# *MUTUAL FRICTION FORCE IN A ROTATING BOSE GAS*

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The mutual friction force associated with the scattering of excitations by vortex filaments is calculated. On the basis of general formulas obtained earlier by the author it is shown that the phonon part of the mutual friction force contains a large transverse component. It is shown that taking additional transverse terms into account improves agreement with experiment for the roton part of the mutual friction force.

A number of papers[1-3] is devoted to the calculation of the mutual friction force in a rotating superfluid on the basis of the Onsager-Feynman model of vortex filaments. The principal difficulty was the appearance in the calculations of a large transverse component (not leading to dissipation) which is not observed experimentally, and also an incorrect temperature dependence. In papers by the present author[4,5] it was shown that the mutual friction force is not directly expressible in terms of the transport cross section for the scattering of excitations by a vortex filament in contrast to the assumption made by authors of earlier papers. Moreover, it is necessary to note that the problem of the scattering of excitations is itself a very complicated one.

The most complicated problem is that of the scattering of short wavelength excitations (rotons). However, certain difficulties exist also in the case of the problem of less interest from the experimental point of view, that of the scattering of phonons, which play a principal role only at very low temperatures. In a recently published paper by Fetter[6] it is noted that the results obtained by Pitaevskiĭ[3] for the problem of phonon scattering are incorrect from a formal point of view since they do not take into account the strong distortion of the incident wave at small distances from the filament.

Fetter has considered small distances in the case of two hydrodynamical models (an empty and a uniformly rotating "core" of a vortex filament). The result for the transport cross section in reference[6] agrees with Pitaevskiĭ's result[3], although the differential cross sections differ somewhat. But the use of the hydrodynamic approximation cannot be justified at small distances from the axis of the filament, and in this sense the model of a weakly nonideal Bose-gas is apparently a more realistic one.

It is necessary to note that expressions for the mutual friction force given in references[6,3] do not take into account an additional term which is not expressible in terms of the transport cross section.

In the present paper we obtain a more accurate general expression for the mutual friction force than obtained previously[4,5], and also investigate the scattering of long wavelength excitations (phonons) by a vortex filament within the framework of the model of a weakly nonideal Bose-gas.

## 1. SOME GENERAL RELATIONS FOR THE PROBLEM OF THE SCATTERING OF EXCITATIONS

We shall start with the well-known Bogolyubov model[7] in accordance with which the second quantization operators $\psi$, $\psi^+$ are taken to be equal to

$$\psi(\mathbf{x}) = \varphi(\mathbf{x}) + \psi_1(\mathbf{x}), \quad \psi^+(\mathbf{x}) = \varphi^*(\mathbf{x}) + \psi_1^+(\mathbf{x}),$$

where $\varphi$ and $\varphi^*$ are ordinary functions (c-numbers) characterizing a Bose-condensate, $\psi_1$ and $\psi_1^+$ are Bose-operators corresponding to the annihilation and creation of noncondensed particles.

For a weakly nonideal Bose-gas when the Born parameter $\xi$ for the two-body interaction $\Phi(x)$ between particles is given by

$$\xi = \Phi(0) r_0^2 m / \hbar^2 \ll 1,$$

where $r_0$ is the characteristic range of the potential $\Phi$, the number of noncondensed particles is small. Therefore, we can assume that the equation for $\varphi$ does not depend in the first approximation on $\psi_1$ and $\psi_1^+$:

$$-i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\varphi - \lambda\varphi + \int \Phi(\mathbf{x}-\mathbf{x}')\,|\varphi(\mathbf{x}')|^2\,\varphi(\mathbf{x})\,d^3x', \tag{1.1}$$

where the constant $\lambda$ is the chemical potential.

Equation (1.1) has been investigated by Pitaevskiĭ[8] and by Gross[9]. It admits the stationary solution

$$\varphi = a(r)e^{i\vartheta}, \quad \lambda = a_0^2\int\Phi(\mathbf{x})\,d^3x, \tag{1.2}$$

which describes an isolated vortex filament. Here $\vartheta$ is the polar angle, r is the distance from the axis of the vortex filament. The function a(r) remains approximately constant and equal to its value $a_0$ at infinity right down to distances of order $r_0$ (for $a_0^2 r_0^3 \sim \xi^{-1}$), and then begins to fall off and tends to zero like r for $r \to 0$.

It can be shown that the operators $\psi_1$ and $\psi_1^+$ can be written in the form[5]

$$\psi_1 = \sum_{\alpha,\,E_\alpha>0} e^{i\vartheta}[c_\alpha\psi_{\alpha E_\alpha}(\mathbf{x}) + c_\alpha^+\overset{+*}{\psi}_{\alpha E_\alpha}(\mathbf{x})],$$

$$\psi_1^+ = \sum_{\alpha,\,E_\alpha>0} e^{-i\vartheta}[c_\alpha\overset{+}{\psi}_{\alpha E_\alpha}(\mathbf{x}) + c_\alpha^+\psi^*_{\alpha E_\alpha}(\mathbf{x})], \tag{1.3}$$

where $c_\alpha$ and $c_\alpha^+$ are the usual Bose annihilation and creation operators. The functions $\psi_{\alpha E_\alpha}$ and $\psi^+_{\alpha E_\alpha}$ satisfy the equation

$$E_\alpha\boldsymbol{\psi}_{\alpha E_\alpha} = \hat{H}_0\boldsymbol{\psi}_{\alpha E_\alpha} + \hat{V}\boldsymbol{\psi}_{\alpha E_\alpha} \tag{1.4}$$

where the vector notation $\boldsymbol{\psi}_{\alpha E_\alpha} = (\psi_{\alpha E_\alpha}, \psi^+_{\alpha E_\alpha})$ has been utilized, while the matrix operator $\hat{H}_0$ is the "Hamiltonian" for free excitations:

$$\hat{H}_0\boldsymbol{\psi} = -\frac{\hbar^2}{2m}\hat{g}\Delta\boldsymbol{\psi} + a_0^2\int\Phi(\mathbf{x}-\mathbf{x}')\,\hat{\delta}\boldsymbol{\psi}(\mathbf{x}')\,d^3x', \tag{1.5}$$

where the matrix operator $\hat{V}$ is the "interaction Hamiltonian" between the excitations and the filament;

$$\hat{V}\boldsymbol{\psi} = -\frac{i\hbar^2}{m}(\nabla\vartheta\cdot\nabla)\boldsymbol{\psi} + \frac{\hbar^2}{2m}\frac{\Delta a}{a}\hat{g}\boldsymbol{\psi} + \int\Phi(\mathbf{x}-\mathbf{x}')\,[a(\mathbf{x}) \times a(\mathbf{x}') - a_0^2]\,\hat{\delta}\boldsymbol{\psi}(\mathbf{x}')\,d^3x', \tag{1.6}$$

with

$$\hat{\delta} = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad \hat{g} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We are interested in the eigenfunctions $\psi_{\mathbf{k}}$ belonging to the continuous spectrum and corresponding to the scattering of excitations by the filament, since only the latter give a contribution to the mutual friction force.

