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CRITICAL FIELDS OF THIN SUPERCONDUCTING FILMS

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Critical fields for a phase transition of the second kind are determined for thin films over the whole temperature range. Both pure films as well as those containing various concentrations of impurities are considered. The reflection of electrons from the surface of the film is assumed to be diffuse.

THIS article is devoted to a determination of the critical fields of thin films (the thickness L of the film is much smaller than the BCS correlation parameter ξ_0) under the assumption of diffuse reflection of the electrons from the surface of the film. As is well known, for thin films the transition from the normal state to the superconducting state in a magnetic field parallel to the film is a phase transition of the second kind. Thus, the critical field as a function of temperature determines the boundary of instability of the normal phase to the appearance of a superconducting correlation in the TH phase plane.

General equations are obtained in Sec. 1, based on a method proposed earlier by the author, allowing one to describe the diffuse reflection of electrons from a surface.[1] In Sec. 2 the critical field for pure samples is determined over the whole temperature range, and in Sec. 3 it is found for films containing various concentrations of impurities.

The problem posed here was recently investigated by a similar method in the article by de Gennes and Tinkham;[2] however, the results obtained there for pure films are incorrect; in particular, the error is related to an unwarranted assumption that the ordering parameter $\Delta(\mathbf{r})$ is constant along the thickness of the film. In addition, de Gennes and Tinkham[2] (following Nambu and Tuan,[3] who investigated the analogous problem for specular reflection) erroneously interpreted the divergence of the critical field of pure films when T → 0 as a "renormalization of the interaction constant." The results obtained in [2] for contaminated films remain valid; however, the equation written out in Sec. 3 of the present article enables us to investigate in a unified way arbitrary concentrations of impurities and to obtain easily all possible limiting cases.

1. GENERAL EQUATIONS

Mathematically the posed problem reduces to the determination of the maximum magnetic field H in which superconductivity exists at a given temperature T, i.e., a nontrivial solution of the following integral equation[1)] exists (see, for example, the article by Gor'kov[4]):

$$\Delta^*(\mathbf{r}_1) = |\lambda| T \sum_{\omega} \int \mathfrak{G}_{\omega}(\mathbf{r}_1, \mathbf{r}_2) \mathfrak{G}_{-\omega}(\mathbf{r}_1, \mathbf{r}_2) \Delta^*(\mathbf{r}_2) d\mathbf{r}_2. \quad (1.1)$$

In this equation, the integartion is carried out over the entire volume of the sample, λ is the four-fermion interaction constant, $\omega = \pi T(2n + 1)$, and $\mathfrak{G}_{\omega}(\mathbf{r}_1, \mathbf{r}_2)$ is the thermodynamic Green's function of an electron for the normal metal in a magnetic field. In the case under consideration of a sample of finite dimensions, it is convenient to express the Green's function in terms of the eigenfunctions $\psi_n(\mathbf{r})$ of an electron in the volume under consideration in the presence of a magnetic field:[1]

$$\mathfrak{G}_{\omega}(\mathbf{r}_1, \mathbf{r}_2) = \sum_n \frac{\psi_n(\mathbf{r}_1)\psi_n^*(\mathbf{r}_2)}{-i\omega - \xi_n} \quad (1.2)$$

Here ξ_n is the energy of the corresponding state, measured from the Fermi surface; the summation is carried out over all states.

Substituting (1.2) in the right side of the integral equation (1.1) and integrating over the volume, we would arrive in the absence of a magnetic field at the matrix elements of $\Delta^*(\mathbf{r})$, for then the eigenfunctions $\psi_n(\mathbf{r})$ could be chosen to be real (see [1]). In the presence of a magnetic field, it is necessary to introduce the complex conjugation operator[2]

$$K\psi(\mathbf{r}) = \psi^*(\mathbf{r}) \quad (1.3)$$

[1)] Of course, such a formulation is equivalent to the determination of the maximum temperature T for which a nontrivial solution of Eq. (1.1) exists in a given magnetic field H.

in order to reduce the right side of Eq. (1.1) to matrix elements. Then Eq. (1.1) takes the form

$$\Delta^*(\mathbf{r}_1) = |\lambda| T \sum_{\omega} \sum_{nm} \frac{1}{(\xi_n - i\omega)(\xi_m + i\omega)} \times \langle n|\Delta^* K|m\rangle \langle m|K^+ \delta(\mathbf{r}_1 - \mathbf{r}) n\rangle. \tag{1.4}$$

