OUTPUT FLUCTUATIONS OF THERMAL RADIATION DETECTORS

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Output fluctuations of radiation detectors having dimensions much larger than the emitted wavelength are considered. The results are valid at any detector temperature. The cases of very large and very small instrumental time constants, of isotropic external radiation, and also of external radiation in a beam of small angular spread are analyzed in detail.

1. INTRODUCTION

PHENOMENA involving quantum fluctuations of the electromagnetic field have received much attention in recent years. Several investigators have studied fluctuations in the outputs of radiation detectors (especially in the visible and infrared regions) in connection with the study of the sensitivity thresholds of the receivers.

Milatz and Van de Velden [1] were the first to suggest that the threshold is determined by Einstein's equation, which in terms of the number of photons contained inside an isothermal cavity, within the frequency interval $\Delta\omega$, is

$$\langle \Delta n^2 \rangle = \langle n \rangle (1 + N/g), \tag{1}$$

where $g = \omega^2 \Delta\omega / \pi^2 c^3$ is the spatial density of field states, $N = \langle n \rangle / V$ is the mean photon density, and V is the volume of the cavity. The first term on the right side is the mean square fluctuation $\langle \Delta n^2 \rangle = \langle n \rangle$ of the number of noninteracting classical particles of mean density N; the second term can be interpreted as the mean square fluctuation in the energy of noninteracting classical waves. Thus (1) relates classical "Poisson" fluctuations to classical "wave" fluctuations.

Using Eq. (1), Lewis [2] obtained an expression for the fluctuations of the energy exchanged between a "black" detector and an isothermal cavity, in agreement with the result obtained in [1]. Jones [3] showed, furthermore, that (1) can be used to obtain an expression for the sensitivity threshold of any radiation detector that is in equilibrium with a cavity, independently of the mechanism employed in the detector. For the mean square number of photons $\langle \Delta m^2 \rangle$ exchanged between the detector and the cavity, Jones obtained

$$\langle \Delta m^2 \rangle = 2 \langle m \rangle (1 + N/g), \tag{2}$$

where $\langle m \rangle = \frac{1}{4} \epsilon c \sigma t N$ is the mean number of photons absorbed in time t by a detector of aperture σ and quantum efficiency ϵ.

These investigations became especially timely following the appearance of improved detectors, especially in the infrared region, [4] having sensitivity thresholds that agreed very accurately with the value computed from (2). These detectors also led to the important conclusion that the sensitivity threshold of any radiation detector depends on its efficiency and not on the microscopic absorption mechanism; this applies to photocells, bolometers, photoconductive cells etc.

Quantum fluctuations of visible light became especially interesting following the experiment of Brown and Twiss, [5] who observed a correlation between the outputs of two photomultipliers irradiated by a thermal source. This correlation results from the wave properties of light, which in particle language are usually associated with the "bunching" of photons.[6] For visible frequencies the "wave" term in (1) and (2) is much smaller than the particle term. Therefore the correlation technique is at the present time the only means of investigating "wave" fluctuations of visible light.

Brown and Twiss developed a detailed theory of the output fluctuations of two photomultipliers irradiated by a thermal source. [7] Unlike the thermodynamic derivation of (1), their theory is based on a detailed analysis of the interaction between light and matter. The predictions of this theory agree well with experimental data. However, when the theory is used to calculate the mean square fluctuations of photomultiplier output we obtain

$$\langle \Delta m^2 \rangle = \langle m \rangle (1 + \varepsilon N/g), \tag{3}$$

which differs from one-half of (2) by having the factor ϵ in the "wave" term. This discrepancy has been discussed for several years,[8-13] and many arguments have been advanced to support both (2) and (3). The authors of both [12] and [13] pointed out, at about the same time, that fluctuations in the incident radiation and in the emission from the detector cannot be considered independent. Therefore the output fluctuations of a cooled "gray" detector do not comprise half of the output fluctuations of the same detector when in equilibrium with the radiation.

In the present work an attempt is made to describe phenomenologically the output fluctuations of a detector for any value of the dielectric constant and at any temperature. We believe that a great advantage of this description lies in the fact that it does not involve any hypotheses that are difficult to confirm, such as a specific hypothetical photoelectron distribution function, and makes it possible to consider detectors at any temperature. The result that will be given in Sec. 3 becomes either (2) or (3) under certain conditions; this confirms the views of certain authors.[12,13] In this section we consider the output fluctuations of a detector having a broad low-frequency passband. In Section 4 we shall consider a "cold" detector irradiated by a beam of waves having a very narrow angle of spread.

It must be emphasized that we confine ourselves here, as did all the aforementioned authors, to radiation from thermal sources. The appearance of powerful coherent light sources has stimulated attempts to apply the aforementioned analysis to the detectors of laser emission. Lack of success here resulted from the fact that although a statistical operator describing laser emission is not known, it is clear that because of saturation effects the operator must be entirely different from the operator for thermal emission.[14,15]

2. DETECTOR OUTPUT

We consider a detector in the shape of a dielectric plate that is much larger than the wavelengths λ corresponding to the considered frequencies. To permit neglect of energy flow through the sides of the detector we shall assume that the plate's transverse dimensions greatly exceed its thickness. Absorption is also so strong that the radiation passing through the plate transversely can be neglected.