We shall make one remark about the model utilized. In the case when the Born parameter $\xi$ is small the Bogolyubov model will be valid also for densities $a_0^2 r_0^3 \sim 1/\xi$ as shown in the paper by Tserkovnikov[10], with the character of the spectrum being qualitatively similar to that for helium. In future we shall simply formally assume that the quantities $\Phi$ and $a_0$ are related in such a way that the equation

$$E_{\mathbf{k}} \equiv \hbar k(\Phi_{\mathbf{k}}a_0^2/m + \hbar^2k^2/4m)^{1/2} = E$$

can have several real roots $k_s > 0$ ($\Phi_{\mathbf{k}}$ is the Fourier transform of $\Phi(\mathbf{x})$, $E_{\mathbf{k}}$ is the Bogolyubov excitation energy at infinity).

In view of certain differences between (1.4) and the usual Schrödinger equation we must for the following discussion obtain the relation between the asymptotic expression for $\psi_{\mathbf{k}}$ and quantities corresponding to the scattering amplitude. In the case of a scattering problem the functions $\psi_{\mathbf{k}}$ are determined by the equation

$$\boldsymbol{\psi}_{\mathbf{k}} = \mathbf{f}_{\mathbf{k}} + \frac{1}{E_{\mathbf{k}} + i\varepsilon - \hat{H}_0}\hat{V}\boldsymbol{\psi}_{\mathbf{k}};$$

$$\mathbf{f}_{\mathbf{k}} = e^{i\mathbf{k}\mathbf{x}}\mathbf{u}(k), \qquad \mathbf{u}(k) = (u(k), v(k)),$$

$$u(k) = \frac{\Phi_{\mathbf{k}}a_0^2}{(2E_{\mathbf{k}}\Phi_{\mathbf{k}}a_0^2 - E_{\mathbf{k}}^2 - \hbar^4k^4/4m^2)^{1/2}},$$

$$v(k) = \frac{E_{\mathbf{k}} - \hbar^2k^2/2m - \Phi_{\mathbf{k}}a_0^2}{\Phi_{\mathbf{k}}a_0^2}u(k). \tag{1.7}$$

The expression for the matrix Green's function can be written in the form

$$G_{ij}(\mathbf{x}-\boldsymbol{\xi}) \equiv (E_{\mathbf{k}} + i\varepsilon - \hat{H}_0)_{ij}^{-1} = \frac{1}{i}\sum_{E_\varkappa}\int_{-\infty}^{\infty}\frac{u_i(\varkappa,E_\varkappa)\tilde{u}_j(\varkappa,E_\varkappa)}{E_{\mathbf{k}} - E_\varkappa + i\varepsilon}\frac{e^{i\varkappa|\mathbf{x}-\boldsymbol{\xi}|}}{|\mathbf{x}-\boldsymbol{\xi}|}\frac{\varkappa\,d\varkappa}{(2\pi)^2}, \tag{1.8}$$

where the summation over E is extended over all possible eigenvalues of $\hat{H}_0$ for a given $\varkappa$, i.e., to $|E_\varkappa|$ and $-|E_\varkappa|$, and

$$\tilde{\mathbf{u}}(\varkappa, E_\varkappa) = \hat{g}\mathbf{u}(\varkappa, E_\varkappa), \quad \mathbf{u}(\varkappa, E_\varkappa) = (u(\varkappa), v(\varkappa)),$$
$$\tilde{\mathbf{u}}(\varkappa, -E_\varkappa) = (-v(\varkappa), u(\varkappa)), \quad \mathbf{u}(\varkappa, -E_\varkappa) = (v(\varkappa), u(\varkappa)). \tag{1.9}$$

In order to obtain the asymptotic representation of $\hat{G}(\mathbf{x}-\boldsymbol{\xi})$ for large values of $|\mathbf{x}-\boldsymbol{\xi}|$, it is convenient to deform the contour of integration over $\varkappa$ in (1.8) into the upper half plane, and in this case the asymptotic behavior will be determined by the residues at the closest singularities, i.e., at points which are roots of the equation $E_{\mathbf{k}} + i\epsilon = E_\varkappa$, such that

$$\varkappa_s = k_s + i\delta_s, \qquad \delta_s = \varepsilon\,\mathrm{sign}\left.\frac{\partial E}{\partial k}\right|_{k_s} > 0. \tag{1.10}$$

Moreover, picking out separately the dependence on the z coordinate along the vortex filament and utilizing the asymptotic representation for the Hankel functions we obtain from (1.7) the asymptotic representation

$$\boldsymbol{\psi}_{\mathbf{k}} = \mathbf{f}_{\mathbf{k}} - \frac{i}{2}e^{ik_zz}\sum_s\frac{u(\varkappa_s, E_k)}{|\partial E/\partial\varkappa|_{\varkappa_s}}\left(\frac{2}{\pi\chi_sr}\right)^{1/2}e^{i(\chi_sr-\pi/4)}\langle\tilde{\mathbf{f}}_\varkappa|\hat{V}|\boldsymbol{\psi}_{\mathbf{k}}\rangle,$$
$$\tilde{\mathbf{f}}_\varkappa = e^{-i\varkappa x}\tilde{\mathbf{u}}(\varkappa, E_\varkappa), \tag{1.11}$$

where $\kappa_{sz} = k_z$, $\chi_s^2 = \kappa_s^2 - k_z^2$, $\chi_s$ is directed along the polar radius.