Here, as indicated by de Gennes and Tinkham,[2] we put

$$\psi_n^*(\mathbf{r}_1)\psi_m(\mathbf{r}_1) = \langle n|\delta(\mathbf{r}_1 - \mathbf{r})|m\rangle. \tag{1.5}$$

As shown earlier,[1] in the quasi-classical approximation the product of two matrix elements reduces to the Fourier component of the correlation function of the corresponding physical quantities, averaged along all possible classical trajectories for the motion of a particle with Fermi velocity v:

$$\langle n|f|m\rangle \langle m|g|n\rangle = \frac{1}{2\pi\nu V} \int_{-\infty}^{+\infty} \overline{f(t) g(t+\tau)} e^{-i(\xi_n - \xi_m)\tau} d\tau, \tag{1.6}$$

where

$$\nu = mp_0 / 2\pi^2 \tag{1.7}$$

is the number of states per unit volume in a unit energy interval, and the bar indicates both the average with respect to the time t as well as the average over all possible classical trajectories.

Since the δ-function

$$f(t) = f_0(t)\delta(\mathbf{r}(t) - \mathbf{r}_1) \tag{1.8}$$

enters into the product of matrix elements on the right side of Eq. (1.4), we obtain by averaging with respect to the time t,

$$\langle n|f|m\rangle \langle m|g|n\rangle = \frac{1}{2\pi\nu} \int_{-\infty}^{+\infty} \overline{f_0(\mathbf{r}_1) g(\mathbf{r}(t))} e^{-i(\xi_n - \xi_m)t} dt, \tag{1.9}$$

where the averaging is carried out over all trajectories passing through the point $\mathbf{r}_1$ at the moment t = 0.

It is not difficult to find the complex conjugation operator in the quasi-classical approximation if one takes into consideration that, in this approximation, the eigenfunctions of an electron in a magnetic field have the form

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) e^{i\varphi(\mathbf{r})}, \tag{1.10}$$

where $\psi_0(\mathbf{r})$ is the eigenfunction in the absence of the magnetic field, and the phase $\varphi(\mathbf{r})$ is determined by the equation

$$\left(\nabla\varphi(\mathbf{r}) - \frac{e}{c}\mathbf{A}(\mathbf{r})\right)\hat{\mathbf{p}}\psi_0(\mathbf{r}) = 0. \tag{1.11}$$

Then, putting $K(\mathbf{r}_1) = 1$, we obtain

$$K(\mathbf{r}(t)) = e^{-2i\varphi(t)}, \tag{1.12}$$

where

$$\varphi(t) = \frac{e}{c} \int_{\mathbf{r}_1}^{\mathbf{r}(t)} \mathbf{A} d\mathbf{l}, \tag{1.13}$$

and the integral is taken along the appropriate classical trajectory.

Taking what has been said into account, after summing over n and m one can reduce the integral equation (1.1) for $\Delta^*(\mathbf{r}_1)$ to the following form:

$$\frac{1}{\nu|\lambda|}\Delta^*(\mathbf{r}_1) = \int d\mathbf{r}_2 \Delta^*(\mathbf{r}_2) \times 4\pi T \sum_{\omega>0} \int_0^\infty dt\, e^{-2\omega t} \overline{e^{2i\varphi(t)} \delta(\mathbf{r}_2 - \mathbf{r}(t))}. \tag{1.14}$$

The expression appearing under the averaging sign represents the probability that a particle emitted from the point $\mathbf{r}_1$ at the moment t = 0, moving along a certain trajectory with Fermi velocity v, reaches the point $\mathbf{r}_2$ at the moment t, having "gained" in this connection the phase $2i\varphi(t)$. Averaging is carried out over all possible classical trajectories.

Since $\Delta^*(\mathbf{r})$ depends only on the z coordinate in the case of a plane film in a parallel magnetic field (the z axis is perpendicular to the plane of the film), we can integrate (1.14) over x and y:

$$\frac{1}{\nu|\lambda|}\Delta^*(z_1) = \int_{-L/2}^{+L/2} K(z_1, z_2)\Delta^*(z_2) dz_2, \tag{1.15}$$

$$K(z_1, z_2) = 4\pi T \sum_{\omega>0} \Phi(\omega; z_1, z_2), \tag{1.16}$$

$$\Phi(\omega; z_1, z_2) = \int_0^\infty dt\, e^{-2\omega t}\, \Phi(t; z_1, z_2), \tag{1.17}$$

$$\Phi(t; z_1, z_2) = \overline{e^{-2i\varphi(t)} \delta(z_2 - z(t))}. \tag{1.18}$$

In the absence of a magnetic field, $\Phi_0(t; z_1, z_2)$ is simply the probability that a particle emitted with equal probability in all directions from the point z_1 at the moment t = 0 is found at the point with coordinate z_2 at the moment t. In the presence of a magnetic field, the phase "gained" by the particle is taken into account.

