ITERATION METHOD IN NONRENORMALIZABLE FIELD THEORY

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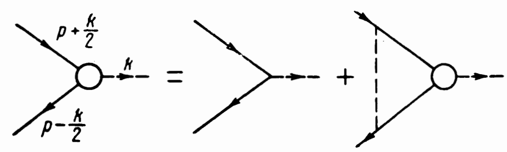
An iteration method in nonrenormalizable field theory is developed, with the example of the linear equation for the vertex function in the theory of the interaction of scalar and vector particles. A correct choice of the zeroth approximation assures the convergence of the iterations and the possibility of applying the Fredholm method to the exact integral equation. A simple modification of perturbation theory enables us to find the expansion of the vertex function in a series of powers of the coupling constant λ and of $\ln \lambda$.

1. INTRODUCTION

IN a previous paper, [1] which will hereafter be referred to as I, a study was made of the Edwards approximate equation for the vertex function in the nonrenormalizable theory of the interaction of scalar and vector particles. This equation is represented in the form of diagrams in the figure. As in I, we shall consider the case $k_\mu = 0$, which decidedly simplifies the calculations.



The Edwards equation for the vertex part $\Gamma_\mu(p, k)$. Dashed lines correspond to free vector particles of mass m, and solid lines to scalar particles of mass M.

The invariant function

$$F(p^2) = \frac{1}{2p^2} p_\mu \Gamma_\mu(p, 0) \tag{1.1}$$

satisfies the equation

$$F = A + K^{(0)}F + K'F; \tag{1.2}$$

$$A = Z + \frac{a\lambda^2}{m^2p^2}\int \frac{d^4q}{(2\pi)^4} \frac{2(pq)^2}{(q^2+M^2)^2}F(q^2), \tag{1.3}$$

$$K^{(0)}F = \frac{a\lambda^2}{m^2p^2}\int \frac{d^4q}{(2\pi)^4}\left[\frac{(p^2-q^2)^2}{(p-q)^2}(pq) - 2(pq)^2\right] \times \frac{F(q^2)}{(q^2+M^2)^2}, \tag{1.4}$$

$$K'F = \frac{4a\lambda^2}{p^2}\int \frac{d^4q}{(2\pi)^4} \frac{p^2q^2(pq)-(pq)^3}{(p-q)^2[(p-q)^2+m^2]} \times \frac{F(q^2)}{(q^2+M^2)^2}. \tag{1.5}$$

Here Z is the vertex renormalization constant, λ is the interaction constant; a = 1 for SU_2 symmetry of the theory, and a = $^3/_2$ for SU_3 symmetry; and p and q are Euclidean four-dimensional momenta. The constant A is in principle fixed by the condition of normalization of the vertex on the mass shell.

We have here broken the kernel in the equation (1.2) up into two parts: 1) the part most singular for large p and q, i.e., the integral in the right member of Eq. (1.3) and the kernel $K^{(0)}$; 2) the less singular part, the kernel K'. The most singular part is of positive dimensions in the variables p and q (for p, q $\to \infty$), and therefore gives power-law divergences when (1.2) is iterated. The less singular part is dimensionless in the large variables p and q, and in this sense is analogous to the corresponding kernels encountered in renormalizable theories.

The fundamental idea of I is that one must choose as the zeroth approximation $F^{(0)}$ to the vertex the solution of the equation with the most singular kernel:

$$F^{(0)} = A + K^{(0)}F^{(0)}. \tag{1.6}$$

For the calculation of the exact function F we then transform Eq. (1.2) by operating on it with the resolvent $R = (1 - K^{(0)})^{-1}$ of the kernel $K^{(0)}$.[1)] Using Eq. (1.6), we get the equation

$$F = F^{(0)} + (1 - K^{(0)})^{-1}K'F = F^{(0)} + RK'F, \tag{1.7}$$

[1)]It has been shown earlier in I that Eq. (1.6) has no solution for A = 0. The existence of the resolvent of the operator $K^{(0)}$ follows from this.

which by iterations gives the correction functions $F^{(n)}$ considered in I. In that paper we showed that for $A \neq 0$ Eq. (1.6) has a unique solution $F^{(0)}(p^2)$ that falls off for $p^2 \to \infty$ and contains a logarithmic branch point of the type $\lambda^2 \ln \lambda^2$.