We see that in accordance with formula (1.11) excitations can be elastically reflected with a different absolute value of the momentum (in the quasiclassical approximation this was noted by Pitaevskiĭ[11]), with the stipulation that it is the group velocity and not the radial momentum that must be positive. It is clear that this result does not depend on the model utilized, but is determined exclusively by the excitation spectrum.

Formula (1.11) gives us the desired relation between the scattering amplitude $\langle \mathbf{f}_\kappa | \hat{V} | \psi_\mathbf{k} \rangle$ and the coefficients in the asymptotic representation for $\psi_\mathbf{k}$. For subsequent discussion it is convenient to carry out a complete separation of variables in cylindrical coordinates (the form of the potential $\hat{V}$ admits such a separation). We introduce the quantities

$$\mathbf{R}_{n\mathbf{k}}(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\vartheta} e^{-ik_z z} \psi_\mathbf{k}(z, \vartheta, r)\, d\vartheta,$$

$$\varphi_{n\kappa}(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{in\vartheta} e^{ik_z z} \tilde{\mathbf{f}}_\kappa(z, \vartheta, r)\, d\vartheta. \qquad (1.12)$$

From this we obtain for the scattering amplitude

$$\langle \tilde{\mathbf{f}}_\kappa | \hat{V} | \psi_\mathbf{k} \rangle = 2\pi\delta(k_z - \kappa_z) \sum_n e^{in\vartheta} \langle \rho_{n\kappa} | \hat{V}_{nk_z} | \mathbf{R}_{n\mathbf{k}} \rangle,$$

where $\hat{V}_{nk_z}$ is the corresponding Fourier component of $\hat{V}$.

The asymptotic representation for the functions (1.12) has in accordance with (1.11) the form

$$\mathbf{R}_{n\mathbf{k}}(r) = u(k) e^{i|n|\pi/2} \frac{1}{(2\pi\chi r)^{1/2}} \left(e^{i(\chi r - \pi/4)} + e^{-i(\chi r - \pi/4 - \pi|n|/2)}\right)$$
$$- i\sum_s \frac{u(\varkappa_s, E_k)}{|\partial E/\partial\varkappa|_{\varkappa_s}} \frac{1}{(2\pi\chi_s r)^{1/2}} e^{i(\chi_s r - \pi/4)} \langle \rho_{n\varkappa_s} | \hat{V}_{nk_z} | \mathbf{R}_{n\mathbf{k}} \rangle,$$
$$\chi^2 = k^2 - k_z^2. \qquad (1.13)$$

The equations for $\mathbf{R}_{n\mathbf{k}} = (R_{n\mathbf{k}}, R_{n\mathbf{k}}^+)$ have in accordance with (1.4), (1.5), and (1.6) the form

$$E_\mathbf{k} R_{n\mathbf{k}} = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} R_{n\mathbf{k}} - k_z^2 R_{n\mathbf{k}} - \frac{n^2}{r^2} R_{n\mathbf{k}}\right]$$
$$+ \frac{\hbar^2}{m}\frac{n}{r^2} R_{n\mathbf{k}} + \frac{\hbar^2}{2m}\frac{1}{ar}\left(\frac{d}{dr} r \frac{da}{dr}\right) R_{n\mathbf{k}}$$
$$+ \int_0^\infty \Phi_{nk_z}(r, r') a(r) a(r') (R_{n\mathbf{k}}(r') + R_{n\mathbf{k}}^+(r')) r' dr',$$

$$E_\mathbf{k} R_{n\mathbf{k}}^+ = \frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} R_{n\mathbf{k}}^+ - k_z^2 R_{n\mathbf{k}}^+ - \frac{n^2}{r^2} R_{n\mathbf{k}}^+\right]$$
$$+ \frac{\hbar^2}{m}\frac{n}{r^2} R_{n\mathbf{k}}^+ - \frac{\hbar^2}{2m}\frac{1}{ar}\left(\frac{d}{dr} r \frac{da}{dr}\right) R_{n\mathbf{k}}^+ - \int_0^\infty \Phi_{nk_z}(r, r') a(r) a(r') (R_{n\mathbf{k}}(r')$$
$$+ R_{n\mathbf{k}}^+(r')) r' dr'. \qquad (1.14)$$

We must find a solution of (1.14) satisfying conditions (1.13) at infinity and bounded at $r \to 0$. Of real interest is the case of small $E_\mathbf{k}$ ("phonons"), and the case when $E_\mathbf{k}$ lies near the minimum of the energy curve ("rotons"). The latter case is extremely complicated, since it requires a knowledge of the exact behaviour of $a(r)$ for $r \lesssim r_0$. We shall consider in detail only the case of small $E_\mathbf{k}$.

## 2. PHONON SCATTERING

It follows from an investigation of (1.1), that $a(r)$ begins to vary significantly at distances $r \lesssim r_0$. For phonons we have

$$E_\mathbf{k} \approx \hbar k (a_0^2 \Phi_0 / m)^{1/2} = \hbar ck \ll \hbar^2 / m r_0^2,$$

and corresponding to this we shall distinguish two regions:

$$\text{I. } r \gtrsim r_0 \zeta \quad (\zeta \gg 1),$$
$$\text{II. } r \ll \chi^{-1},$$

having a common part $\chi^{-1} \gg r \gg r_0$ which we shall denote as region III.