The complete expression for $\Phi(\omega, z_1, z_2)$ in the case under consideration of a field parallel to the film (vector potential $A_x = Hz$, $A_z = A_y = 0$) is given in the Appendix, Eq. (A.1). The limiting values of this function for different relationships between the parameters are of the greatest interest.

Region I:

$$\Phi(\omega; z_1, z_2) = \frac{1}{2\omega L} \exp\left[-\frac{eH}{c}\left(\frac{L^2}{2} - z_1^2 - z_2^2\right)\right],$$

$$\frac{\omega L}{v} \ll \frac{eH}{c} L^2, 1; \tag{1.19a}$$

region II:

$$\Phi(\omega; z_1, z_2) = \Phi_0(\omega; z_1, z_2) - \frac{v}{32}\left(\frac{eH}{c}\right)^2 \frac{1}{\omega^2}\left(\frac{L^2}{2} + 2z_1z_2\right),$$

$$\frac{eH}{c}L^2 \ll \frac{\omega L}{v} \ll 1; \qquad (1.19b)$$

region III:

$$\Phi(\omega; z_1, z_2) = \Phi_0(\omega; z_1, z_2) - \frac{v^2}{6\omega^3}\left(\frac{eH}{c}\right)^2 z^2\delta_\omega(z_1 - z_2),$$

$$\frac{\omega L}{v} \gg 1, \frac{eH}{c}L^2. \qquad (1.19c)$$

Φ_0 is the corresponding function in the absence of the field. It has the property

$$\int_{-L/2}^{+L/2} \Phi_0(\omega; z_1, z_2)\, dz_2 = \frac{1}{2\omega}, \qquad (1.20)$$

which follows from the indicated physical interpretation of $\Phi_0(t)$ and Eq. (1.17). $\delta_\omega(z_1 - z_2)$ is a δ-function that falls off exponentially over distances on the order of v/ω. We note that the function $\Phi_0(\omega)$ has a similar character for $\omega L/v \gg 1$:

$$\Phi_0(\omega; z_1, z_2) \approx \frac{1}{2\omega}\delta(z_1 - z_2) + \frac{v^2}{24\omega^3}\delta''(z_1 - z_2). \qquad (1.21)$$

Here it is appropriate to indicate the limits of applicability of the expressions obtained for the function $\Phi(\omega;\ z_1,\ z_2)$. As is evident from Eqs. (2.1)—(2.2), because of the exponential decrease of the corresponding expressions for small values of μ, the major contribution is made by the electrons for which

$$\mu = \cos\theta \gtrsim \max(eHL^2/c,\ \omega L/v).$$

Our use of the quasi-classical approximation is valid so long as quantization of the projection of the electron momentum on the normal to the film surface is not important, i.e., so long as $\mu p_0 \gg L^{-1}$. Comparing with the previous estimate, we obtain the following condition for applicability of the expressions obtained for $\Phi(\omega;\ z_1,\ z_2)$:

$$\max(eHL^2/c,\ \omega L/v) \gg 1/p_0L, \qquad (1.22)$$

which gives for region I, expression (1.19a),

$$eHL^2/c \gg 1/p_0L, \qquad (1.22a)$$

and for region II, expression (1.19b),

$$\omega L/v \gg 1/p_0L; \qquad (1.22b)$$

condition (1.22) is always satisfied in region III.

2. PURE FILMS

The results obtained in the preceding section enable us to determine the critical field of pure films in which the mean free path of an electron is infinite. For relatively strong fields, when $eHL^2/c \sim 1$, values of the function $\Phi(\omega)$ for which $\omega \lesssim v/L$, given by expression (1.19a), give the major contribution to the kernel $K(z_1,\ z_2)$ given by Eq. (1.16), but when $\omega \gg v/L$ the function $\Phi(\omega)$ has the δ-function character (1.19c), so that the corresponding part of the sum contributes essentially to the left side of the integral equation (1.15). Thus the kernel $K(z_1,\ z_2)$ turns out to be degenerate (correct to terms of logarithmic order), and we can easily find the solution of the integral equation:

$$\Delta^*(z) \sim \exp\left\{-\frac{eH}{c}\left(\frac{L^2}{4} - z^2\right)\right\}. \qquad (2.1)$$

It is obvious therefore that for such fields the parameter $\Delta^*(z)$ depends significantly on z, reaching its maximum value at the boundary and its minimum in the middle of the film.

