus we shall seek the solution in the form (5), and no transformations containing two-dimensional, let alone three-dimensional functions remain. This means that the entire arbitrariness in the metric (5) will be physical. Of the equations (1) let us first of all consider the equations $R_{ab} = 0$. Instead of these, it is more convenient to solve the equations $\bar{R}_{ab} = 0$, where $\bar{R}_{ab} = 2R_{ab}/g_{33}$, since the latter are solved with respect to the highest derivatives of g_{ab} in the coordinate x^3. Substituting (5) in $\bar{R}_{ab}$ leads to the expansion

$$\bar{R}_{ab} = \bar{R}_{ab}^{(1)} t + \bar{R}_{ab}^{(2)} t^2 + \ldots,$$

and the corresponding calculations show that the equations $\bar{R}_{ab}^{(k)} = 0$ for $k = 1, 2, 3, \ldots$ have the form

$$g_{ab,33}^{(k)} - k^2 g_{ab}^{(k)} - \psi_{ab}^{(k)} = 0, \tag{18}$$

where the functions $\psi_{ab}^{(k)}$ are constructed from the following components:

$$g_{ab}^{(1)},\ g_{ab}^{(2)}\ \ldots,\ g_{ab}^{(k-1)},\ q; \qquad g_{\alpha 3}^{(2)},\ g_{\alpha 3}^{(3)},\ \ldots,\ g_{\alpha 3}^{(k)}. \tag{19}$$

This implies that for given $g_{\alpha 3}$ the functions $\psi_{ab}^{(k)}$ in Eqs. (18) are free terms independent of $g_{ab}^{(k)}$, and the solutions of the equations have the form

$$g_{ab}^{(k)} = \Phi_{ab}^{(k)}(x^1, x^2)e^{kz} + F_{ab}^{(k)}(x^1, x^2)e^{-kz} + j_{ab}^{(k)}, \tag{20}$$

where the first two terms are the solutions of the corresponding homogeneous equations, where $\Phi_{ab}^{(k)}$ and $F_{ab}^{(k)}$ are arbitrary two-dimensional functions and $j_{ab}^{(k)}$ are particular solutions of the inhomogeneous equations which are completely determined by the quantities $\psi_{ab}^{(k)}$, i.e., the functions (19). This means that $j_{ab}^{(k)}$ contains a two-dimensional arbitrariness through $\Phi_{ab}^{(n)}$ and $F_{ab}^{(n)}$ up to and including the order $k - 1$, but not higher. Substituting the metric (5) in the equations $R_{\alpha 3} = 0$, we obtain the expansion where we substitute in $R_{\alpha 3}$ the expressions for the components g_{ab}.

$$R_{\alpha 3} = R_{\alpha 3}^{(1)} t + R_{\alpha 3}^{(2)} t^2 + \ldots,$$

If the equations $R_{\alpha 3}^{(k)}$ are written down explicitly, one sees easily that they completely determine all $g_{\alpha 3}^{(k)}$ in terms of $\Phi_{ab}^{(n)}$ and $F_{ab}^{(n)}$, and no new arbitrariness is introduced. For clarity, we write down the whole system of tensor equations in the leading orders in the time, viz., $R_{ab} = 0$ in the orders t^{-1} and t^0, $R_{a3} = 0$ in the order t^1, and $R_{33} = 0$ in the orders t^1 and t^2. We introduce the notation

$$\begin{gathered} g_{ab}^{(1)} = b_{ab}, \quad g_{ab}^{(2)} = c_{ab}, \quad g_{a3}^{(3)} = \psi_a, \quad g_{33}^{(4)} = \omega, \\ b_{ab,3} = \beta_{ab}, \quad c_{ab,3} = \eta_{ab}. \end{gathered}$$