Unlike the aforementioned investigations, we consider fluctuations not in the number of absorbed photons, but in the energy absorbed during a period of time determined by the passband of the low-frequency output filter. Accordingly, it is reasonable to represent the mean output current of the device as follows:

$$\langle I(t)\rangle = \int \mathbf{n}\langle \mathbf{S}\rangle F(\omega+\omega')\, e^{i(\omega+\omega')t}\, d\omega\, d\omega'\, d\mathbf{r},$$

$$\mathbf{S} = \mathbf{S}(\omega,\omega') = \frac{c}{8\pi}([\mathbf{E}\mathbf{H}'] - [\mathbf{H}'\mathbf{E}]); \qquad (4)^*$$

here the primed quantities are functions of ω', $F(\omega)$ is the characteristic of a low-frequency filter, and $\mathbf{dr}$ is an area element of the detector. For the correlation function we have

$$\Psi_I = \langle I(t)I(t')\rangle - \langle I(t)\rangle^2$$
$$= \int \langle(\mathbf{nS})(\mathbf{nS}_1)\rangle_0 F(\omega+\omega')F(\omega_1+\omega_1')$$
$$\times \exp\{i[(\omega+\omega')t + (\omega_1+\omega_1')t']\}\, d\omega \ldots d\omega_1'\, d\mathbf{r}\, d\mathbf{r}', \qquad (5)$$

with $\mathbf{S}_1 = \mathbf{S}(\omega_1, \omega_1')$, and for the central moment of Hermitian operators we have introduced the notation

$$\langle AB\rangle_0 = {}^1\!/_2\langle AB + BA\rangle - \langle A\rangle\langle B\rangle. \qquad (6)$$

We have previously shown [16] that these central moments in a transparent medium can be calculated with the aid of classical field equations. It is easily seen that the presence of an absorbing detector does not change the situation. Indeed, Eqs. (4) and (5) are not changed when integration is shifted from the surface of the detector to a very close-lying plane. However, it is possible to use the formulas of the present work for the fields of black body radiation in this plane (in a vacuum).

We require a simple consequence of Eq. (21) of [16], and an expression for the spectral density of extraneous (non-electromagnetic) electric induction $\mathbf{D}$ [17] †

$$\langle D_i D_k'\rangle = i\hbar \operatorname{cth}\frac{\hbar\omega}{2kT}(\varepsilon_{ki}^* - \varepsilon_{ik})\,\delta(\omega+\omega')\,\delta(\mathbf{r}-\mathbf{r}').$$

Let the fields E, E', E_1, and E_1' be excited by identical sources at the temperature T. Then, since the field equations are linear their central moment is represented correctly by

$$\langle EE'E_1E_1'\rangle_0 = [\langle EE_1\rangle\langle E'E_1'\rangle$$
$$+ \langle EE_1'\rangle\langle E'E_1\rangle]\,\alpha(\omega, T, \omega', T),$$
$$\alpha(\omega, T, \omega', T') = 1 + \operatorname{cth}^{-1}\frac{\hbar\omega}{2kT}\operatorname{cth}^{-1}\frac{\hbar\omega'}{2kT'}. \qquad (7)$$

If E and E', on the one hand, and E_1 and E_1', on the other hand, are excited by different sources

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*$[\mathbf{EH}'] \equiv \mathbf{E}\times\mathbf{H}'$.

† cth ≡ coth.

having the respective temperatures T and T′, we have

$$\langle EE'E_1E_1'\rangle_0 = 0$$

$$\langle EE_1E'E_1'\rangle_0 = \langle EE'\rangle\langle E_1E_1'\rangle\alpha(\omega, T, \omega_1, T'). \qquad (8)$$

Thus in the phenomenological theory Ψ_I can be interpreted as the correlation function of the classical quantity

$$I = \frac{c}{4\pi}\int \mathbf{n}[\mathbf{EH}]F(\omega+\omega')e^{i(\omega+\omega')t}\,d\omega\,d\omega'\,d\mathbf{r},$$

and the fourth order moments can be calculated from (7) and (8).

We now expand the field strengths in Fourier space integrals (retaining the notation **E**, **H** for the spectral amplitudes), and then integrate with respect to dr, i.e., over the detector surface z = 0. Assuming for the sake of simplicity that the plate is a square of side L, we obtain

$$I = \frac{c}{4\pi}\int \mathbf{n}[\mathbf{EH}']F(\omega+\omega')e^{i(\omega+\omega')t}\frac{\sin(\varkappa_1+\varkappa_1')L/2}{\varkappa_1+\varkappa_1'} \times\frac{\sin(\varkappa_2+\varkappa_2')L/2}{\varkappa_2+\varkappa_2'}d\omega\,d\omega'\,d\boldsymbol{\varkappa}\,d\boldsymbol{\varkappa}', \qquad (9)$$

where κ is the projection of the vector **k** sign ω (k is the wave vector and $k = \omega/c$) on the plane $z = 0$, $E = E(\omega, \boldsymbol{\kappa})$ and $\mathbf{H}' = \mathbf{H}(\omega', \boldsymbol{\kappa}')$ are the plane wave amplitudes. The latter comprise the amplitudes of 1) the incident waves ($\mathbf{E}_{inc}$, $\mathbf{H}_{inc}$), 2) the reflected waves ($\mathbf{E}_0$, $\mathbf{H}_0$), and 3) the thermal field of the detector ($\mathbf{E}_t$, $\mathbf{H}_t$). Rytov [17] has shown that the last of these components contains, in addition to the radiation field, a quasi-stationary field that drops off rapidly with increasing distance from the surface and does not participate in the energy transfer of the system. We shall henceforth assign the energy of the quasi-stationary field to the internal energy of the detector; $\mathbf{E}_t$ and $\mathbf{H}_t$ will represent only the wave part of this field.