Next we find, correct to terms of logarithmic order, the equation which determines the limit of instability of the normal phase (i.e., the relation between the critical field and the temperature):

$$f\left(\frac{eH}{c}L^2\right) = \ln\frac{v}{LT_c} \Big/ \ln\frac{v}{LT}, \qquad \frac{eH}{c}L^2 \sim 1, \qquad (2.2)$$

$$f(x) = \int_0^1 \exp\left\{-\frac{x}{2}(1 - \xi^2)\right\}d\xi = \begin{cases} 1 - x/3, & x \ll 1 \\ 1/x, & x \gg 1 \end{cases}. \qquad (2.3)$$

The function f(x) is expressed in terms of the probability integral of imaginary argument.

It follows from Eq. (2.2) that the critical field increases logarithmically as $T \to 0$. For very large fields ($eHL^2/c \gg 1$), we can obtain by a variational method from the general expression (A.1) for $\Phi(\omega)$ the following expression for the critical field:

$$\frac{eH}{c}L^2 = \ln\frac{v}{LT}\Big/\ln\left(\frac{v}{LT_c}\,\frac{eH}{c}L^2\right). \qquad (2.4)$$

Just as (2.2), this formula is valid to within a coefficient of the order of unity inside the logarithm sign.

It follows from Eq. (2.2) that as $T \to T_c$ the function $f(eHL^2/c) \to 1$, and therefore eHL^2/c becomes small. Then it is evident from (2.1) that for weak fields ($eHL^2/c \ll 1$) the magnitude of the "gap" does not depend on z. So long as the field is not too small ($1 \gg eHL^2/c \gg LT/v$), values of $\Phi(\omega)$ in region I (1.19a) ($\omega L/v \ll eHL^2/c$) make as before the major field-dependent contribution to the sum (1.16). Then the critical field is determined from the following equation:

$$\frac{eH}{c}L^2 = 3\left(1 - \frac{T}{T_c}\right)\Big/\left[\ln\left(\frac{4\gamma}{\pi}\,\frac{v}{LT}\,\frac{eH}{c}L^2\right) - \frac{7}{3}\right] \qquad (2.5)$$

(the numerical factor inside the argument of the logarithm is found from the exact expression (A.1) for $\Phi(\omega)$).

Equation (2.5) is valid in the temperature range

$$1 \gg 1 - T/T_c \gg L/\xi_0.$$

Upon further increase of the temperature, the field decreases so much that eHL^2/c becomes smaller than LT/v, so that region I makes no contribution to the sum in (1.16), and region II (1.19b) ($eHL^2/c \ll \omega L/v \ll 1$), gives the major field-dependent contribution. Hence, for $1 - (T/T_c) \ll L/\xi_0$ we obtain

$$\left(\frac{eH}{c}\right)^2 = \frac{128}{\pi}\,\frac{1 - T/T_c}{vL^3/T_c}. \tag{2.6}$$

In this temperature region the critical field varies with the thickness of the film like $L^{-3/2}$, whereas at lower temperatures H depends on the thickness for the most part like L^{-2}.

From what was said at the end of the preceding section with regard to the validity of the quasi-classical approximation, it follows that the results (2.2), (2.4), and (2.5) for the critical field are valid so long as $eHL^2/c \gg 1/p_0L$ (see (1.22a)). This condition may be violated only near T_c in the case $L^2 \ll \xi_0/p_0$, the result (2.6) in the immediate neighborhood of T_c is applicable, as follows from (1.22b), for not too thin films ($L^2 \gg \xi_0/p_0$).