Raising and lowering of indices and covariant differentiation are carried out with the help of $g_{ab}^{(0)} = q\delta_{ab}$. Then these equations take the form

$$\begin{gathered} (R_{ab}, t^{-1}) \quad b_{ab,33} - b_{ab} = 0, \quad (R_{ab}, t^0) \quad c_{ab,33} - 4c_{ab} \\ + {}^1/{}_2\beta\beta_{ab} - {}^1/{}_2 b b_{ab} - 2D_{ab} - \beta_{bc}\beta_a^c + b_{bc}b_a^c = 0, \\ (R_{3a}, t^1) \quad \psi_a + {}^1/{}_3\beta_{a;c}^c - {}^1/{}_3\beta_{;a} = 0, \\ (R_{33}, t^1) \quad b_{,33} - b = 0, \quad (R_{33}, t^2) \quad \omega + {}^1/{}_3 c \\ - {}^1/{}_6 b_a^c b_c^a - {}^1/{}_{12}\beta_a\beta_c^a - {}^1/{}_6(\eta - b_c^a\beta_a)_{,3} = 0. \end{gathered} \tag{21}$$

Here $\beta = \beta_a^a$, $c = c_a$, etc., and D_{ab} is the two-dimensional Ricci tensor corresponding to $g_{ab}^{(0)}$.

The equation $R_{33} = 0$ in order t^1 is a consequence of $R_{ab} = 0$. The remaining equations have the same structure, as was shown above, with $j^{(1)}_{ab} = 0$.

Thus the solution can be written in the form

$$g_{ab} = q\delta_{ab} + \sum_{k=1}^{\infty} t^k [\Phi^{(k)}_{ab}(x^1, x^2) e^{kz} + F^{(k)}_{ab}(x^1, x^2) e^{-kz} + j^{(k)}_{ab}],$$

$$g_{a3} = \sum_{k=3}^{\infty} t^k g^{(k)}_{a3}, \qquad g_{33} = t^2 + \sum_{k=4}^{\infty} t^k g^{(k)}_{33}. \tag{22}$$

We must now satisfy the equations $R^0_\alpha = 0$ and $R^0_0 = 0$ in all remaining orders besides those considered in (12) to (14). For this purpose we take recourse to the Bianchi identity, which for $R_{\alpha\beta} = 0$ can be written in the form

$$\frac{1}{2}\dot{B} + \frac{g}{4g}B + \frac{g_{,\alpha}}{g^2}M^{\alpha\beta}D_\beta - \frac{1}{g}M^{33}D_{3,3} - \frac{1}{g}M^{33}_{,3}D_3$$

$$-\frac{1}{g}(\overline{M^{\alpha\beta}D_\beta})_{,\alpha} \equiv 0, \tag{23}$$

$$\dot{D_\alpha} + \frac{g_{,\alpha}}{4g}B - \frac{1}{2}B_{,\alpha} \equiv 0.$$

$$B = \sqrt{g}R^0_0, \quad D_\alpha = \sqrt{g}R^0_\alpha, \quad g = |g_{\alpha\beta}|. \tag{24}$$

Here $M^{\alpha\beta}$ are the minors corresponding to the elements $g_{\alpha\beta}$. The bar over the last term in (23) indicates that in the summation the term with $\alpha = \beta = 3$ is to be omitted, since it has already been written down explicitly.

It is easy to see that for the metric (22) the functions entering in the identity have the following expansions:

$$B = B^{(1)}t + B^{(2)}t^2 + \ldots, \qquad D_\alpha = D^{(1)}_\alpha t + D^{(2)}_\alpha t^2 + \ldots,$$

$$g = q^2t^2 + g^{(3)}t^3 + \ldots, \qquad g_{,3} = g^{(3)}_{,3}t^3 + \ldots,$$

$$M^{33} = q^2 + M^{33}_{(1)}t + \ldots, \qquad M^{33}_{,3} = M^{33}_{(1),3}t + \ldots,$$

$$M^{\alpha\beta} = M^{\alpha\beta}_{(2)}t^2 + M^{\alpha\beta}_{(3)}t^3 + \ldots, \qquad \alpha.\beta \neq 3.$$