We now note that an appreciable contribution to (9) comes only from the integration regions where $|\omega + \omega'| \lesssim \Delta\Omega$ ($\Delta\Omega$ is the low-frequency filter band) and $|\boldsymbol{\kappa} + \boldsymbol{\kappa}'| \lesssim 1/L$. This means that in k-space the vector **k′** (or its reflection in the plane of the detector) must lie inside the solid angle $\delta\chi = \delta\psi\delta\theta$ around **k**, where the angle $\delta\theta$ is measured in the meridional plane, and $\delta\psi$ is measured in a plane perpendicular to the latter and passing through **k**. For these angles we easily obtain

$$\delta\psi \approx \lambda/L, \qquad \delta\theta \lesssim \sqrt{\Delta\Omega/\omega} + \sqrt{\lambda/L},$$

from which it follows that by virtue of the foregoing assumptions $\lambda/L \ll 1$ and $\Delta\Omega/\omega \ll 1$ these angles are small. We therefore represent each amplitude in (9) as the sum of two components (1 and 2), which are orthogonal to the wave vector and to each other. Then, with a relative error not exceeding $\delta\theta$, we can substitute in the integrand of (9),

$$\sum_{i,k=1}^{2}\mathbf{n}[\mathbf{E}^i\mathbf{H}'^k] \cong \sum_{i=1}^{2}\mathbf{n}[\mathbf{E}^i\mathbf{H}^i].$$

Let the electric vector of the waves after the first polarization process be parallel to the detector plane. Then in the reflected field we have

$$\mathbf{E}_0^1 = R_1\mathbf{E}_{inc}^1, \quad \mathbf{H}_0^1 = k^{-1}R_1[\tilde{\mathbf{k}}\mathbf{E}_{inc}^1],$$

$$\tilde{\mathbf{k}} = (k_1, k_2, -k_3), \qquad k_3 \geqslant 0,$$

where R_1 is the reflection coefficient for the first polarization. We have here

$$\mathbf{n}[\mathbf{E}^1\mathbf{H}'^1] = \mathbf{n}[(\mathbf{E}_{inc}^1 + (1+R_1) + \mathbf{E}_t^1),$$

$$(\mathbf{H}_{inc}'^1 + \frac{R_1'}{k'}[\tilde{\mathbf{k}}'\mathbf{E}_{inc}'^1] + H_t'^1)].$$

Since E^1 is perpendicular to **k**, **k′**, $\tilde{\mathbf{k}}$, and $\tilde{\mathbf{k}}'$, with a relative error of the order $\delta\theta$ we set

$$\mathbf{n}[\mathbf{E}_{inc}^1\mathbf{H}_{inc}'^1] \cong \frac{k_3'}{k'}E_{inc}^1E_{inc}'^1, \quad \mathbf{n}[\mathbf{E}_{inc}^1\mathbf{H}_t'^1] \cong \frac{k_3'}{k}E_{inc}^1E_t'^1$$

etc. Therefore for the first polarization we have

$$\mathbf{n}[\mathbf{E}^1\mathbf{H}'^1] = \cos\theta[E_{inc}^1E_{inc}'^1(1-R_1')(1+R_1) - 2R_1E_{inc}^1E_t'^1 - E_t^1E_t'^1], \qquad (10)$$

where $\cos\theta = k_3/k$ ($0 \le \theta \le \pi/2$).

Symmetrizing (10) with respect to ω and ω' and adding a similarly simplified expression for the second polarization, we find that the integrand in (9) can be put in the approximate form

$$\mathbf{n}[\mathbf{EH}'] = \sum_{i=1}^{2}\cos\theta[E_{inc}^iE_{inc}'^i(1-R_iR_i') - E_t^iE_t'^i - 2R_iE_{inc}^iE_t'^i]. \qquad (11)$$

3. OUTPUT FLUCTUATIONS OF A RECEIVER IN EQUILIBRIUM WITH RADIATION

In the case of radiative equilibrium orthogonally polarized waves are statistically independent, so that

$$\langle(\mathbf{n}[\mathbf{EH}'])(\mathbf{n}[\mathbf{E}_1\mathbf{H}_1'])\rangle_0 = \sum_{i=1}^{2}\langle(\mathbf{n}[\mathbf{E}^i\mathbf{H}'^i])(\mathbf{n}[\mathbf{E}_1^i\mathbf{H}_1'^i])\rangle_0$$

$$= \cos\theta\cos\theta_1\sum_{i=1}^{2}[(1-R_iR_i')$$

$$\times(1-R_{i1}R_{i1}')\langle E_{\mathrm{inc}}{}^iE_t{}'^iE_{\mathrm{inc}\,1}{}^iE_{t1}{}'^i\rangle_0$$
$$+\langle E_t{}^iE_t{}'^iE_{t1}{}^iE_{t1}{}'^i\rangle_0+4R_iR_{i1}\langle E_{\mathrm{inc}}{}^iE_t{}'^iE_{\mathrm{inc}\,1}{}^iE_{t1}{}'^i\rangle_0]. \tag{12}$$

The fourth order moments can be represented by (7) and (8).