It was pointed out in I that an arbitrary iteration $F^{(n)}$ of Eq. (1.7) exists, but the convergence of the iteration series

$$F = \sum_{n=0}^{\infty} F^{(n)} \tag{1.8}$$

was not investigated. Moreover, the expansion of the function $F^{(0)}$ for small λ was found only in the case $M = 0$. We shall now examine the properties of Eq. (1.7) and of its iteration solution (1.8) in more detail. In particular this will enable us to make a calculation of the expansion of the exact function F for small λ.

2. THE RESOLVENT OF THE KERNEL $K^{(0)}$

As was found in I, after integration over the angle variables in a four-dimensional spherical coordinate system the kernel $K^{(0)}$ in Eq. (1.6) can be written in the form

$$K^{(0)}(x, y) = \frac{g^2}{12}\frac{y^2}{(y+M^2)^2}\left[\left(\frac{y^2}{x^2} - 2\frac{y}{x}\right)\vartheta(x-y) + \left(\frac{x^2}{y^2} - 2\frac{x}{y}\right)\vartheta(y-x)\right], \tag{2.1}$$

$$g^2 = 3a\lambda^2/8m^2\pi^2, \quad x = p^2, \quad y = q^2. \tag{2.1a}$$

The presence of the ϑ function allows us to reduce (cf. I) this integral equation to a differential equation

$$\frac{1}{x}\frac{d^2}{dx^2}\left[\frac{1}{x}\frac{d^2}{dx^2}(x^2F^{(0)})\right] + \frac{g^2F^{(0)}}{x(x+M^2)^2} = \frac{1}{x}\frac{d^2}{dx^2}\left[\frac{1}{x}\frac{d^2}{dx^2}(x^2A)\right] \tag{2.2}$$

with the boundary conditions

$$F^{(0)}(x) \underset{x\to\infty}{\to} 0; \tag{2.3a}$$

$$F^{(0)}(x) \text{ bounded for } x \to 0 \tag{2.3b}$$

We note that if in (1.6) we replace the constant A by an arbitrary function f(x), the corresponding function $F^{(0)}(x)$ will satisfy (2.2) with A replaced by f(x). We make use of this fact in calculating the resolvent. In fact, the resolvent is determined by the relation $F^{(0)} = Rf$. Therefore to determine R it suffices to find the solution of Eq. (2.2) with the boundary conditions for an arbitrary function f.

Rewriting the equation for $F^{(0)}$ in the form

$$\frac{1}{x}\frac{d^2}{dx^2}\left\{\frac{1}{x}\frac{d^2}{dx^2}[x^2(\bar{F}^{(0)} - f)]\right\} + \frac{g^2(\bar{F}^{(0)} - f)}{x(x+M^2)^2} = -\frac{g^2 f}{x(x+M^2)^2}, \tag{2.4}$$

we express the solution $F^{(0)}$ in terms of f by means of the Green's function G(x, y) of the boundary problem (2.2)—(2.3) (for A = 0):

$$\bar{F}^{(0)}(x) = f(x) - g^2\int_0^\infty dy \frac{G(x, y)}{y(y+M^2)^2} f(y). \tag{2.5}$$

Knowing the linearly independent solutions F_i of the homogeneous equation obtained from (2.2) by setting A = 0, we can easily construct the Green's function (cf. e.g., the book by Ince[2]):

$$G(x, y) = \vartheta(x-y)\sum_{i=1,2} F_i(x)\frac{W_i(y)}{W(y)} - \vartheta(y-x)\sum_{i=3,4} F_i(x)\frac{W_i(y)}{W(y)}. \tag{2.6}$$

Here W(y) is the Wronskian determinant of the solutions $F_i(y)$, and $W_i(y)$ are the algebraic complements of the elements $d^3F_i(y)/dy^3$ in the determinant W(y). The linearly independent solutions have the asymptotic forms

$$F_{1,2}(x) \underset{x\to\infty}{\approx} (g^2x)^{-3/8}\exp[-4e^{\pm i\pi/4}(g^2x)^{1/4}],$$

$$F_{3,4}(x) \underset{x\to\infty}{\approx} (g^2x)^{-3/8}\exp[4e^{\pm i\pi/4}(g^2x)^{1/4}]. \tag{2.7}$$

In addition, the solutions $F_{3,4}(x)$ satisfy the condition (2.3b). (That it is possible to choose the linearly independent solutions in this way follows from the results of I.)