Using these expansions, we find that if the functions $B^{(1)}, B^{(2)}, \ldots, B^{(k)}$ vanish then the functions $D^{(1)}_\alpha, D^{(2)}_\alpha, \ldots, D^{(k+1)}_\alpha$ vanish identically [as follows from (24)] and for $B^{(k+1)}$ and $D^{(k+2)}_3$ we have the identity

$$(k+2)D^{(k+2)}_3 - {}^1/_2 B^{(k+1)}_{,3} \equiv 0. \tag{25}$$

Here the identity (23) is automatically satisfied up to and including the order t^{k-1}, and in order t^k it yields

$$(k/2 + 1)B^{(k+1)} - D^{(k+2)}_{3,3} \equiv 0. \tag{26}$$

From (25) and (26) we obtain the identity

$$B^{(k+1)}_{,33} - (k+2)^2 B^{(k+1)} \equiv 0. \tag{27}$$

or

$$B^{(k+1)} \equiv \gamma(x^1, x^2)e^{(k+2)z} + \lambda(x^1, x^2)e^{-(k+2)z}, \tag{28}$$

where the functions γ and λ are constructed from $\Phi^{(n)}_{ab}$ and $F^{(n)}_{ab}$ in all orders up to and including $k + 2$.

It follows from (28) that the vanishing of $B^{(k+1)}$ requires two conditions: $\gamma = 0$ and $\lambda = 0$. Then we obtain from (25) $D^{(k+2)}_\alpha \equiv 0$, and the process starts over again.

Thus, if we start the preceding discussion with $k = 0$, we find that in order to satisfy the equations $R^0_\alpha = 0$ and $R^0_0 = 0$, we must impose two additional conditions on the functions $\Phi^{(k)}_{ab}$ and $F^{(k)}_{ab}$ in each order in the time t^k. The explicit form of these conditions can be found from a direct determination of the function B with the metric (22) and setting that equal to zero. Since B contains only time derivatives, these relations are algebraic. With the notation introduced earlier, we find, for example, that the condition $B^{(1)} = 0$ reduces to

$$\omega + {}^1/_3 c - {}^1/_{12} b_c{}^a b_a{}^c = 0. \tag{29}$$

After solving the system (21) and substituting the solution in (29), the latter reduces to

$$\gamma(x^1, x^2)e^{2z} + \lambda(x^1, x^2)e^{-2z} = 0,$$

where γ and λ are determined through $\Phi^{(1)}_{ab}$, $F^{(1)}_{ab}$ $\Phi^{(2)}_{ab}$, $F^{(2)}_{ab}$, q.

We note that the system (21) together with (29) is a complete system for the determination of the metric in the following approximation:

$$g_{ab} = q\delta_{ab} + b_{ab}t + c_{ab}t^2,$$

$$g_{a3} = \psi_a t^3, \qquad g_{33} = t^2 + \omega t^4. \tag{30}$$

We now give the exact general solution of the system (21), (29):

$$b_{ab} = \Phi^{(1)}_{ab}e^z + F^{(1)}_{ab}e^{-z},$$

$$c_{ab} = \Phi^{(2)}_{ab}e^{2z} + F^{(2)}_{ab}e^{-2z} - {}^1/_2 D_{ab} + {}^1/_4(2\Phi^{(1)}_{bc}F^{(1)c}{}_a$$

$$+ 2F^{(1)}_{bc}\Phi^{(1)c}{}_a - \Phi^{(1)}F^{(1)}_{ab} + F^{(1)}\Phi^{(1)}_{ab}), \tag{31}$$

$$\psi_a = {}^1/_3[(F^{(1)c}{}_{a;c} - F^{(1)}_{;a})e^{-z} - (\Phi^{(1)c}{}_{a;c} - \Phi^{(1)}{}_{;a})e^z],$$

$$\omega = -{}^1/_6(F^{(1)c}{}_a\Phi^{(1)a}{}_c - F^{(1)}\Phi^{(1)} - D);$$

$$4\Phi^{(2)} - \Phi^{(1)c}{}_a\Phi^{(1)a}{}_c = 0, \qquad 4F^{(2)} - F^{(1)c}{}_a F^{(1)a}{}_c = 0. \tag{32}$$

Here $\Phi = \Phi^c_c$, $F = F^c_c$, D is the two-dimensional scalar curvature, and the tensor operations are carried out with the help of $q\delta_{ab}$. The two-dimensional functions $F^{(k)}_{ab}$ and $\Phi^{(k)}_{ab}$ satisfy (32). These relations are additional conditions following from (29), and it is due to them that the function ω was obtained independent of the coordinate z. It should also be noted that there are no additional conditions for the first order functions $F^{(1)}_{ab}$ and $\Phi^{(1)}_{ab}$, and all six of them remain arbitrary.