Let a detector of temperature T_2 be placed in equilibrium with radiation of temperature T_1. Then, from (7) we have

$$\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}}{}'^iE_{\mathrm{inc}\,1}{}^iE_{\mathrm{inc}\,1}{}'^i\rangle_0$$
$$=[\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}1}{}^i\rangle\langle E_{\mathrm{inc}}{}'^iE_{\mathrm{inc}\,1}{}'^i\rangle$$
$$+\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}1}{}'^i\rangle\langle E_{\mathrm{inc}}{}'^iE_{\mathrm{inc}1}{}^i\rangle)\ \alpha(\omega,T_1,\omega',T_1), \tag{13}$$

where, for example,

$$\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}1}{}'^i\rangle=\frac{\hbar\omega}{8\pi^2c\cos\theta}\mathrm{cth}\frac{\hbar\omega}{2kT_1}\delta(\varkappa+\varkappa_1')\delta(\omega+\omega_1'). \tag{14}$$

As a result we have

$$\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}}{}'^iE_{\mathrm{inc}1}{}^iE_{\mathrm{inc}\,1}{}'^i\rangle_0$$
$$=\frac{\hbar^2\omega\omega'}{c^2(8\pi^2)^2\cos^2\theta}\left[\mathrm{cth}\frac{\hbar\omega}{2kT_1}\mathrm{cth}\frac{\hbar\omega'}{2kT_1}+1\right]$$
$$\times[\delta(\varkappa+\varkappa_1)\delta(\varkappa'+\varkappa_1')\delta(\omega+\omega_1)\delta(\omega'+\omega_1')$$
$$+\delta(\varkappa+\varkappa_1')\delta(\varkappa'+\varkappa_1)\delta(\omega'+\omega_1)\delta(\omega+\omega_1')]. \tag{15}$$

Also, with the aid of (8), we obtain

$$\langle E_{\mathrm{inc}}{}^iE_t{}'^iE_{\mathrm{inc}1}{}^iE_{t1}{}'^i\rangle_0=\langle E_t{}'^iE_{t1}{}'^i\rangle\langle E_{\mathrm{inc}}{}^iE_{\mathrm{inc}1}{}^i\rangle$$
$$\times\alpha(\omega',T_2,\omega,T_1). \tag{16}$$

The first factor in the right-hand side of this equation is proportional to the detector emission intensity. Therefore from Kirchhoff's law we obtain

$$\langle E_t{}^iE_{t1}{}'^i\rangle=\frac{A_i\hbar\omega}{8\pi^2c\cos\theta}\mathrm{cth}\frac{\hbar\omega}{2kT_2}\delta(\varkappa+\varkappa_1')\delta(\omega+\omega_1'), \tag{17}$$

where $A_i = 1 - |R_i|^2$ is the energy coefficient of absorption. Finally, applying (7) and (17) to $\langle E_t^iE_t'^iE_{t1}^iE_{t1}'^i\rangle$, we find that the latter expression is equal to (14) multiplied by A_iA_i', with T_1 replaced by T_2.

Substituting (14)–(17) into (12), integrating over ω_1, ω_1', κ, and κ_1', substituting the notation $\mathbf{r}-\mathbf{r}'=\rho$, $t-t'=\tau$, and writing

$$\int\exp\,[i(\varkappa+\varkappa')\rho]d\mathbf{r}\,d\mathbf{r}'=\sigma(2\pi)^2\delta(\varkappa+\varkappa'), \tag{18}$$

(in virtue of the condition $\lambda/L \ll 1$), where σ is the area of the detector, we obtain

$$\Psi_I=\frac{\hbar^2\sigma}{128\pi^4}\int\omega\omega'|F(\omega+\omega')|^2\exp[i(\omega+\omega')\tau]$$
$$\times\sum_{i=1}^{2}\Big[|1-R_iR_i'|^2\Big(\mathrm{cth}\frac{\hbar\omega}{2kT_1}\mathrm{cth}\frac{\hbar\omega'}{2kT_1}+1\Big)$$
$$+A_iA_i'\Big(\mathrm{cth}\frac{\hbar\omega}{2kT_2}\mathrm{cth}\frac{\hbar\omega'}{2kT_2}+1\Big)$$
$$+2A_i'(1-A_i)\Big(\mathrm{cth}\frac{\hbar\omega}{2kT_1}\mathrm{cth}\frac{\hbar\omega'}{2kT_2}+1\Big)\Big]\,d\omega\,d\omega'\,d\varkappa. \tag{19}$$

To calculate this integral we must know how the complex reflection coefficients R_i depend on the frequency and on the angle θ in the entire frequency interval. To facilitate the calculation and to obtain understandable expressions, we introduce a different (physically realizable) limitation on the high frequency band designated by $\Delta\omega$. We recall that an optical band of width $\Delta\omega \gtrsim 10^9$ cps can be obtained from thermal sources. For any real instrument the low-frequency band width is clearly many times smaller. In the rf region both $\Delta\omega \gg \Delta\Omega$ and $\Delta\omega \ll \Delta\Omega$ are possible; we shall consider these two extreme cases.