We shall construct solutions of the system (1.14) in each of these regions requiring that they should coincide in the common portion, in just the same way as is done in the case of an investigation of the behavior of phases in the limit of small momenta.[12] We first consider the region I where we can set $a = a_0$, and, moreover, in view of the smallness of $\mathbf{k}$ we can assume that $R_n$ and $R_n^+$ are slowly varying functions of r, and this enables us to take $R_n$ and $R_n^+$ outside the integral sign (in future we shall omit the subscript $\mathbf{k}$ from $\mathbf{R}_{n\mathbf{k}}$). In this region the fundamental equations (1.14) take on the form

$$\left(-\frac{n\hbar^2}{mr^2} + E_\mathbf{k}\right)(R_n + R_n^+) = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}(R_n - R_n^+)\right.$$
$$\left. - k_z^2 (R_n - R_n^+) - \frac{n^2}{r^2}(R_n - R_n^+)\right],$$
$$\left(-\frac{n\hbar^2}{mr^2} + E_\mathbf{k}\right)(R_n - R_n^+) = -\frac{\hbar^2}{2m}\left[\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}(R_n + R_n^+)\right.$$
$$\left. - k_z^2 (R_n + R_n^+) - \frac{n^2}{r^2}(R_n + R_n^+)\right]$$
$$+ 2a_0^2 \Phi_0 (R_n + R_n^+). \qquad (2.1)$$

From the last equation it follows that

$$R_n + R_n^+ \approx \frac{(-n\hbar^2/mr^2 + E_\mathbf{k})}{2a_0^2 \Phi_0}(R_n - R_n^+) \ll R_n - R_n^+, \qquad (2.2)$$

since one can neglect terms involving derivatives. From this one can easily obtain the equation for the quantity $\rho_n = R_n - R_n^+$:

$$\rho_n'' + \frac{1}{r}\rho_n' + \left(\chi^2 - \frac{\nu^2(n)}{r^2}\right)\rho_n = 0,$$

$$\chi^2 = k^2 - k_z^2, \qquad \nu^2(n) = n^2 + 2nk\hbar / (a_0^2\Phi_0 m)^{1/2} \qquad (2.3)$$

with the solutions

$$\rho_n = c_1(n)J_\nu(\chi r) + c_2(n)Y_\nu(\chi r), \qquad (2.4)$$

where $J_\nu$, $Y_\nu$ are respectively the Bessel and the Neumann functions.

Comparing the asymptotic expansion (2.6) with the asymptotic representation (1.12) one can easily obtain the relations (for small k there exists only the one root $k_s = k$):

$$c_1(n) = [u(k) - v(k)]\left[e^{i|n|\pi/2}\cos\left((|n| - \nu)\frac{\pi}{2}\right) - \frac{i}{2}\frac{k}{\partial E/\partial k}e^{i\nu\pi/2}\Lambda_{n\mathbf{k}}\right],$$

$$c_2(n) = [u(k) - v(k)]\left[e^{i|n|\pi/2}\sin\left((|n| - \nu)\frac{\pi}{2}\right) + \frac{1}{2}\frac{k}{\partial E/\partial k}e^{i\nu\pi/2}\Lambda_{n\mathbf{k}}\right],$$

$$\Lambda_{n\mathbf{k}} = \langle\rho_{n\mathbf{k}}|\hat{V}_{nk_z}|\mathbf{R}_{n\mathbf{k}}\rangle. \qquad (2.5)$$

In order to determine the quantity $\Lambda_{n\mathbf{k}}$ of interest to us it is necessary to construct the solutions in region II. In region II we can neglect the quantities $k_z^2$ and $E_\mathbf{k}$ and construct the solution of (1.14) satisfying the conditions of being bounded at r = 0. In order to avoid complications associated with the integral terms of (1.14) we shall in determining the order of the ratio $c_2(n)/c_1(n)$ for $|n| \geq 2$ assume that $\Phi(x) = \Phi_0\delta(\mathbf{x})$, although the result is independent of this assumption. Then Eqs. (1.14) are converted into a system of two equations of the second order and the corresponding solution in region II has the form

$$R_n = A_1(n)R_{1n}(r) + A_2(n)R_{2n}(r), \qquad (2.6)$$

where $R_{1n}$ and $R_{2n}$ are independent of **k**.

The asymptotic representation (2.6) for $r \gg r_0$ (i.e., in region III) must coincide with the asymptotic representation (2.4) for $r \ll \chi^{-1}$. Among the four fundamental solutions of (1.14) with k = 0 two have the nature of powers, one is exponentially increasing and one is exponentially damped in region III. Conditions of smooth joining require the absence of the exponentially increasing solution in (2.6), and this gives a linear relationship between $A_1(n)$ and $A_2(n)$ with a coefficient independent of **k**. Equating terms of power form in (2.6) with the corresponding power terms in (2.4) one can easily conclude that

$$A_{1,2}(n) \sim \chi^{|n|}c_1(n), \qquad c_2(n) \sim \chi^{2|n|}c_1(n) \qquad (2.7)$$

(the exponentially damped terms can be neglected).

Thus, for $|n| \gg 2$ small distances from the filament axis do not play any important role and one can simply set $c_2(n) = 0$ so that in accordance with (2.5) we have

$$\Lambda_{n\mathbf{k}} \approx \frac{2}{k}\frac{\partial E}{\partial k}\sin\left[(|n| - \nu)\frac{\pi}{2}\right] \qquad (|n| \geqslant 2). \quad (2.8)$$

But the cases of n = 0 and n = ±1 are special cases and do not obey the general formulas (2.7), so that we must discuss them separately.

In order to find the corresponding solutions of the fundamental equations (1.14) in region II we shall utilize certain properties of Eqs. (1.1). It can be easily seen that the linearization of this equation near the stationary solution leads to none other than Eq. (1.5) for the quantities

$$\psi_0 = \delta\varphi e^{-i\vartheta}, \qquad \psi_0^+ = \delta\varphi^* e^{i\vartheta}.$$

We make use of the fact that (1.1) admits of gauge transformations, i.e., the quantity $\varphi_0 = \varphi(\mathbf{x})e^{i\alpha}$ is a solution of (1.1) for arbitrary constant $\alpha$; from this, carrying out a variation with respect to $\alpha$, we obtain the solutions in region II for n = 0:

$$R_0 = A_0 a(r), \qquad R_0^+ = -A_0 a(r). \qquad (2.9)$$

Joining in region II the solution (2.9) with the solution (2.4) we obtain

$$c_2(0) = 0. \qquad (2.10)$$

We see that the partial wave with n = 0 in general makes no contribution to the scattering amplitude (up to terms of order **k**), since $\nu(0) = 0$, and according to (2.5) $\lambda_{0\mathbf{k}} = 0$.

Thus, in spite of the fact that in accordance with the general results contained in (2.7) n = 0 should, seemingly, give the principal contribution to scattering, nevertheless due to the form of the solution (2.9) in the domain of small r, this contribution turns out to be smaller than that from $n \geq 2$. This circumstance is not related to the specific form of the equations for $\psi_\mathbf{k}$, but is a consequence of gauge invariance.

