ON THE PERMISSIBLE RATE OF DECREASE OF THE FORWARD SCATTERING AMPLITUDE AT HIGH ENERGIES

R. WIT [1)]

Joint Institute for Nuclear Research

Submitted to JETP editor January 16, 1965

J. Exptl. Theoret. Phys. (U.S.S.R.) **49**, 538-541 (August, 1965)

The permissible rate of decrease of the forward scattering amplitude is estimated on the basis of the Phragmén-Lindelöf theorem and the theory of Herglotz functions. As an example for the application of the results, a proof is given of the incompatibility of the two-particle unitarity condition and analyticity in the Lee model.

1. Recently, much attention has been paid to the circumstance that, owing to unitarity, the imaginary part of the scattering amplitude is positive for

$$0 \leqslant t \leqslant 4M^2 \tag{1}$$

and physical values of the energy.[1] Together with analyticity, this leads to interesting results in the region of high energies.[2]

On the other hand, there has also been a successful development in another direction based on some theorems of the general theory of analytic functions connected with the concept of harmonic measure.[3] Here the Phragmén-Lindelöf theorem (PLT in the following) plays a fundamental role. It turned out that in this approach the positiveness of the imaginary part of the scattering amplitude leads to important results.[4] We should like again to address ourselves to the possibilities contained in this method.

Recently, Jin and Martin[5] have studied the question of the restrictions on the rate of decrease of the forward scattering amplitude imposed by unitarity, analyticity, and crossing symmetry. They arrived at the conclusion that

$$\lim_{s\to\infty} s^2 |T(s, 0)| (\ln s)^{1/2} = \infty \tag{2}$$

or

$$\lim_{s\to\infty} |T(s, 0)| (\ln s)^{1/2} = \infty, \tag{3}$$

if $T(s, 0)$ has an additional zero.

2. We shall show in the present paper that the results (2) and (3), which are obtained as necessary conditions for the convergence of certain integrals (about which it is known beforehand, from the general theory of Herglotz H functions, that they are bounded[6]), are not the best possible conditions. With this aim, we consider the amplitude for the scattering of π^+ mesons by protons

$$T(z) = {}^1/_2 [T^+(z) + T^-(z)], \tag{4}$$

for which the Cauchy formula takes the form [2)]

$$T(z) = T(1) + \frac{f^2}{M} \frac{z^2 - 1}{z^2 - \omega_B{}^2} \frac{1}{1 - \omega_B{}^2} + \frac{2}{\pi}(z^2 - 1) \int_1^\infty \frac{x \operatorname{Im} T(x)\, dx}{(x^2 - 1)(x^2 - z^2)} \tag{5}$$

where $z = \omega + iy$; ω is the energy of the incident meson in the lab system and is proportional to the invariant s.

For $\operatorname{Im} u > 0$, $u = z^2$, we see from (5) that $\operatorname{Im} T(u) > 0$ and therefore, $T(u)$ is a Herglotz H function. It follows from this[6] that for $\epsilon < \arg u < \pi - \epsilon$

$$c / |u| \leqslant |T(u)| \leqslant c' |u|. \tag{6}$$

Now the question arises whether $T(\omega)$ can decrease more rapidly than $1/\omega^2$ at infinity, i.e., whether

$$\lim_{\omega\to+\infty} \omega^2 T(\omega + i0) = 0. \tag{7}$$

As already noted, microcausality imposes on the increase of T the restriction

1)Jagiellonian University, Cracow, Poland.

2)In deriving this formula, one assumes, besides crossing symmetry and the Schwarz-Riemann lemma, that $\lim_{\omega\to\infty} T(\omega + i0)/\omega^2 = 0$ such that the corresponding integrals converge, and

$$|T(z)| < Ae^{\epsilon|z|},$$

for $\operatorname{Im} z > 0$, which follows from the causality condition.[7]

$$|T(u)| < A \exp(\varepsilon|u|^{1/2}), \qquad \operatorname{Im} u > 0. \tag{8}$$

Then condition (7) is fulfilled in the upper half-plane by the PLT,[3] i.e., we have

$$\lim_{|u|\to\infty} uT(u) = 0 \qquad \text{for } \operatorname{Im} u > 0. \tag{9}$$

The asymptotic equality (9) is in clear contradiction with (6), and thus (7) cannot be fulfilled.[3)]

In analogy to the foregoing one can easily show the finiteness of the number of zeros of $T(\omega)$, if it is bounded by a polynomial at high energies.[5] [4)] Some remarks connected with the existence of zeros of $T(z)$ are collected in the Appendix.

3. Let us show now how our discussion can be applied to the proof of the incompatibility of unitarity and analyticity in the Lee model (another proof was given several years ago by Ter-Martirosyan[12]). We recall that in this model the scattering amplitude satisfies the following equation:

$$T(z) = \frac{g^2}{\varepsilon_0 - z} + \frac{1}{\pi}\int_1^\infty \frac{dx(x^2-1)^{1/2}}{x-z}|T(x)|^2. \tag{10}$$

Here the unitarity condition

$$\operatorname{Im} T(\omega + i0) = (\omega^2 - 1)^{1/2}|T(\omega)|^2, \quad \omega > 1, \tag{11}$$

is taken into account, which also implies that at infinity, $T(\omega)$ must have the form $\varphi(\omega)/\omega$, where $|\varphi(\omega)|$ is bounded.