3. CONTAMINATED FILMS

Knowing the function $\Phi(t;\, z_1, z_2)$ for clean films, one can solve the problem of the critical field of thin contaminated films. Starting from the physical interpretation of this function (see Sec. 1), we obtain the following integral equation for the analogous function $\Phi_\tau(t;\, z_1, z_2)$ for films containing impurities, assuming that the scattering by impurities is isotropic:

$$\Phi_\tau(t;\, z_1, z_2) = \Phi(t;\, z_1, z_2)\,e^{-t/\tau} + \int_0^t \frac{dt'}{\tau} \int_{-L/2}^{+L/2} dz'\,\Phi(t';\, z_1, z') \times e^{-t'/\tau}\Phi_\tau(t - t';\, z', z_2). \tag{3.1}$$

The first term on the right hand side of this equation corresponds to the motion of a particle from the point z_1 to z_2 without collision with impurities; therefore, it is equal to the corresponding function $\Phi(t)$ for pure films multiplied by the probability that a collision with an impurity does not take place during the time t. The second integral term takes multiple collisions into account under the assumption that the first collision with an impurity after the flight of the particle from the point z_1 takes place at the point with coordinate z', at the moment of time t'. If we take the Laplace transform,

$$\Phi_\tau(\omega;\, z_1, z_2) = \int_0^\infty \Phi_\tau(t;\, z_1, z_2)\,e^{-2\omega t}\,dt, \tag{3.2}$$

then the equation takes the form

$$\Phi_\tau(\omega;\, z_1, z_2) = \Phi\left(\omega + \frac{1}{2\tau};\, z_1, z_2\right) + \frac{1}{\tau}\int_{-L/2}^{+L/2} dz'\,\Phi\left(\omega + \frac{1}{2\tau};\, z_1, z'\right)\Phi_\tau(\omega;\, z', z_2). \tag{3.3}$$

The kernel $K(z_1, z_2)$ of the integral equation (1.15) is constructed according to (1.16) out of the function $\Phi_\tau(\omega)$ determined in this manner.

The impurity concentration, i.e., the relation between the mean free path and the other parameters ξ_0 and L of our problem, substantially influences the behavior and value of the critical field.

A. Let us consider the case of a mean free path which is large in comparison with the BCS correlation parameter ξ_0 ($l \gg \xi_0$). When the field is not too small ($eHL^2/c \gg L/v\tau$), then for small values of ω ($\omega L/v \ll eHL^2/c$) it is easy to solve the integral equation (3.3) for $\Phi_\tau(\omega)$ if it is recognized that $\Phi(\omega)$ in region I is determined by expression (1.19a). We have

$$\Phi_\tau(\omega;\, z_1, z_2) = \frac{\tau}{2\omega\tau + 1 - f(eHL^2/c)} \times \exp\left[-\frac{eH}{c}\left(\frac{L^2}{2} - z_1^2 - z_2^2\right)\right], \tag{3.4}$$

(the function f(x) is defined by formula (2.3)).

For large fields ($eHL^2/c \sim 1$) the "gap" depends on the z coordinate [see Eq. (2.1)], and with account of the fact that $1 - f \sim 1$ we obtain, with logarithmic accuracy, the equation for the critical field when $T \ll 1/\tau$ [2]

$$f\left(\frac{eH}{c}L^2\right) = \ln\frac{v}{LT_c}\Big/\ln\frac{v\tau}{L}, \tag{3.5}$$

and for $T \gg 1/\tau$ we obtain the same results (2.2), (2.5), and (2.6) as for pure films. Thus the logarithmic increase of the field stops when $T \sim 1/\tau$, and the critical field tends to the finite limit (3.5) as $T \to 0$.

B. $l \sim \xi_0$, $1 - (T/T_c) \ll 1$. Equation (3.5) holds so long as the value of the critical field given by it satisfies the condition $eHL^2/c \sim 1$. For smaller fields [$1 \gg eHL^2/c \gg Lv^{-1}(T + \tau^{-1})$], using Eq. (3.4) and the limiting value of f(x) for small arguments [see Eq. (2.3)], and recognizing that $\Phi_\tau = \Phi$ for

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[2] For films with such a large mean free path that $eHL^2/c \gg 1$ at low temperatures, one can obtain the equation for the critical field in the temperature range $T \ll 1/\tau$ from Eq. (2.4) by replacing T by $1/\tau$.

large values of ω ($L\omega/v \sim eHL^2/c \gg L/v\tau$), we obtain the following equation for the critical field:

$$\left(1-\frac{1}{3}\frac{eH}{c}L^2\right)\left(\ln\frac{T}{T_c}-\psi\left(\frac{1}{2}\right)+\psi\left(\frac{1}{2}+\frac{1}{12\pi T\tau}\frac{eH}{c}L^2\right)\right)$$
$$+\frac{1}{3}\frac{eH}{c}L^2\left(\ln\frac{4\gamma}{\pi}\frac{eHLv}{cT_c}-\frac{7}{3}\right)=0. \qquad (3.6)$$