Similarly, considering the invariance of (1.1) under translations in x space, and also the Galilean invariance, one can obtain the solution of (1.14) in region II up to terms of the first order in $E_\mathbf{k}$ (cf., reference[8]):

$$R_{\pm1} = \frac{A_{\pm1}}{2}\left(\mp\frac{a}{r} + \frac{da}{dr}\right) - A_{\pm1}\frac{mE_\mathbf{k}}{2\hbar^2}ra,$$

$$R_{\pm1}^+ = \frac{A_{\pm1}}{2}\left(\pm\frac{a}{r} + \frac{da}{dr}\right) + A_{\pm1}\frac{mE_\mathbf{k}}{2\hbar^2}ra. \qquad (2.11)$$

Joining this to the solution (2.4) we obtain

$$c_2(\pm 1) = \mp \frac{\pi\hbar^2\chi^2}{4mE_{\mathbf{k}}} c_1(\pm 1). \qquad (2.12)$$

It is of interest to note that no characteristics at all of the behavior of a(r) at short distances appear in this formula. From this, utilizing (2.5), we obtain for the scattering amplitude

$$\Lambda_{\pm 1,\,\mathbf{k}} = -\frac{2}{k}\frac{\partial E}{\partial k}\left[\sin\left((1-\nu(\pm 1))\frac{\pi}{2}\right) \pm \frac{\pi\hbar^2\chi^2}{4mE_{\mathbf{k}}}\right]. \qquad (2.13)$$

From formula (2.13) it can be seen that n = ±1 give a contribution to the scattering amplitude of the same order as do the remaining |n| ≥ 2. Formula (2.13) is also in fact not connected with the specific form of (1.14).

Thus, we have constructed the approximate solutions $R_n$ and $R_n^+$ and we have obtained expressions for the partial amplitudes on the assumption that the propagation vector of the incident excitations is small.

We note that for all values of n, with the exception of n = ±1, the result for the scattering amplitude can be obtained by perturbation theory, since $\nu$ is close to |n|, and sin $((|n| - \nu)\pi/2)$ can be expanded in terms of **k**, and we can restrict ourselves to the first term. This leads to the same result as the theory which considers the term with $n\hbar^2/mr^2$ in (1.14) as a perturbation and neglects all the other quantities describing the interaction between the excitations and the vortex filament. Therefore, the total scattering amplitude can be written in the form

$$\langle \tilde{\mathbf{f}}_\varkappa | \hat{V} | \psi_{\mathbf{k}} \rangle = 2\pi\delta(k_z - \varkappa_z)\left[\Lambda(\vartheta) - \frac{i\pi\chi^2}{mkE_k}\frac{\partial E}{\partial k}\sin\vartheta\right] \qquad (2.14)$$

The quantity $\Lambda(\vartheta)$ is determined by the Born approximation:*

$$2\pi\delta(k_z - \varkappa_z)\Lambda(\vartheta) = \frac{\hbar^2}{m}\langle \tilde{\mathbf{f}}_\varkappa | \nabla\vartheta\cdot\nabla | \mathbf{f}_{\mathbf{k}}\rangle$$
$$= -\frac{i\hbar^2}{m}(2\pi)^2\delta(k_z - \varkappa_z)\frac{[\boldsymbol{\mu}\boldsymbol{\varkappa}]\,\mathbf{k}}{|\mathbf{k}-\boldsymbol{\varkappa}|^2}(u(\varkappa)\,u(k) - v(\varkappa)\,v(k)),$$

where $\boldsymbol{\mu}$ is the matrix vector for the circulation of $\nabla\vartheta$ about the filament. Writing the second term of (2.14) also in vector form, we obtain finally (taking into account the fact that $\kappa$ = k and, consequently, $u(k)u(\kappa) - v(\kappa)v(k) = 1$)

$$\langle \tilde{\mathbf{f}}_\varkappa | \hat{V} | \psi_{\mathbf{k}} \rangle = -2\pi^2\frac{\hbar^2}{m}\delta(k_z - \varkappa_z)\left[\frac{2k^2}{|\mathbf{k}-\boldsymbol{\varkappa}|^2} - 1\right]\frac{[\boldsymbol{\mu}\boldsymbol{\varkappa}]\,\mathbf{k}}{k^2}. \qquad (2.15)$$

It is of interest to note that this formula corresponds exactly to the hydrodynamic formula for the scattering of sound by a vortex filament obtained by perturbation theory[3] without taking into account the distortion of the incident wave at small distances, although within the framework of the model of a weakly nonideal Bose-gas we had to take such distortion into account (for n = ±1).

## 3. CALCULATION OF THE MUTUAL FRICTION FORCE

In accordance with the work of the present author[4,5] the mutual friction force for a weakly nonideal Bose-gas is expressed by the formula

$$F_j = -\int n(\mathbf{k}) \lim_{\substack{\rho\to\infty \\ \varepsilon\to 0}} \Big\{ \int\limits_{\Omega_{\rho\varepsilon}} \left[\frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial x_j}(\hat{V}\psi_{\mathbf{k}}) + (\hat{\tilde{V}}\tilde{\psi}_{\mathbf{k}})\frac{\partial\psi_{\mathbf{k}}}{\partial x_j}\right] d^2x$$
$$+ \frac{i\hbar^2}{2m}\oint \tau_j\left(\tilde{\psi}_{\mathbf{k}}\frac{\partial\psi_{\mathbf{k}}}{\partial N} - \frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial N}\psi_{\mathbf{k}}\right)d\vartheta + \frac{\hbar^2}{2m}\oint\frac{1}{r}(\tilde{\psi}_{\mathbf{k}}\hat{g}\psi_{\mathbf{k}})\,d\tau_j$$
$$+ \frac{i\hbar^2}{2m}\oint \tau_l\left(\tilde{\psi}_{\mathbf{k}}\frac{\partial\psi_{\mathbf{k}}}{\partial x_l} - \frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial x_l}\psi_{\mathbf{k}} + 2i\frac{1}{r}\tau_l\tilde{\psi}_{\mathbf{k}}\hat{g}\psi_{\mathbf{k}}\right)d\tau_j\Big\}\frac{d^3k}{(2\pi)^3}, \qquad (3.1)$$

where $\Omega_{\rho\epsilon}$ is a circle of radius $\rho$ with a deleted circle of radius $\epsilon$, $\tau$ is the unit vector along the tangent to the contour, $\tilde{\psi}_{\mathbf{k}} = \hat{g}\psi^*_{\mathbf{k}}$ is the vector function conjugate to $\psi_{\mathbf{k}}$, $\hat{\tilde{V}}$ is the operator Hermitian conjugate to $\hat{V}$, $n_{\mathbf{k}} = \langle c_{\mathbf{k}}c^+_{\mathbf{k}}\rangle$ are the average occupation numbers of the excitations incident on the filament, and the contour integrals are taken along the boundaries of the region $\Omega_{\rho\epsilon}$, N is the vector along the external normal.