A. $\Delta\omega \gg \Delta\Omega$. Since our results will subsequently be compared with (2) and (3), we may assume in this case that the detector absorbs only in a band $\Delta\omega$ that is much narrower than the carrier; (2) and (3) apply to this condition. Retaining only rapidly varying functions of the frequency in the integrand of (19), setting $d\kappa = k^2\cos\theta do$, and integrating over positive frequencies, we obtain

$$\Psi_I=\frac{\hbar^2\sigma\omega^4}{16\pi^4c^2}\sum_{i=1}^{2}\int|F(\omega-\omega')|^2\cos(\omega-\omega')\tau$$
$$\times[A_i^2n_1(n_1+1)+A_iA_i'n_2(n_2+1)$$
$$+A_i'(1-A_i)(2n_1n_2+n_1+n_2)]\,d\omega\,d\omega'\cos\theta\,do, \tag{20}$$

where $n_\alpha = \frac{1}{2}[\coth(\hbar\omega/2kT_\alpha)-1]$ is the mean occupation number of an oscillator at the temperature T_α and $|1 - R_i^*R_i'| \cong A_i$ when $\Delta\omega \gg \Delta\Omega$.

We denote the absorption coefficient at the carrier frequency by $\tilde{A}_i$. With a rectangular shape of $\tilde{A}_i$ and $F(\omega)$ as functions of frequency, we have

$$\int_0^\infty A_i^2|F(\omega-\omega')|^2\cos(\omega-\omega')\tau\,d\omega\,d\omega'$$
$$=\tilde{A}_i\int_0^\infty A_i|F(\omega-\omega')|^2\cos(\omega-\omega')\tau\,d\omega\,d\omega'$$
$$\cong\int_0^\infty A_iA_i'|F(\omega-\omega')|^2\cos(\omega-\omega')\tau\,d\omega\,d\omega'$$
$$\cong 2\tilde{A}_i^2\frac{\Delta\omega\sin\Delta\Omega\tau}{\tau} \tag{21}$$

For isotropic radiation the efficiency of the detector can reasonably be defined as

$$\varepsilon = \frac{1}{2}\int (\bar{A}_1 + \bar{A}_2)\cos\theta\, do \Big/ \int \cos\theta\, do$$
$$= \frac{1}{2\pi}\int (\bar{A}_1 + \bar{A}_2)\cos\theta\, do. \tag{22}$$

This equation, in conjunction with (21), leads to

$$\Psi_I = \frac{\hbar^2\sigma\omega^4\varepsilon}{4\pi^3c^2}\,\frac{\Delta\omega\sin\Delta\Omega\tau}{\tau}\Big[n_1+n_2+\frac{\mu}{2\pi\varepsilon}(n_1^2+n_2^2)$$
$$+2\Big(1-\frac{\mu}{2\pi\varepsilon}\Big)n_1n_2\Big], \tag{23}$$
$$\mu = \int (\bar{A}_1^2 + \bar{A}_2^2)\cos\theta\, do. \tag{24}$$

The first term within the square brackets in (23) is the sum of "particle" fluctuations in the incident radiation and the characteristic radiation of the detector; the second term is the sum of the "wave" fluctuations. The last term results from interference between the radiation of the detector and external radiation reflected by the latter. This term will, of course, vanish when one of the interfering components is absent (either $n_1 = 0$ or $n_2 = 0$):

$$1 - \frac{\mu}{2\pi\varepsilon} = \frac{1}{2\pi}\sum_{i=1}^{2}\int \bar{A}_i(1-\bar{A}_i)\cos\theta\, do = 0,$$

i.e., when $\bar{A}_i \equiv 1$. This represents the idealized case of a nonreflecting (perfectly black) detector. When the detector is at room temperature these "interference" fluctuations can make a considerable contribution to Ψ_I only at radio frequencies and for small detector efficiency ($\epsilon \lesssim n_2$).

It is easily seen that (2) and (3) are actually limiting cases of (23). Before proceeding to the proof, we note that the angular dependence of the photoelectric yield was neglected in the derivation of (3). If the results are to be compared, we must therefore set $\mu = 2\pi\epsilon^2$ in (23). Setting the current output I equal to $m\hbar\omega/t$ (where m is the difference between the numbers of photons absorbed and emitted in a time t), we obtain the dispersion of m: $\langle \Delta m^2\rangle = \Psi_I(0)t^2/(\hbar\omega)^2$. Substituting for $\Psi_I(0)$ from (23) and making the customary assumption $t = \pi/\Delta\Omega$, we obtain

$$\langle\Delta m^2\rangle = \langle m\rangle[n_1\varepsilon(1-n_2/n_1)^2 + n_2/n_1 + 2n_2 + 1]. \tag{25}$$

This equation is reduced to (2) for $T_1 = T_2 = T$, and to (3) for a "cold" detector ($T_2 = 0$, $T_1 = T$).