Hence, in the region $1 \gg 1-(T/T_c) \gg L/\min(\xi_0, l)$

$$\frac{eH}{c}L^2=3\left(1-\frac{T}{T_c}\right)\Big/\left[\ln\left(\frac{4\gamma}{\pi}\frac{eH}{c}\frac{Lv}{T_c}\right)-\frac{7}{3}+\frac{\pi}{8T_c\tau}\right]. \qquad (3.7)$$

Of course, in the limiting case of very slight impurity concentrations ($l \gg \xi_0$) this expression goes over into the corresponding expression (2.5) for pure films.

Upon further increase of the temperature $[1-(T/T_c) \ll L/\min(\xi_0, l)]$ the contribution to the kernel $K(z_1, z_2)$ is made by those values of $\Phi_T(\omega; z_1, z_2)$ which are determined from $\Phi(\omega; z_1, z_2)$ in region II (1.19b). In the absence of a field, Eq. (1.20) holds for arbitrary values of ω and similarly

$$\int_{-L/2}^{+L/2}\Phi_{0\tau}(\omega; z_1, z_2)\,dz_1 \equiv \Phi_{0\tau}(\omega)=\frac{1}{2\omega}. \qquad (3.8)$$

Since $\Phi(\omega; z_1, z_2)$ in region II (1.19b), when integrated over one of the coordinates, does not depend on the other coordinate

$$\int_{-L/2}^{+L/2}\Phi(\omega; z_1, z_2)\,dz_2 \equiv \Phi(\omega), \qquad \frac{eH}{c}L^2 \ll \frac{\omega L}{v} \ll 1, \qquad (3.9)$$

$\Phi_T(\omega; z_1, z_2)$ has the same property when

$$\frac{eH}{c}L^2 \ll \frac{L}{v}\left(\omega+\frac{1}{2\tau}\right) \ll 1.$$

Then from (3.3) we obtain the following equation which determines $\Phi_T(\omega)$ in this region:

$$\Phi_\tau(\omega)=\frac{\Phi(\omega+1/2\tau)}{1-\tau^{-1}\Phi(\omega+1/2\tau)}, \qquad (3.10)$$

$$\Phi_\tau(\omega)=\frac{\frac{1}{2}\left[1-\frac{v}{16}\left(\frac{eH}{c}\right)^2\frac{L^3\tau}{2\omega\tau+1}\right]}{\left[\omega+\frac{v}{32}\left(\frac{eH}{c}\right)^2\frac{L^3}{2\omega\tau+1}\right]},$$

$$\frac{eH}{c}L^2 \ll \frac{L}{v}\left(\omega+\frac{1}{2\tau}\right) \ll 1. \qquad (3.11)$$

In the case under consideration $l \sim \xi_0$, $1-(T/T_c) \ll L/\min(\xi_0, l)$, the second term in the denominator of (3.11) is smaller than the first; hence, for the critical field we obtain the same expression (2.6) as for pure films.

C. The mean free path satisfies the condition $\xi_0 \gg l \gg \sqrt{\xi_0 L}$. Then from expression (3.6) we obtain the equation of the critical field over the whole temperature range $1-(T/T_c) \gg L/l$,

$$\ln\frac{T}{T_c}=\psi\left(\frac{1}{2}\right)-\psi\left(\frac{1}{2}+\frac{1}{12\pi T\tau}\frac{eH}{c}L^2\right). \qquad (3.12)$$

In particular, for $T \ll T_c$

$$\frac{eH}{c}L^2=\frac{3\pi}{\gamma}T_c\tau, \qquad (3.13)$$

and for $1 \gg 1-(T/T_c) \gg L/l$

$$\frac{eH}{c}L^2=\frac{24}{\pi}\tau T_c(1-T/T_c), \qquad (3.14)$$

which, of course, coincides with the appropriate limiting value of formula (3.7).

In the temperature range $1-(T/T_c) \ll L/l$, the result, as indicated above, coincides with the result (2.6) for pure films.