The further transition to the formula containing the transport cross section is not accurate in references[4,5], since the first term in the figure brackets of formula (3.1) reduces to the transport cross section only in case when the operator $\hat{V}$ has no singularities. In order to take these singularities into account we shall transform the first two terms of formula (3.1) utilizing equations for $\psi_{\mathbf{k}}$ and $\tilde{\psi}_{\mathbf{k}}$:

$$\int\left[\frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial x_j}(\hat{V}\psi_{\mathbf{k}}) + (\hat{\tilde{V}}\tilde{\psi}_{\mathbf{k}})\frac{\partial\psi_{\mathbf{k}}}{\partial x_j}\right]d^2x = -\frac{1}{2}\oint[\tilde{\psi}_{\mathbf{k}}(\hat{V}\psi_{\mathbf{k}})$$
$$+ (\hat{\tilde{V}}\tilde{\psi}_{\mathbf{k}})\psi_{\mathbf{k}}]\,r d\tau_j + \frac{\hbar^2}{2m}\oint\left(\frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial N}\hat{g}\frac{\partial\psi_{\mathbf{k}}}{\partial x_j}\right.$$
$$\left.+ \frac{\partial\tilde{\psi}_{\mathbf{k}}}{\partial x_j}\hat{g}\frac{\partial\psi_{\mathbf{k}}}{\partial N}\right)r d\vartheta + \frac{\hbar^2}{4m}\oint\frac{\partial^2}{dx_l^2}(\tilde{\psi}_{\mathbf{k}}\hat{g}\psi_{\mathbf{k}})\,r d\tau_j. \qquad (3.2)$$

Thus, we have reduced all the two-dimensional integrals to contour integrals. In the evaluation of

*$[\mu\,\kappa] \equiv \mu \times \kappa$.

the integrals along the small circles one must take into account the fact that in accordance with the investigation of the singularities of equations (1.4) $\psi_{\mathbf{k}}, \widetilde{\psi}_{\mathbf{k}}$ have unbounded derivatives at $r \to 0$ and only the quantities

$$\widetilde{\theta} = (e^{-i\vartheta}\widetilde{\psi}_{\mathbf{k}},\ e^{i\vartheta}\widetilde{\psi}_{\mathbf{k}}^{+}),\ \theta = (e^{i\vartheta}\psi_{\mathbf{k}},\ e^{-i\vartheta}\psi_{\mathbf{k}}^{+})$$

have bounded derivatives. In accordance with (1.4), the quantities

$$\hat{V}\psi_{\mathbf{k}} - \frac{\hbar^2}{2m}\hat{g}\Delta\psi_{\mathbf{k}}$$

and their complex conjugates are bounded. From this we can easily find that the integrals over the small circumferences appearing in (3.1) mutually cancel in the limit $\epsilon \to 0$, and the expression for $F_j$ can be written in the form

$$F_j = F_j^{(1)} + F_j^{(2)}; \tag{3.3}$$

$$F_j^{(1)} = \lim_{\rho\to\infty}\int n(\mathbf{k})\Big\{\frac{1}{2}\oint_{r=\rho}\Big[\widetilde{\psi}_{\mathbf{k}}\Big(\hat{V} - \frac{\hbar^2}{2m}\hat{g}\Delta\Big)\psi_{\mathbf{k}} + \Big(\Big(\hat{\tilde{V}} - \frac{\hbar^2}{2m}\hat{g}\Delta\Big)\widetilde{\psi}_{\mathbf{k}}\Big)\psi_{\mathbf{k}}\Big] r\,d\tau_j - \frac{\hbar^2}{2m}\oint_{r=\rho}\frac{\partial\widetilde{\psi}_{\mathbf{k}}}{\partial x_l}\hat{g}\frac{\partial\psi_{\mathbf{k}}}{\partial x_l} r\,d\tau_j - \frac{\hbar^2}{2m}\oint_{r=\rho}\Big(\frac{\partial\widetilde{\psi}_{\mathbf{k}}}{\partial N}\hat{g}\frac{\partial\psi_{\mathbf{k}}}{\partial x_j} + \frac{\partial\widetilde{\psi}_{\mathbf{k}}}{\partial x_j}\hat{g}\frac{\partial\psi_{\mathbf{k}}}{\partial N}\Big) r\,d\vartheta, \tag{3.4}$$

$$F_j^{(2)} = \lim_{\rho\to\infty}\int n(\mathbf{k})\Big\{\frac{i\hbar^2}{2m}\oint_{r=\rho}\tau_j\Big(\widetilde{\psi}_{\mathbf{k}}\frac{\partial\psi_{\mathbf{k}}}{\partial N} - \frac{\partial\widetilde{\psi}_{\mathbf{k}}}{\partial N}\psi_{\mathbf{k}}\Big)d\vartheta + \frac{i\hbar^2}{2m}\oint_{r=\rho}\tau_l\Big(\widetilde{\psi}_{\mathbf{k}}\frac{\partial\psi_{\mathbf{k}}}{\partial x_l} - \frac{\partial\widetilde{\psi}_{\mathbf{k}}}{\partial x_l}\psi_{\mathbf{k}}\Big)d\tau_j - \frac{\hbar^2}{2m}\oint_{r=\rho}\frac{1}{r}\big(\widetilde{\psi}_{\mathbf{k}}\hat{g}\psi_{\mathbf{k}}\big)d\tau_j\Big\}\frac{d^3k}{(2\pi)^3}. \tag{3.5}$$