It follows from (10) that $T(z)$ a) is a Herglotz H function, b) has only one cut along the real axis for $\omega \in (1, \infty)$, and c) satisfies the reality condition

$$T(z) = T^*(z^*).$$

Let us consider the function

$$\Phi(z) = (z - \varepsilon_0)T(z),$$

which has all the properties enumerated above, and for which the unitarity condition takes the form

$$\operatorname{Im}\Phi(\omega) = |\Phi(\omega)|^2(\omega^2-1)^{1/2}/(\omega-\omega_0), \quad \omega \geqslant 1. \tag{12}$$

If $\Phi(\omega)$ has a finite limit at infinity,[5)] this limit must be real, since the sign of $\operatorname{Im}\Phi(\omega - i0)$ is opposite that of $\operatorname{Im}\Phi(\omega + i0)$ for $\omega > 1$. Therefore we obtain from (12)

$$\lim_{\omega\to\infty}\Phi(\omega + i0) = 0. \tag{13}$$

The relation (13) can be proved even if $\Phi(\omega)$ has no finite limit at infinity. Indeed, for a Herglotz H function, for which $\operatorname{Im} H(\omega) = 0$ for $\omega \leq \omega_0$, one of the following integrals must converge:[12]

$$\int^\infty \frac{\operatorname{Im} H(\omega)}{\omega} d\omega, \qquad \int^\infty \frac{\operatorname{Im} H(\omega)}{|H(\omega)|^2}\frac{d\omega}{\omega}.$$

For the function $\Phi(\omega)$ the second integral diverges because of (12). Therefore we must require that

$$\lim_{\omega\to\infty}\Phi(\omega)(\ln\omega)^{1/2} = 0. \tag{14}$$

The equations (13) and (14) reduce thus to

$$\lim_{\omega\to\infty}\omega T(\omega + i0) = 0,$$

which must not be the case, as shown earlier. This is the proof of the incompatibility of the unitarity condition and analyticity in the Lee model (without form factors).

4. Since the boundedness by a polynomial plays an essential role in the proof[13] of the inequality

$$\sigma_{el}(\ln\omega)^2 > \text{const}\,(\sigma_{tot})^2, \tag{15}$$

this inequality (15) is another of the results obtained from the theory of harmonic measure.

As is seen from our discussion, the PLT affords, besides simplifications in the course of the proofs, also important improvements of the results obtained earlier.

I thank I. T. Todorov and Nguyen Van Hieu for useful discussions and also B. Slowinski for a careful reading of the manuscript of this paper.

APPENDIX

$T(u)$, being a Herglotz H function, is real for $u \in (-\infty, \omega_B^2)$ and has a positive derivative. Therefore, $T(u)$ only has a zero on this half-axis if

$$\lim_{u\to-\infty} T(u) = M < 0$$

or

$$\lim_{u\to-\infty} T(u) = -\infty,$$

since near the pole it is positive for the values of u considered. If

$$\lim_{u\to\infty} T(u) = 0,$$

then we obtain by the PLT

$$\lim_{u\to-\infty} T(u) = 0$$

and $T(u)$ has no zeros an the real axis. Hence one can impose stronger restrictions on the rate

---

[3)]This was shown by Jin and Martin[5] assuming polynomial behavior of $T(\omega)$ at infinity.

[4)]On this point cf. also[8-10].

[5)]Owing to the two-particle unitarity condition, it suffices for this that there exist a limit to the imaginary or the real part of $T(\omega)$ alone.

of decrease of $T(\omega)$ even in the other case, analyzed by Jin and Martin.

[1] A. Martin, Lecture at the Scottish Summer School of Theoretical Physics, Edinburgh, Scotland, 1963.

[2] T. Kinoshita, Restriction Imposed by Unitarity and Analyticity on the Behavior of the Scattering Amplitude at High Energy, LNS Cornell University, Ithaca, N. Y., 1964.

[3] N. N. Meĭman, JETP **43**, 2277 (1962), Soviet Phys. JETP **16**, 1609 (1963).

[4] Logunov, Nguyen van Hieu, and Todorov, Ann. Phys. (N. Y.) **31**, 203 (1965).

[5] Y. S. Jin and A. Martin, Phys. Rev. **135**, B1369 (1964).

[6] J. A. Shonat and J. D. Tamarkin, The Problem of Moments, American Mathematical Society, N. Y., 1943.

[7] N. N. Meĭman, preprint ITEF No. 252, 1964. JETP **46**, 1502 (1964), Soviet Phys. JETP **19**, 1018 (1964).

[8] Ansel'm, Gribov, Danilov, Dyatlov, and Shekhter, JETP **41**, 619 (1961), Soviet Phys. JETP **14**, 444 (1962).

[9] S. Aramaki, Progr. Theoret. Phys. **28**, 479 (1963).

[10] J. Namyslowski and R. Wit, Acta Phys. Pol. **23**, 197 (1963).

[11] K. A. Ter-Martirosyan, JETP **37**, 1005 (1959), Soviet Phys. JETP **10**, 714 (1960).

[12] S. Weinberg, Phys. Rev. **124**, 2049 (1961).

[13] A. Martin, Nuovo cimento **29**, 993 (1963).

Translated by R. Lipperheide
72