D. The de Gennes-Tinkham case[2] ($\sqrt{\xi_0 L} \gg l \gg L$). Here there is no contribution Φ_T corresponding to Φ in region I (1.19a) in the entire temperature interval. In region II $[eHL^2/c \ll Lv^{-1}\{1+(1/2\tau)\}]$ the function Φ_T is determined by expression (3.11), in the denominator of which one can, in this case, neglect $2\omega\tau$ in comparison with unity since for $\omega \sim 1/\tau$ the second term of the denominator is much smaller than the first. The second term of the numerator is always negligibly small. Then, for the entire temperature range, the equation for the critical field has the form

$$\ln\frac{T}{T_c}=\psi\left(\frac{1}{2}\right)-\psi\left(\frac{1}{2}+\frac{vL^3}{64\pi T}\left(\frac{eH}{c}\right)^2\right). \qquad (3.15)$$

We stress the fact, mentioned in [2], that in this case the critical field does not depend on the mean free path. For $T \ll T_c$, formula (3.15) gives

$$\left(\frac{eH}{c}\right)^2=\frac{16\pi}{\gamma}\frac{T_c}{vL^3}, \qquad (3.16)$$

and for $1-(T/T_c) \ll 1$ it gives the same result as for pure films in a narrow region near T_c [see Eq. (2.6)].

E. Very dirty films (Maki's case,[5] $L \gg l$). Here the function $\Phi_T(\omega; z_1, z_2)$ is determined by expression (1.19c) for $\Phi(\omega; z_1, z_2)$ in region III. Taking its δ-function character into consideration over the distances $\sim L$ which are of importance to us, it is advisable to convert the integral equation (3.3) for Φ_T to a differential equation:[3)]

[3)] One can show that if the scattering by impurities is not isotropic, then the free flight transport time enters into this expression as τ. Equations analogous to (3.17)–(3.21), representing a generalization to arbitrary temperatures of the Ginzburg-Landau equations without the nonlinear term, were obtained in another way by Maki.[6]

$$\left(\omega+\frac{2v^2\tau}{3}\left(\frac{eHz_1}{c}\right)^2+\frac{v^2\tau}{6}\frac{\partial^2}{\partial z_1^2}\right)\Phi_\tau(\omega;z_1,z_2)$$
$$=\frac{1}{2}\delta(z_1-z_2) \qquad (3.17)$$

with the boundary conditions

$$\frac{\partial}{\partial z_1}\Phi_\tau(\omega;z_1,z_2)\Big|_{z_1=\pm L/2}=0 \qquad (3.18)$$

(the same equation holds with regard to the other variable z_2).

It is possible to write the solution of this equation in the form

$$\Phi_\tau(\omega;z_1,z_2)=\sum_k\frac{\varphi_k(z_1)\varphi_k(z_2)}{v^2\tau\varepsilon_k/6+\omega}, \qquad (3.19)$$

where φ_k and ϵ_k are the eigenfunctions and eigenvalues of the equation

$$\left(\frac{\partial^2}{\partial z^2}+4\left(\frac{eH}{c}\right)^2z^2\right)\varphi_k=\varepsilon_k\varphi_k(z) \qquad (3.20)$$

with the boundary conditions

$$\frac{\partial\varphi_k}{\partial z}\Big|_{z=\pm L/2}=0. \qquad (3.21)$$

Substituting the thus obtained Φ_T into expression (1.16) for the kernel K, we find that $\Delta^*(z)$ must be the eigenfunction of the differential equation (3.20) with the corresponding boundary conditions (3.21). For the case of sufficiently thin films, when $eHL^2/c \ll 1$ over the whole temperature range, the critical field is determined by the equation

$$\ln\frac{T}{T_c}=\psi\left(\frac{1}{2}\right)-\psi\left(\frac{1}{2}+\frac{v^2\tau}{36\pi T}\left(\frac{eH}{c}L\right)^2\right),$$
$$l\ll L\ll\left(\frac{l\xi_0}{1-T/T_c}\right)^{1/2} \qquad (3.22)$$

which, in the limit of temperatures close to T_c, gives the result of the Ginzburg-Landau theory:[7]

$$\frac{eH}{c}=\left(\frac{72}{\pi}\frac{T_c}{v^2\tau}\right)^{1/2}\frac{(1-T/T_c)^{1/2}}{L} \qquad (3.23)$$

and for $T \ll T_c$

$$\frac{eH}{c}=\left(\frac{9\pi T_c}{\gamma v^2\tau}\right)^{1/2}\frac{1}{L} \qquad (3.24)$$