It is not hard to verify that the right member of (2.6) is invariant under an arbitrary nonsingular linear transformation

$$F_i'(x) = \sum_{j=1,2} c_{ij}F_j(x), \quad i = 1,2;$$

$$F_k'(x) = \sum_{l=3,4} d_{kl}F_l(x), \quad k = 3,4. \tag{2.8}$$

We shall make use of this last remark later in studying the behavior of the function G(x, y) in various regions of variation of the variables x and y.

The previously obtained relation (2.5) gives the following representation for the resolvent:

$$R(x, y) = \delta(x-y) - g^2\frac{G(x, y)}{y(y+M^2)^2}. \tag{2.9}$$

In what follows we shall also use a different representation for R, which is obtained from the direct solution of an equation of the type (2.2):

$$Rf=\int_0^\infty dy\,G(x,y)\frac{1}{y}\frac{d^2}{dy^2}\left[\frac{1}{y}\frac{d^2}{dy^2}(y^2f(y))\right]. \qquad (2.10)$$

In both cases we have first to study the function G(x, y).

By means of the formulas (2.6) and (2.7) we can easily find the asymptotic behavior of the Green's function G(x, y) for large values of x and y:

$$G(x\ y)\underset{\substack{x\to\infty\\ y\to\infty}}{\approx}\frac{x^{-3/8}y^{7/8}}{4g^{3/2}}\Big\{\vartheta(x-y)\exp\Big[4e^{i\pi/4}\Big((g^2y)^{1/4}-(g^2x)^{1/4}\Big)+i\frac{\pi}{4}\Big]+\vartheta(y-x)\exp\Big[4e^{i\pi/4}((g^2x)^{1/4}-(g^2y)^{1/4})+i\frac{\pi}{4}\Big]+\text{c.c.}\Big\}. \qquad (2.11)$$

To determine the asymptotic behavior of the Green's function for small x and y we use the transformation (2.8) and choose linearly independent solutions $F_i'(x)$ and $F_k'(x)$ which behave as follows for small x:

$$F_1'(x)\approx(g^2x)^{-2},\quad F_2'(x)\underset{x\to 0}{\approx}1/g^2x;\quad F_3'(x)\underset{x\to 0}{\approx}g^2x,\quad F_4'(x)\approx(g^2x)^2. \qquad (2.12)$$

Omitting the simple calculations, we write out the final expression for the Green's function for $x\to 0$ and $y\to 0$:

$$G(x,y)\underset{\substack{x\to 0\\ y\to 0}}{\approx}\frac{1}{12}\Big[\vartheta(x-y)\Big(2\frac{y^4}{x}-\frac{y^5}{x^2}\Big)+\vartheta(y-x)(2xy^2-x^2y)\Big]. \qquad (2.13)$$

By analogous calculations one can find the behavior of G(x, y) for $x\to 0$, $y\to\infty$ or $x\to\infty$, $y\to 0$. Since the resolvent is expressed in terms of the Green's function, our asymptotic formulas for G determine the asymptotic properties of the resolvent R.

3. PROOF OF THE CONVERGENCE OF THE ITERATIONS

In this section it will be shown that the kernel K = RK′ of Eq. (1.7) is quadratically integrable. Using the known properties of the zeroth approximation $F^{(0)}$, we then establish the possibility of applying the Fredholm method to this equation, and in addition prove the convergence of the iteration series (1.8) for sufficiently small values of λ.