In order to express $F_j^{(1)}$ in terms of the transport cross section we introduce the quantity

$$\hat{V}_\rho = \begin{cases}\hat{V}, & r \leqslant \rho\\ 0, & r > \rho\end{cases}$$

and we consider $F_j^{(1)}$ as the corresponding limit for $\rho \to \infty$, where for $\psi_{\mathbf{k}}, \widetilde{\psi}_{\mathbf{k}}$ we can take the corresponding solutions of (1.4) and the equation conjugate to it with $\hat{V} = \hat{V}_\rho$. In this case the second derivatives of $\psi_{\mathbf{k}}, \widetilde{\psi}_{\mathbf{k}}$ will have a discontinuity at $r = \rho$. However, it can be easily seen that in virtue of (1.4) the integrands in formula (3.4) are continuous, and we can in replacing $\hat{V}$ by $\hat{V}_\rho$ assume that the integrals in (3.4) will be evaluated over a circumference at infinity.

From physical considerations it is clear that this quantity can be expressed directly in terms of the transport cross section for scattering by a potential $\hat{V}_\rho$. In order to obtain the corresponding formula it is convenient to use the Fourier transforms of formulas (1.7) and to consider the corresponding volume integral. For lack of space we shall not dwell on this in detail and shall give the final result:

$$F_j^{(1)} = \int n(\mathbf{k})\lim_{\rho\to\infty}\Big\{\int\frac{\delta(E_\varkappa - E_k)}{\delta(k_z - \varkappa_z)}(k_j - \varkappa_j)\,|\langle \mathbf{f}_\varkappa{}^*|\hat{g}\hat{V}_\rho|\psi_{\mathbf{k}}\rangle|^2 \times \frac{d^3\varkappa}{(2\pi)^3}\Big\}\frac{d^3k}{(2\pi)^3}. \tag{3.6}$$

In the evaluation of the limit which determines the quantity $F_j^{(2)}$ we encounter a difficulty associated with the slow falling off of the interaction potential V and the corresponding nonintegrable singularity of the scattering amplitude at small angles. If this singularity were absent, then it can be easily seen that only the incident wave would give a contribution to $F_j^{(2)}$ since the limit of the remaining terms would be equal to zero due to the more rapid decrease for $\rho \to \infty$, and correspondingly we would have

$$F_j^{(2)} = \frac{2\pi\hbar^2}{m}\int n(\mathbf{k})[\mathbf{k}\boldsymbol{\mu}]_j\frac{d^3k}{(2\pi)^3}. \tag{3.7}$$

A rigorous investigation of the limit in (3.5) can be carried out in the following manner. After the separation of variables and the carrying out of integration over the angles, Eq. (3.5) reduces to certain sums containing $\mathbf{R}_n(\rho)$. Since $\mathbf{R}_n(\rho)$ tends to zero for $\rho \to \infty$ and fixed n, then in these sums only large $n \sim \rho$ are significant. For large n for the solution of (1.14) the quasiclassic approximation is valid in accordance with which $\mathbf{R}_n$ can be written in the form

$$\mathbf{R}_n \approx \mathbf{u}(k)J_{\mu_n}(\chi\rho)e^{i|\delta_n(k)|\operatorname{sign} n}e^{i|n|\pi/2}, \quad \mu_n = |n| + \frac{2}{\pi}\delta_n,$$

$$\delta_n = \frac{k\partial k}{\partial E}\frac{\hbar^2\pi}{2m}\operatorname{sign} n.$$

If we set $\delta_n = 0$, we shall obtain simply an expansion of the incident plane wave and the result (3.7) for $F_j^{(2)}$. Thus, the difference between Eqs. (3.7) and (3.5) can be written in the form of a sum over n terms with $\delta_n = 0$ and $\delta_n \neq 0$. This sum can be conveniently transformed into a contour integral over n surrounding the real semiaxis, if we introduce in the integrand a discontinuous factor cot $n\pi$ The contour can be deformed into a straight line parallel to the imaginary axis. Introducing a change of variables, it is possible to reduce the desired difference to a sum of integrals of the form*

$$\int_C[\operatorname{ctg} n\pi - \operatorname{ctg}(n - \delta)\pi]J_nJ_{n+\alpha}\,dn,$$

which tend to zero for $\rho \to \infty$, because the expression in square brackets tends to zero rapidly with increasing |n| along the path of integration. For lack of space we omit a more detailed proof which requires the use of estimates for Bessel functions with complex values of the subscript.

*ctg = cot.

Finally, the mutual friction force has the form in accordance with (3.6), (3.7) and (3.3),

$$\mathbf{F} = \int n(\mathbf{k}) \lim_{\rho\to\infty} \left\{ \int \frac{\delta(E_k - E_\varkappa)}{\delta(k_z - \varkappa_z)} |\langle \mathbf{f}_\varkappa^* | g\hat{V}_\rho | \psi_\mathbf{k}\rangle|^2 \times (\mathbf{k} - \boldsymbol{\varkappa}) \frac{d^3\varkappa}{(2\pi)^3} \right\} \frac{d^3k}{(2\pi)^3} + 2\pi \frac{\hbar^2}{m} \int n(\mathbf{k}) [\mathbf{k}\boldsymbol{\mu}] \frac{d^3k}{(2\pi)^3}, \quad (3.8)$$

which reduces to the final formula given earlier[4,5] if in it we set

$$\Pi_j(\mathbf{k}) = -\mathbf{p}_j^0 = -\hbar\mathbf{k}, \qquad R_j(\mathbf{k}) = 0.$$

Thus, the expression for the retarding force differs from the value obtained on the basis of the transport cross section[1-3] by an additional purely transverse component. For small values of the average velocity of the incident excitations **u** we can write

$$\mathbf{F} = -\int \frac{\partial n}{\partial E_k} (\hbar\mathbf{k}\mathbf{u}) \lim_{\rho\to\infty} \left\{ \int \frac{\delta(E_k - E_\varkappa)}{\delta(k_z - \varkappa_z)} (k - \varkappa) \times |\langle \mathbf{f}_\varkappa^* | g\hat{V}_\rho | \psi_\mathbf{k}\rangle|^2 \frac{d^3\varkappa}{(2\pi)^3} \right\} \frac{d^3k}{(2\pi)^3} + \frac{2\pi\hbar}{m} [\mathbf{u}\boldsymbol{\mu}] \rho_n, \quad (3.9)$$

where $\rho_n$ is the density of the normal component.