Thus we see that the metric (22) contains in its components g_{ab} four two-dimensional arbitrary functions in each order except the linear order, where there are six, and the zeroth order, where there is one function q. The infinite number of arbitrary functions in this solution is completely understandable. The point is that the equations $R_{ab} = 0$ can be solved for the highest derivatives of g_{ab} with respect to the coordinate z, and can then be solved with expanding g_{ab} in the time. It can be shown that in this case they will satisfy the Cauchy-Kovalevskiĭ theorem, and the components g_{ab} must then of course admit an arbitrariness of the form

$$g_{ab}|_{z=0} = \Phi_{ab}(x^1, x^2, t), \qquad \frac{\partial}{\partial z} g_{ab}|_{z=0} = F_{ab}(x^1, x^2, t).$$

Expanding now in the time coordinate, we find of course, that the arbitrary functions Φ_{ab} and F_{ab} lead to a two-dimensional arbitrariness in each order.

3. CONCLUSION

Let us now show that our solution of the gravitational equations is general and hence, describes an arbitrary gravitational field in the vacuum. The criterion for the generality of the solution is the presence of four arbitrary functions of three variables. Owing to the symmetry of the theory in all four coordinates, it is immaterial on which three variables these functions will depend, on all three space variables or on two space variables and the time variable. Both cases are physically equivalent.

Let us consider the solution (22) and show that it contains exactly four arbitrary functions of the variables x, y, and t. We shall consider only the components g_{ab}, since g_{a3} and g_{33} are completely determined by the arbitrariness in g_{ab} and contain no "own" functions. Using $j^{(1)}_{ab} = 0$, we write g_{ab} in the form

$$g_{11} = q + \sum_{k=1}^{\infty} t^k (\Phi_{11}^{(k)} e^{kz} + F_{11}^{(k)} e^{-kz}) + \sum_{k=2}^{\infty} t^k j_{11}^{(k)},$$

$$g_{12} = \sum_{k=1}^{\infty} t^k (\Phi_{12}^{(k)} e^{kz} + F_{12}^{(k)} e^{-kz}) + \sum_{k=2}^{\infty} t^k j_{12}^{(k)},$$

$$g_{22} = q + t(\Phi_{22}^{(1)} e^{z} + F_{22}^{(1)} e^{-z}) + \sum_{k=2}^{\infty} t^k (\Phi_{22}^{(k)} e^{kz} + F_{22}^{(k)} e^{-kz}) + \sum_{k=2}^{\infty} t^k j_{22}^{(k)}. \tag{33}$$

It was shown earlier that not all six functions $\Phi^{(k)}_{ab}$ and $F^{(k)}_{ab}$ are arbitrary in each order k, since the equations $R^0_\alpha = 0$ and $R^0_0 = 0$ impose on them two algebraic conditions in each order, beginning with k = 2. Using these, we can express all functions $\Phi^{(k)}_{22}$ and $F^{(k)}_{22}$ (k = 2, 3, ...) through $\Phi^{(1)}_{22}$, $F^{(1)}_{22}$, q, $\Phi^{(k)}_{11}$, $\Phi^{(k)}_{12}$, $F^{(k)}_{11}$, $F^{(k)}_{12}$ (k = 1, 2, ...). Since, as explained above, in the linear order in the time all six functions $\Phi^{(1)}_{ab}$ and $F^{(1)}_{ab}$ are arbitrary, the component g_{22} can be written in the form

$$g_{22} = q + t(\Phi_{22}^{(1)} e^{z} + F_{22}^{(1)} e^{-z}) + t^2 G_{22}, \tag{34}$$

where the time dependent function G_{22} is completely determined by q, $\Phi^{(1)}_{22}$, $F^{(1)}_{22}$, $\Phi^{(k)}_{11}$, $\Phi^{(k)}_{12}$, $F^{(k)}_{11}$, $F^{(k)}_{12}$ (k = 1, 2, ...), which remain arbitrary.