We first study the kernel K′(x, y). Performing the integration over the angles in (1.5), we get for K′(x, y) the expression

$$K'(x,y)=\frac{g^2}{12}\frac{y^2}{(y+M^2)^2}[h(\xi)-h(\xi_0)], \qquad (3.1)$$

$$h(\xi)=3-12\xi^2+8\xi^4-8\xi(\xi^2-1)^{3/2},$$

$$\xi=\frac{x+y+m^2}{2\sqrt{xy}},\quad \xi_0=\frac{x+y}{2\sqrt{xy}}. \qquad (3.1a)$$

In the case in which $x+y\gg m^2$, it is not hard to find a convenient asymptotic representation for K′(x, y):

$$K'(x,y)\approx\frac{g^2m^2}{3}\frac{y^2}{(y+M^2)^2}\times\Big[\frac{y}{x^2}\vartheta(x-y)+\frac{x}{y^2}\vartheta(y-x)\Big]. \qquad (3.2)$$

In the case $x+y\ll m^2$ we get a different representation for (3.1):

$$K'(x,y)\underset{x+y\ll m^2}{\approx}\frac{g^2}{12M^4}\times\Big[\Big(2\frac{y^3}{x}-\frac{y^4}{x^2}\Big)\vartheta(x-y)+(2xy-x^2)\vartheta(y-x)\Big]. \qquad (3.3)$$

It follows from the formula (2.10) for the resolvent that the kernel $\overline{K}$ can be written in the form

$$\overline{K}(x,y)=\int_0^\infty dz\,G(x,z)\frac{1}{z}\frac{d^2}{dz^2}\Big[\frac{1}{z}\frac{d^2}{dz^2}(z^2K'(z,y))\Big]. \qquad (3.4)$$

The asymptotic representations (2.11) and (3.2) enable us to find the asymptotic behavior of $\overline{K}$ for $x\to\infty$, $y\to\infty$:

$$\overline{K}(x,y)\underset{\substack{x\to\infty\\ y\to\infty}}{\approx}\frac{g^{3/2}m^2}{4}x^{-3/8}y^{-7/8}\Big\{\vartheta(x-y)\exp\Big[4e^{i\pi/4}((g^2y)^{1/4}-(g^2x)^{1/4})+i\frac{3\pi}{4}\Big]+\vartheta(y-x)\exp\Big[4e^{i\pi/4}((g^2x)^{1/4}-(g^2y)^{1/4})+i\frac{3\pi}{4}\Big]+\text{c.c.}\Big\}. \qquad (3.5)$$

Similarly, we find from (2.13) and (3.3) that for $x\to 0$, $y\to 0$:

$$\overline{K}(x,y)\underset{\substack{x\to 0\\ y\to 0}}{\approx}\frac{g^2}{12M^4}\Big\{\Big(2\frac{y^3}{x}-\frac{y^4}{x^2}\Big)\vartheta(x-y)+(2xy-x^2)\vartheta(y-x)\Big\}. \qquad (3.6)$$

We shall not write out here the easily obtainable asymptotic expressions for $\bar{K}(x, y)$ in the regions $x \to 0$, $y \to \infty$ and $x \to \infty$, $y \to 0$. It is not hard to verify that inclusion of these regions does not change the results presented below.

When the asymptotic properties of the kernel $\bar{K}$ which we have found are used, it can be shown without difficulty that its norm $\|K\|$ is finite, i.e., that

$$\|\bar{K}\|^2 = \int_0^\infty dx \int_0^\infty dy\, |\bar{K}(x, y)|^2 < \infty. \tag{3.7}$$

Since $F^{(0)}(x)$ is a quadratically integrable function, this allows us [3] to apply the Fredholm method to Eq. (1.7). In what follows we shall examine in more detail the possibilities opened up by this result. For the study of the convergence of the iteration series it is essential, however, to know the behavior of the norm of the kernel K for small values of the coupling constant λ. Simple calculations given in the Appendix lead to the following rough estimate:

$$\|\bar{K}\|^2 < C\lambda^4 |\ln \lambda|, \tag{3.8}$$

where C is a dimensionless constant which does not depend on λ. It follows from this that for sufficiently small values of λ the inequality $\|K\| < 1$ will be satisfied, and this is sufficient for the convergence of the iteration series (1.8). We note that the convergence is uniform in x in the interval $0 \le x \le \infty$.