In the case of phonon scattering the quantity $F_j^{(1)}$ can be evaluated by means of formula (2.15)

$$F_j^{(1)} = -\int \frac{\partial n}{\partial E_\mathbf{k}} \hbar(\mathbf{k}\mathbf{u}) \int (k_j - \varkappa_j) \delta(E_\mathbf{k} - E_\varkappa) \frac{\pi^2\hbar^4}{m^2} \times (2\pi)^2 \delta(k_z - \varkappa_z) \left\{ \left( \frac{2k^2}{|\mathbf{k} - \boldsymbol{\varkappa}|^2} - 1 \right) \frac{[\boldsymbol{\mu}\boldsymbol{\varkappa}]\,\mathbf{k}}{k^2} \right\}^2 \times \frac{d^3\varkappa}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} \approx \frac{76}{7} \frac{\Theta^5}{m^2c^7\hbar^2} u_j, \quad (3.10)$$

where c is the velocity of light, Θ is the temperature in energy units, m is the mass of the atoms. The coefficient in (3.10) differs somewhat from that calculated in reference[3], since in that article phonons incident in a direction which is not perpendicular to the axis of the vortex filament are discussed incorrectly. Moreover, we have here utilized the fact that for a symmetric cut-off of the potential $\hat{V}$ the lifting force (which, generally speak ing, diverges logarithmically) in the expression for $F_j^{(1)}$ (i.e., the component perpendicular to **u**) is equal to zero.

Finally for the phonon part of the mutual friction force we obtain in accordance with (3.9)

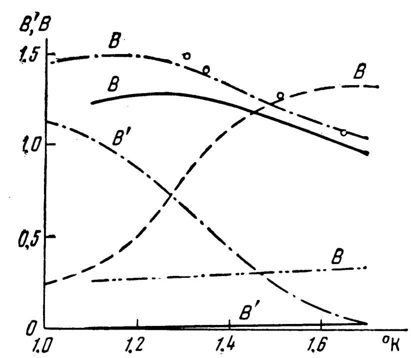
$$\mathbf{F} = 10.8 \frac{\Theta^5}{m^2c^7\hbar^2} \mathbf{u} + \frac{8{,}24}{\pi} \frac{\Theta^4}{\hbar^2mc^5} [\mathbf{u}\boldsymbol{\mu}]. \quad (3.11)$$

We see that the transverse component falls off with temperature more slowly than does the longitudinal one. Formula (3.11) can be utilized directly for HeII, since no quantities characteristic of the model appear in it. It would be of interest to carry out the corresponding experiments in HeII, since due to the large value of the phonon transverse force it could, apparently, be noticed at higher temperatures than the longitudinal phonon component.

Since the correction to the mutual friction force does not depend on the model utilized, formula (3.9) enables us also to recalculate the old results on the evaluation of the retarding force for HeII carried out by E. Lifshitz and Pitaevskiĭ in the quasiclassical approximation[2]. In doing this one must take into account, as has been done first by Hall and Vinen[1], different hydrodynamic corrections assuming that the expression (3.9) gives the force **f** per unit length of the filament in the case of the scattering of rotons moving near the filament with the velocity $\mathbf{u} = \mathbf{v}_R - \mathbf{v}_L$. Utilizing for the first term in (3.10) the expression obtained in reference[2] we have

$$f = D(\mathbf{v}_R - \mathbf{v}_L), \; D = \frac{2{,}4\pi\hbar}{m} \rho_n \left( \frac{2\mu\Theta}{p_0^2} \right)^{1/2}, \qquad D' = 0, \quad (3.12)$$

where $\mu$ is the effective mass of the roton, $p_0$ is the roton momentum. The expression (3.12) differs from that obtained in reference[2] by the absence of the transverse component which exceeded the longitudinal one by a factor of approximately 10.




The coefficients B and B′: dotted curve – without taking into account the additional cross section of the filament in accordance with reference[2]; dashed curve – taking into account the additional cross section $\sigma_0 \approx 10$ Å in accordance with reference[2]; dash-dotted curve – taking into account corrections for the transverse component in accordance with the present paper, but without the additional cross section of the filament; the solid curve – according to the present work for a dimensionless coefficient $\alpha = 5$ (or $\sigma_0 \approx 4$ Å). The points denote the results of the experiments of Hall and Vinen[1].

The calculation, in accordance with Hall and Vinen,[1] of the coefficients B and B′ utilizing formula (3.12) for f, leads to a value of B which is approximately by a factor four smaller than the experimental value, and gives an extremely small value of B′ (smaller than B by a factor of approximately 100), which is in accord with the experimental facts. However, it is necessary to note that the use of the quasiclassical approximation in this case is not quite rigorous, since for harmonics of not very large order n small distances from the axis of the filament of the order of the roton wavelength (~4 Å) play a role, for which neither quasiclassical considerations, nor the expressions utilized in reference[2] for the description of the interaction between the excitations and the filament are valid. In this sense the disagreement between the results of the calculations of reference[2] and experiment must be seen rather in the large value of B′ and the incorrect temperature dependence, than in the difference between the value of B and the experimental one.

Formula (3.12) partially removes these defects. And if we assume that an additional absorption of the momentum of excitations takes place, we can achieve a satisfactory qualitative agreement with experiment assuming that the corresponding cross section is symmetric with respect to the filament axis and equal to 4 Å instead of 10 Å in reference[2]. This agrees with the concept of a possible inaccuracy of the quasiclassical approximation. Moreover, there exist inaccuracies in the evaluation of the coefficients B and B′ in accordance with reference[1], and in particular the quantity f enters into the calculations with an undetermined coefficient $\alpha \sim 1$; if we take $\alpha = 5$, and not $\alpha = 1$, as we have done above, then we can also obtain good quantitative agreement with the experimental data for B. The results of the calculations are shown on the graph.

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