We now expand the exponents in g_{11} and g_{22} in powers of z. Then these components can be written in the form

$$g_{11} = q + \sum_{k=1}^{\infty} t^k (\Phi_{11}^{(k)} + F_{11}^{(k)}) + z \sum_{k=1}^{\infty} t^k (\Phi_{11}^{(k)} - F_{11}^{(k)}) k + tG_{11},$$

$$g_{12} = \sum_{k=1}^{\infty} t^k (\Phi_{12}^{(k)} + F_{12}^{(k)}) + z \sum_{k=1}^{\infty} t^k (\Phi_{12}^{(k)} - F_{12}^{(k)}) k + tG_{12}, \tag{35}$$

where the t and z dependent functions G_{11} and G_{12} are completely determined by the preceding terms and the functions q, $\Phi^{(1)}_{22}$ and $F^{(1)}_{22}$. Reforming the sums and differences in these expressions, we can bring them to the form

$$g_{11} = q + \sum_{k=1}^{\infty} t^k D_{11}^{(k)} + z \sum_{k=1}^{\infty} t^k B_{11}^{(k)} + tG_{11},$$

$$g_{12} = \sum_{k=1}^{\infty} t^k D_{12}^{(k)} + z \sum_{k=1}^{\infty} t^k B_{12}^{(k)} + tG_{12}, \tag{36}$$

where all coefficients of the expansion $D^{(k)}_{11}$, $B^{(k)}_{1}$, $D^{(k)}_{12}$, $B^{(k)}_{12}$ (k = 1, 2, ...) are arbitrary two-dimensional functions of the variables x and y, and this being the case, these expansions are arbitrary functions of the three variables x, y, and t. They are, to be sure, not completely arbitrary, since the expansions do not contain the free time-independent terms. This will be discussed below.

Finally, the components g_{ab} can be written in the following form, which clearly exhibits the physical arbitrariness:

$$g_{11} = D_{11}(x, y, t) + zB_{11}(x, y, t) + tG_{11},$$
$$g_{12} = D_{12}(x, y, t) + zB_{12}(x, y, t) + tG_{12},$$
$$g_{22} = q + t[\Phi_{22}^{(1)} e^z + F_{22}^{(1)} e^{-z}] + t^2 G_{22}. \tag{37}$$

It can now be said that G_{11}, G_{12}, G_{22} and also g_{a3} and g_{33} are completely determined by the functions D_{11}, B_{11}, D_{12}, B_{12}, $\Phi_{22}^{(1)}$, $F_{22}^{(1)}$, q and the entire physical arbitrariness of the solution is determined by three four-dimensional functions of x, y, t, viz., D_{11}, B_{11}, D_{12}, and B_{12} and three two-dimensional functions of x, y, viz., $\Phi_{22}^{(1)}$, $F_{22}^{(1)}$, and q. The possibility of such a form of the solution has already been noted earlier, at the end of the preceding section, on the basis of other considerations of more general character.

The functions D_{ab} and B_{ab} in the solution (37) are not completely arbitrary, as already mentioned. They satisfy the conditions

$$D_{11}(x, y, 0) = q, \quad B_{11}(x, y, 0) = 0,$$
$$D_{12}(x, y, 0) = 0, \quad B_{12}(x, y, 0) = 0.$$

However, these conditions do not constitute any essential physical restriction. They are connected with our choice of coordinate system, which in turn was made possible by the simultaneity of the singularity. They imply that the closer we come to the singularity at t = 0, the less we can deform the space-time, since arbitrary three-dimensional deformations are incompatible with the two-dimensionality of the caustic, and arbitrary two-dimensional deformations (of x and y) are not physical and can be removed by a special choice of the coordinate system.

Writing the solution (22) in the form (37), we therefore see that it contains the necessary amount of arbitrariness to be the general solution of the gravitational equations in vacuum.

The authors express their deep gratitude to Professor E. M. Lifshitz for a discussion.

[1] E. M. Lifshitz and I. M. Khalatnikov, UFN **80**, 391 (1963), Soviet Phys. Uspekhi

[2] Lifshitz, Sudakov, and Khalatnikov, JETP **40**, 1847 (1961), Soviet Phys. JETP **13**, 1298 (1961).

Translated by R. Lipperheide
128