As was shown earlier (in I), the iteration $F^{(n)}(x)$ falls off more rapidly than $F^{(n-1)}(x)$ as $x \to \infty$. Owing to the uniform convergence of the iteration series (for sufficiently small values of λ), its asymptotic behavior is determined by that of the zeroth approximation $F^{(0)}(x)$. This result is of importance for the justification of the passage to a Euclidean metric in Eq. (1.2). This change was accomplished by means of a rotation of the path of integration through the angle $\pi/2$, in the complex planes of the variables p_0 and q_0. [4] In I we proved that such a transformation is possible for the zeroth-order equation (1.6). The fact that the asymptotic behavior of the exact solution $F(x)$ is the same as that of the zeroth-order solution $F^{(0)}(x)$ enables us to justify the rotation of the path for the complete equation (1.2).

We note that the possibility of applying the Fredholm method greatly facilitates the investigation of the analytic properties of the function $F(x)$. There is no difference in principle between this problem and that treated earlier, [5] and we shall not concern ourselves with it here. In concluding this section we merely emphasize that the Fredholm method can be applied for arbitrary finite λ, and this enables us to study the properties of (1.2) for the vertex function in the case of strong coupling.

4. THE EXPANSION OF THE SOLUTION FOR SMALL VALUES OF THE COUPLING CONSTANT

This section is devoted to the calculation of the radiative corrections to the vertex function $F(x)$ for small values of λ. In I the expansions of $F^{(0)}(x)$ was calculated in the approximation $M = 0$:

$$F^{(0)}(x) \underset{M=0}{=} F_0^{(0)}(x) = A\left[1 + \frac{g^2x}{6}\ln(m^2g^2) + \frac{g^2x}{6}\left(\ln\frac{x}{m^2} + 4\gamma - \frac{10}{3}\right) + o(g^2m^2)\right]. \tag{4.1}$$

[$\gamma = 0.577\ldots$ is the Euler constant, and the constant g^2 is connected with λ^2 by Eq. (2.1a)]. We stated there without proof that neither making $M \ne 0$ nor including the correction kernel K' would affect the nonanalytic term $(g^2x/t)\ln(m^2g^2)$. We shall now prove this assertion, and at the same time calculate the terms of order g^2 in the exact solution $F(x)$. The method used in principle allows us to write out the complete expansion of the vertex function F in a series of powers of λ^2 and $\ln \lambda^2$. The terms of higher order in λ^2, however, contain contributions not only from the diagrams shown in the figure, but also from diagrams of other classes. Therefore we confine ourselves to the calculation of the terms of orders $\lambda^2 \ln \lambda^2$ and λ^2.

For this purpose we use the representation of the resolvent in the form (2.9) and write Eq. (1.7) in the form

$$F(x) = F^{(0)}(x) + \int_0^\infty dy\, K'(x, y)F(y) - g^2\int_0^\infty dy \int_0^\infty dz\, \frac{G(x, z)K'(z, y)}{z(z + M^2)^2}F(y). \tag{4.2}$$

This equation can be solved by iterations, when we take as known the expansion of the zeroth-approximation function $F^{(0)}$. It is obvious that only the first and second terms in the right member of (4.2) can contribute to the orders in which we are interested; the contribution of the third term is proportional to λ^4.