It follows from (23) that detector output fluctuations are determined by the integral sensitivity only when the detector is in equilibrium with radiation. In other cases (especially at radio frequencies) Ψ_I can exhibit strong dependence on the angular sensitivity of the detector.

B. $\Delta\omega \ll \Delta\Omega$. If we in this case also limit the spectral sensitivity of the detector, the behavior of the function $1 - R_i^* R_i'$ inside the band $\Delta\Omega$ becomes important, i.e., we must take into account the frequency dependence of both the modulus and argument of R_i. We shall simplify the problem somewhat by assuming that incident external radiation passes through a filter with the energy characteristic $\Phi(\omega)$ (the $\Delta\omega$ band), but that the reflection coefficients vary little within the $\Delta\Omega$ band.

Within the framework of the present theory, we must take into account not only the filtered external radiation, but also the characteristic radiation of the light filter. We assume that the latter is located sufficiently far from the detector so that it does not affect the emission from the latter. Consideration of this "zero-point" energy flow from a "cold" filter does not mean, of course, that any real energy transfer occurs, but is simply a formal consequence of the symmetrized mean values [see Eq. (4)] that do not vanish in the vacuum state and are used to represent second order moments. When this "zero-point" flux is taken into account we have for the incident field

$$\mathbf{E}_{\mathrm{inc}}{}^i = \mathbf{E}^i + \mathbf{E}_f{}^i,$$

where $\mathbf{E}^i$ is the filtered field strength and $\mathbf{E}_f^i$ is the "zero-point" filter emission that is independent of $\mathbf{E}^i$. For the central moment of the incident field we now have

$$\langle E_{\mathrm{inc}}{}^i E_{\mathrm{inc}}{}'^i E_{\mathrm{inc}\,1}{}^i E_{\mathrm{inc}\,1}{}'^i\rangle_0 = \langle E^i E'^i E_1{}^i E_1{}'^i\rangle_0$$
$$+ \langle E_f{}^i E_f{}'^i E_{f1}{}^i E_{f1}{}'^i\rangle_0$$
$$+ \langle (E^i E_f{}'^i + E_f{}^i E'^i)(E_1{}^i E_{f1}{}'^i + E_{f1}{}^i E_1{}'^i)\rangle_0. \tag{26}$$

Equations (7) and (8) are used to express the right-hand side of (26) in terms of the moments $\langle E^i E_1'^i\rangle$ and $\langle E_f^i E_{f1}'^i\rangle$. For the first of these we obviously have, from (13),

$$\langle E^i E_1{}'^i\rangle = \frac{\Phi(\omega)\hbar\omega}{8\pi^2c\cos\theta}\,\mathrm{cth}\,\frac{\hbar\omega}{2kT_1}\,\delta(\varkappa+\varkappa_1')\,\delta(\omega+\omega_1'). \tag{27}$$

In order to obtain $\langle E_f^i E_{f1}'^i\rangle$ we note that for $T = 0$ the behavior of the light filter (which is also at zero temperature) does not change the vacuum mean $\langle E_{\mathrm{inc}}^i E_{\mathrm{inc}\ 1}'^i\rangle|_{T=0}$. Therefore,

$$\langle E_{\mathrm{inc}}{}^i E_{\mathrm{inc}\ 1}{}'^i\rangle|_{T=0} = \langle E^i E_1{}'^i\rangle|_{T=0} + \langle E_f{}^i E_{f\,1}{}'^i\rangle$$
$$= \frac{\hbar\omega}{8\pi^2c\cos\theta}\,\mathrm{sign}\,\omega\,\delta(\varkappa+\varkappa_1')\,\delta(\omega+\omega_1').$$

Using (27), we then obtain

$$\langle E_f{}^i E_{f1}{}'^i\rangle = (1-\Phi(\omega))\frac{\mathrm{sign}\,\omega}{8\pi^2c\cos\theta}\,\delta(\varkappa+\varkappa_1')\,\delta(\omega+\omega_1'). \tag{28}$$

Repeating the calculations in part A, except for the fact that $\Phi(\omega)$ and $F(\omega)$ are now considered to be rapidly vanishing functions of ω and that the reflection coefficients are constant in the interval $\Delta\Omega$, we obtain

$$\Psi_I = \Psi_I{}^0 + \frac{\hbar^2\omega^4\sigma}{16\pi^4 c^2}\{L_1\mu n_1{}^2 + 2L_2[\pi\varepsilon n_1 + 2n_1 n_2(2\pi\varepsilon - \mu)]\}, \tag{29}$$

where Ψ_I^0 is the correlation function of the output signal in the absence of external radiation:

$$\Psi_I{}^0 = \frac{\hbar^2\sigma}{128\pi^4}\sum_{i=1}^{2}\int \omega\omega'|F(\omega+\omega')|^2 \exp[i(\omega+\omega')\tau]$$
$$\times\Big[A_i A_i'\Big(\operatorname{cth}\frac{\hbar\omega}{2kT_2}\operatorname{cth}\frac{\hbar\omega'}{2kT_2}+1\Big)$$
$$+ 2A_i'(1-A_i)\Big(\operatorname{sign}\omega \operatorname{cth}\frac{\hbar\omega'}{2kT_2}+1\Big)\Big]\,d\omega\, d\omega'\, d\varkappa; \tag{30}$$

$$L_1 = \int_0^\infty \Phi(\omega)\Phi(\omega')\,|F(\omega-\omega')|^2\cos(\omega-\omega')\tau\, d\omega\, d\omega',$$

$$L_2 = \int_0^\infty \Phi(\omega)\,|F(\omega-\omega')|^2\cos(\omega-\omega')\tau\, d\omega\, d\omega'. \tag{31}$$

When all the frequency characteristics are rectangular and $\Delta\omega \ll \Delta\Omega$, we obtain

$$L_1 = 4\frac{\sin^2(\Delta\omega\tau/2)}{\tau^2},$$
$$L_2 = 4\frac{\sin\Delta\omega\tau}{\tau}\,\frac{\sin(\Delta\omega\tau/2)}{\tau}. \tag{32}$$

The mean square fluctuation becomes

$$\Psi_I(0) = \Psi_I{}^0(0) + \frac{\hbar^2\omega^4\sigma\varepsilon n_1\Delta\omega}{4\pi^3c^2}\Big[\frac{\mu\Delta\omega n_1}{4\pi\varepsilon} + \Delta\Omega\Big[1+2n_2\Big(1-\frac{\mu}{2\pi\varepsilon}\Big)\Big]\Big]. \tag{33}$$

Unlike the case represented by (23), the "wave" fluctuations are here determined only for a narrower $\Delta\omega$ band. The expression for the dispersion of the "particle" and "interference" fluctuations, despite the different correlation functions, remains the same as previously and depends only on the band product $\Delta\omega\Delta\Omega$.

4. COHERENT DETECTOR IRRADIATION

We shall show in this section that the phenomenological theory leads to certain expressions that were previously obtained by means of a semiclassical analysis.[8,11,18] The latter studies were based on an assumed Poisson distribution for the number of photoelectrons ejected in a time t from the surface of a cathode at zero temperature; the mean number was proportional to the time integral of the instantaneous incident light intensity. This hypothesis leads to correct results, although we do not believe that it has been justified rigorously.

In speaking of coherent radiation we shall henceforth refer simply to the fulfillment of the condition that the detector area lies entirely inside the coherence area $\lambda^2/\Delta o$ of a narrow incident beam, where Δo is the solid angle of the beam. We note that the coherence area $\lambda^2/\Delta o$ coincides with the cross section of an elementary cell in the phase space of the beam.

We assume that the incident beam is discriminated from the equilibrium radiation by a suitable instrument. This is convenient, since all calculations are simplified because the individual Fourier components are uncorrelated. A simple analysis also shows that when $\sigma \ll \lambda^2/\Delta\omega$ the absence of a delta correlation between the separate Fourier components does not affect the final result. Reasoning as in the derivation of (28), we arrive at the conclusion that for the second order moments of the incident field we may use the expressions

$$\langle E^i E_1'^i\rangle = \frac{\Sigma(\theta)\Phi(\omega)\hbar\omega}{8\pi^2 c\cos\theta}\operatorname{ctg}\frac{\hbar\omega}{2kT}\delta(\varkappa+\varkappa_1')\delta(\omega+\omega_1'),$$
$$\langle E_f{}^i E_{f1}'^i\rangle = (1-\Sigma(\theta)\Phi(\omega))\frac{\hbar\omega}{8\pi^2c\cos\theta}\operatorname{sign}\omega\,\delta(\varkappa_1+\varkappa_1')$$
$$\times\,\delta(\omega+\omega_1'), \tag{34}^*$$

where $\Phi(\omega)$ is the frequency characteristic and $\Sigma(\theta)$ is the angular characteristic of the instrument. The latter is taken to have the form

$$\Sigma(\theta) = \begin{cases}1, & \theta \leqslant \Delta\theta\\ 0, & \theta > \Delta\theta.\end{cases}$$

The second equation in (34) pertains to zero-point fluctuations in the incident field, and to an "excited" light filter and collimator at zero temperature.

From (34) together with (12)—(17) we obtain for the detector at zero temperature

$$\Psi_I = \frac{\hbar^2}{512\pi^6}\sum_{i=1}^{2}\int \omega\omega'|F(\omega+\omega')|^2\exp[i(\omega+\omega')\tau]$$
$$\times\{\ldots\}\,d\omega\,d\omega'\,d\varkappa\,d\varkappa'\,d\mathbf{r}\,d\mathbf{r}',$$
$$\{\ldots\} = |1-R_iR_i'|^2\Phi(\omega)\Sigma(\theta)\Big[\Phi(\omega')\Sigma(\theta')$$
$$\times\Big(\operatorname{cth}\frac{\hbar\omega}{2kT}\operatorname{cth}\frac{\hbar\omega'}{2kT}+1\Big)$$