Provided that

$$L\gg\left(\frac{l\xi_0}{1-T/T_c}\right)^{1/2}$$

the film behaves like a bulk sample, and the phase transition of the second kind occurs at the upper critical field H_{c2} of dirty alloys (the correct value of which was determined over the whole temperature range by Maki[6]) or at the surface field H_{c3} of Saint James and de Gennes[8]. 4)

In conclusion, let us go into the applicability of the obtained expressions, following from the validity of the quasi-classical approximation. What was said at the end of Sec. 2 with regard to very pure films remains in effect for ideally pure films (case A). Expressions (3.7) and (3.12)—(3.14) are applicable when $eHL^2/c \gg 1/p_0L$ [see (1.22a)]. This condition may be violated near T_c [formulas (3.7), (3.14)] if

$$L^2\ll\min(\xi_0,\ l)\,/\,p_0, \qquad (3.25)$$

and (3.13) is valid far from T_c for $L \gg \xi_0/lp_0$. Expression (2.6) in a narrow region near T_c for slightly contaminated films (cases A—B) and expression (3.15) for case Γ are valid for $L^2 \gg \min(\xi_0,\ l)/p_0$, as follows from (1.22b).

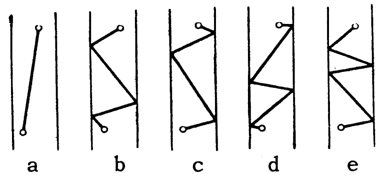
The author thanks L. P. Gor'kov for a discussion of the obtained results.

APPENDIX

The general expression for $\Phi(\omega;\ z_1,\ z_2)$ has the form

$$\Phi(\omega;z_1,z_2)=\frac{1}{2v}\left\{\int_0^1\frac{d\mu}{\mu}\exp\left(-\frac{2\omega}{v\mu}|z_1-z_2|\right)\right.$$
$$\times J_0\left(\frac{\sqrt{1-\mu^2}}{\mu}(z_1^2-z_2^2)\frac{eH}{c}\right)+2\Psi\left(\frac{2\omega L}{v}\right)$$
$$\times[F(z_1)F(z_2)+F(-z_1)F(-z_2)]+2\Psi\left(\frac{2\omega L}{v}\right)$$
$$\left.\times\psi\left(\frac{2\omega L}{v}\right)[F(z_1)F(-z_2)+F(-z_1)F(z_2)]\right\}. \qquad (A.1)$$

The first term on the right hand side is obtained upon averaging over trajectories that pass directly through the points with coordinates z_1 and z_2 (see Fig. a). The following terms correspond to averaging over the trajectories shown in Figs. b, c, d, and e. The sign of the argument of F depends on whether the first collision after the flight from



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[4] The intermediate case $L \sim \sqrt{l\xi_0}\,[1-(T/T_c)]^{-1/2}$ was considered near T_c by Abrikosov.[9] It is not difficult to generalize his result to arbitrary temperatures, using Eqs. (3.17)–(3.21).

the point z_1, or the last collision before hitting the point z_2, took place on the right or left side of the film. The function F(z) is given by

$$F(z)=\int_0^1 d\mu \exp\left[-\frac{2\omega}{v\mu}\left(\frac{L}{2}+z\right)\right]$$
$$\times J_0\left(\frac{\sqrt{1-\mu^2}}{\mu}\left(\frac{L^2}{4}-z^2\right)\frac{eH}{c}\right). \qquad (A.2)$$

The functions Ψ and ψ are obtained as the result of summing the series of diffuse multiple reflections from the walls:

$$\psi(x)=2\int_0^1 e^{-x/\mu}\,\mu d\mu=\begin{cases}1-2x, & x\ll 1,\\ 2x^{-1}e^{-x}, & x\gg 1,\end{cases} \qquad (A.3)$$

$$\Psi(x)=\frac{1}{1-\psi^2(x)}=\begin{cases}1/4x, & x\ll 1,\\ 1, & x\gg 1.\end{cases} \qquad (A.4)$$

These functions do not depend on the field, since the phase "gained" in connection with the motion of a particle from one side of the lamina to the other is equal to zero. We note that

$$F(L/2)={}^1\!/\!_2\psi(2\omega L/v), \qquad (A.5)$$

and in the absence of a field

$$\int_{-L/2}^{+L/2} F_0(z)\,dz=\frac{v}{4\omega}\left(1-\psi\left(\frac{2\omega L}{v}\right)\right). \qquad (A.6)$$

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