Let us first calculate the necessary terms of the expansion of $F^{(0)}(x)$, using the expansion (4.1) for $F_0^{(0)}(x)$. It is not hard to verify that $F^{(0)}(x)$ satisfies an equation analogous to (4.2):

$$F^{(0)}(x) = F_0^{(0)}(x) + \int_0^\infty dy\, K_0'(x, y)F^{(0)}(y) - g^2\int_0^\infty dy \int_0^\infty dz\, \frac{G_0(x, z)K_0'(z, y)}{z^3}F^{(0)}(y), \tag{4.3}$$

where we have introduced the following notations:

$$K_0'(x,y) = K^{(0)}(x,y) - K^{(0)}(x,y)\,|_{M=0}$$

$$= -\frac{g^2M^2}{12}\frac{2y+M^2}{(y+M^2)^2}\left[\left(\frac{y^2}{x^2}-2\frac{y}{x}\right)\vartheta(x-y)\right.$$

$$\left.+\left(\frac{x^2}{y^2}-2\frac{x}{y}\right)\vartheta(y-x)\right],$$

$$G_0(x,y) = G(x,y)\,|_{M=0}. \tag{4.4}$$

Accordingly, the corrections to the terms calculated above are given by the expression

$$A\int_0^\infty dy\,(K'(x,y)+K_0'(x,y)). \tag{4.5}$$

It is not hard to verify that

$$K'+K_0' = \frac{g^2}{12}\frac{y^2}{(y+M^2)^2}h(\xi)+\frac{g^2}{12}\left[\left(2\frac{y}{x}-\frac{y^2}{x^2}\right)\vartheta(x-y)\right.$$

$$\left.+\left(2\frac{x}{y}-\frac{x^2}{y^2}\right)\vartheta(y-x)\right]. \tag{4.6}$$

Omitting straightforward but tedious calculations, we give the final form of the expansion of the function F in the case M = m (which we consider to make the notation simpler):

$$F(x) = A\left\{1+\frac{g^2x}{6}\ln(g^2m^2)-\frac{g^2x}{6}\left(4\gamma+\frac{10}{3}\right)+\frac{2g^2m^2}{3}\right.$$

$$+\frac{g^2}{12}(x-2m^2)\left(\frac{x+4m^2}{x}\right)^{1/2}$$

$$\times\ln\left[\frac{(x+2m^2)[x(x+4m^2)]^{1/2}+(x+2m^2)^2}{2m^4}-1\right]$$

$$\left.+o(g^2)\right\}. \tag{4.7}$$

The results presented in this section show that for the calculation of the terms which are nonanalytic in the coupling constant one needs the exact solution of a rather simple differential equation (given in I), while the inclusion of subsequent corrections is not more complicated than ordinary perturbation theory and reduces to the calculation of convergent integrals.

5. CONCLUSION

In conclusion we discuss the main results of this paper and the preceding paper (see I). We give a brief characterization of the general scheme of solution of approximate linear equations in nonrenormalizable theories, on the basis of the detailed investigation of the equation for the vertex function.

The first step is to separate the kernel of the integral equation into the most singular part and a less singular part. The principle of this separation is formulated in the Introduction. Our approach here is essentially different from that proposed by Pais and Feinberg, [6] whose main point was to keep in the equation for the zeroth approximation the part of the kernel that gives the highest degree of divergence in each order of perturbation theory. According to the recipe of Pais and Feinberg we would thus have to take instead of the kernel $K^{(0)}$ defined by Eq. (1.4) the simpler expression

$$\frac{a\lambda^2}{m^2p^2}\int\frac{d^4q}{(2\pi)^4}\frac{(pq)}{(p-q)^2}F(q^2).$$

For the corresponding zeroth approximation we would then get a simpler second-order differential equation. This equation has a solution which satisfies the necessary boundary conditions. There is, however, no resemblance between the asymptotic behavior of this solution for large values of x and the correct asymptotic form. In particular, for this solution the rotation of the path of integration needed for the transition to Euclidean momentum vectors is impossible. Therefore it is obvious that in this case the iteration series cannot be convergent. Roughly speaking, the main shortcoming of the method of Pais and Feinberg is that the terms they have dropped affect the asymptotic behavior for large momenta to the same degree as those they include. Our rule simply reduces to recognition that the variables p and q are on an equal footing in the kernel, and assures the correct asymptotic behavior of the solution at infinity.