*ctg ≡ cot.

$$+2(1-\Phi(\omega')\Sigma(\theta'))\left(\operatorname{sign}\omega'\operatorname{cth}\frac{\hbar\omega}{2kT}+1\right)\Big]$$
$$+2A_i'(1-A_i)\Phi(\omega)\Sigma(\theta)\left(\operatorname{cth}\frac{\hbar\omega}{2kT}\operatorname{sign}\omega'+1\right). \quad (35)$$

Since $\sigma^2 \ll \lambda^2/\Delta o$, with $(\kappa + \kappa')\rho \ll 1$ inside the coherence area, we have

$$\int \Sigma(\theta)\Sigma(\theta')\exp[i(\varkappa+\varkappa')\rho]\,dr\,dr' \cong \sigma^2\Sigma(\theta)\Sigma(\theta'). \quad (36)$$

Furthermore,

$$\int \Sigma(\theta)\exp[i(\varkappa+\varkappa')\rho]\,dr\,dr' \cong 4\pi^2\sigma\Sigma(\theta)\delta(\varkappa'),$$
$$\int \Sigma(\theta)\Sigma(\theta')\,d\varkappa\,d\varkappa' \cong k^2k'^2(\Delta o)^2,$$
$$\int \Sigma(\theta)\delta(\varkappa')\,d\varkappa\,d\varkappa' \cong k^2\Delta o. \quad (37)$$

Assuming that the reflection coefficients vary very little in the frequency intervals $\Delta\omega$ and $\Delta\Omega$, we let $|1 - R_i^*R_i'|^2 \equiv A^2$. Moreover, for the sake of simplicity we shall consider linearly polarized radiation ($A_2 = 0$). Then, substituting (36) and (37) into (35), if all frequency characteristics are rectangular we obtain

$$\Psi_I = \frac{A\hbar^2\sigma\Delta on}{4\pi^2\lambda^2}\left[L_1 n\frac{\sigma\Delta oA}{\lambda^2}+L_2\right], \quad (38)$$

where L_1 and L_2 are given by (31). When written for the number of photons absorbed during a time t, this formula gives for the intensity of fluctuations

$$\langle\Delta\nu^2\rangle = \frac{t\langle\nu\rangle}{2\pi\Delta\omega}\left[\frac{2\pi L_1(0)}{t\Delta\omega}\langle\nu\rangle + L_2(0)\right], \quad (39)$$

where $\langle \nu \rangle = \sigma\Delta oAt\Delta\omega n/2\pi\lambda^2$ is the mean number of photons absorbed from the beam during a time $t = \pi/\Delta\Omega$.

For $\Delta\omega \ll \Delta\Omega$ we have

$$\langle\Delta\nu^2\rangle = \langle\nu\rangle(\langle\nu\rangle+1), \quad (40)$$

which coincides with Mandel's equation for linearly polarized light.[11] The case $\Delta\omega \gg \Delta\Omega$ has been analyzed by Purcell,[8] who obtained

$$\langle\Delta\nu^2\rangle = \langle\nu\rangle\left(\frac{2\pi\gamma}{\Delta\omega t}\langle\nu\rangle+1\right), \quad (41)$$

where γ is a constant, of the order of unity, that depends on the line shape. We note that for $\Delta\omega \gg \Delta\Omega$ we have $L_1(0) = L_2(0) = 2\Delta\omega\Delta\Omega$; we then obtain from (39)

$$\langle\Delta\nu^2\rangle = \langle\nu\rangle\left(\frac{2\pi}{\Delta\omega t}\langle t\rangle+1\right), \quad (42)$$

which coincides with (41) when $\gamma = 1$.

Mandel [11] has given a simple interpretation of (40) and (41) in particle language. When we recall that for the incident beam an elementary cell is of the order $c/\Delta\omega$, it is clear that in the case $\Delta\omega \gg \Delta\Omega$ photons from $t\Delta\omega$ cells impinge on the detector in a time t. The photoelectron distribution therefore obeys Bose-Einstein statistics for the number ν of particles distributed in $t\Delta\omega$ cells [see Eq. (42)]. When $\Delta\omega \ll \Delta\Omega$, photons from a single cell strike the detector in a time t, and the photoelectron distribution obeys Bose-Einstein statistics for the number ν of indistinguishable particles [see Eq. (40)].

When we set $A_1 = \alpha A_2$ it is not difficult to extend (39) to the case of external radiation with any degree of polarization. We shall not present this result but shall confine ourselves to mentioning that it agrees completely with that obtained by Wolf.[18]

We have here confined ourselves to a detector with a flat surface. However, all our results can clearly be extended to detectors of any shapes that permit the use of surface reflection formulas. The problem is especially simple in the case of isotropic external radiation. It is then sufficient to divide the detector surface into separate plane elements having dimensions that are much larger than the correlation radius of the fields (i.e., the wavelength), whose fluctuations are thus mutually independent. Therefore σ can be taken to represent the entire detector area in all formulas of Sec. 3.

The foregoing method will enable us, of course, not only to study the output fluctuations of a single detector, but also to calculate the mutual correlation function of output signals from two detectors. In the special case of "cold" detectors the result agrees with that obtained in [7].

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