The subsequent steps, described in detail in the present paper, allow us to study the solution for arbitrary values of the coupling constant. In the case of weak coupling the solution can be calculated to any degree of accuracy by means of a modified perturbation theory which takes into account the nonanalytic dependence on the coupling constant. These calculations give an expansion in powers of λ^2 and $\ln\lambda^2$. 2)

The method we have described can be applied to a wide range of problems in various nonrenormalizable field theories. In particular, it can be used to study the scattering amplitude in nonrenormalizable theories. In this case the problem of finding the zeroth approximation also reduces to a differential equation with boundary conditions. (We have previously [8] considered similar equations in the nonrelativistic theory of scattering by a singular potential). All of the further steps are

2)Lee[7] was the first to call attention to the possibility of the appearance of terms of the form $\lambda^2\ln\lambda^2$ in nonrenormalizable theories.

also analogous to those considered here. We note that our restriction here to the case $k_\mu = 0$ is in principle unimportant, and use of $k_\mu \neq 0$ brings in only technical difficulties.

It seems to us that there is great promise in the possibility demonstrated here of applying the Fredholm method to equations of the type of (1.7). For $k^2 \neq 0$ the Fredholm denominator will depend on k^2 and other parameters of the problem, and its zeros will determine the energies of the bound states of the system. Thus in principle there is a possibility of solving the problem of bound states in a nonrenormalizable theory without the introduction of supplementary parameters (for cut-off, subtractions, and so on).

The writers express their sincere gratitude to N. N. Bogolyubov for fruitful discussions.

APPENDIX

To derive the estimate (3.8) we examine in more detail the structure of the kernel $\overline{K}$ for $x + y \gg m^2$. We choose a constant L of the dimensions of mass so that $m^2, M^2 \ll L^2 \ll m^2/\lambda^2$. This is always possible for sufficiently small λ. We now break the region of integration in (3.7) into two parts:

$$1)\ x < L^2,\ y < L^2, \quad 2)\ x > L^2 \text{ or } y > L^2.$$

For small λ the representation (2.13) for the Green's function holds in the region $x, y \lesssim L^2$. Therefore the integral over the first region can be estimated in the following way:

$$\int_0^{L^2} dx \int_0^{L^2} dy\, |\overline{K}(x, y)|^2 < \lambda^4 C_1. \qquad (A.1)$$

In fact, λ^4 occurs in this integral only as a factor, while the integrand and the limits of the integration do not depend on λ.

In the second region the representation (3.2) holds for K'. Using the representation (3.4) and making some simple calculations, we find that in this region

$$\overline{K}(x, y) = \frac{a\lambda^2}{8\pi^2}\frac{1}{y^2}\left\{-\frac{\partial^2}{\partial y^2}G(x, y) + \frac{4}{y}\frac{\partial}{\partial y}G(x, y)\right\}. \qquad (A.2)$$

We note that in the second region the Green's function $G(x, y)$ depends on g^2 in the following way:

$$G(x, y) \approx g^{-6}\overline{G}(u, v),$$

where $\overline{G}(u, v) = G(g^2x, g^2y)$ and we have introduced the dimensionless variables $u = g^2x$, $v = g^2y$. Therefore simple calculations lead to the following expression:

$$\int\limits_{\substack{x>L^2\\ \text{or}\\ y>L^2}} dx \int dy\, |\overline{K}(x, y)|^2 = \left(\frac{a\lambda^2}{8\pi^2}\right)^2 \iint\limits_{\substack{u>g^2L^2\\ \text{or}\\ v>g^2L^2}} du\, dv\, |\varphi(u, v)|^2; \qquad (A.3)$$

$$\varphi(u, v) = \frac{1}{v^2}\left\{-\frac{\partial^2}{dv^2}\overline{G}(u, v) + \frac{4}{v}\frac{\partial}{\partial v}\overline{G}(u, v)\right\}. \qquad (A.4)$$

Knowing that the integral in (A.3) converges, and using Eq. (2.13), which is still valid for $x, y \sim L^2$, we find that (A.3) gives the following estimate:

$$\int dx \int dy\, |\overline{K}(x, y)|^2 < \lambda^4 (C_2 |\ln L^2 g^2| + C_3). \qquad (A.5)$$

From (A.1) and (A.5) we get the estimate (3.8).

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