

GRAVITATIONAL COLLAPSE OF NONSYMMETRIC AND ROTATING MASSES

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Collapse of nonsymmetric and rotating masses is considered. It is shown that the characteristic pattern of gravitational self-closing valid for the spherical case also holds in the general case. Moreover, collapse of a nonrotating body leads to a t^{-1} damping of quadrupole and higher field moments for an external observer. The field of a collapsing rotating body changes in a different manner. Metric changes related to a rotating local inertial system approach a nonvanishing constant. However, qualitatively the collapse picture remains the same as in the spherical case. Static nonspherical solutions of the Einstein equations are also investigated and in particular the properties of the $g_{00} = 0$ Schwarzschild surface in these solutions are analyzed.

1. INTRODUCTION

AS is well known, stars with $M > \sim 1.6M_{\odot}$ have an evolution such that they contract without limit. The theory of this phenomenon, for a simple model of a spherical body, has by now been clarified to a considerable extent (see the review^[1], where references to original papers can be found). A characteristic feature of the process is the gravitational self-closing of a body, manifest in the fact that after contraction to a critical dimension, the gravitational field of the body emits neither radiation nor information. This critical dimension is determined by the gravitational radius $R_g = 2Gm/c^2$, where G is Newton's gravitational constant, c —the velocity of light, and m —the mass of the body.

In close relation with self-closing is the fact that, from the point of view of a remote observer, on approaching R_g the evolution slows down, and the observed picture approaches asymptotically (as $t \rightarrow \infty$) a certain limiting state, which, however, is not at all an equilibrium state. This apparent stoppage is a result of the slowing down of the time in the strong gravitational field, and for a contracting body the Doppler effect only intensifies this deceleration as seen by a remote observer. Thus, the apparent stoppage of the contraction is brought about by the same factors as the red shift of the emitted spectrum and self-closing.

This raises the question whether the picture is general, whether it has any special connection with the symmetry of the problem, and whether the deductions remain in force also in the general nonspherically symmetrical case. The statement that the picture remains qualitatively the same also in a nonspherically symmetrical collapse was advanced

earlier^[2] (see also^[3] concerning the stability of the Schwarzschild solution). In this paper we present a proof of this far from obvious statement.

By way of a first attempt at finding the asymptotic nonspherical solution, it is natural to seek the stationary solutions by starting from the assumption that the collapse is seen by an external observer as a monotonic process and that as $t \rightarrow \infty$ all the $\partial/\partial t \rightarrow 0$. This assumption is proved in Sec. 3.

An analysis of the static solution outside the body shows that the deviation from the spherical solution, which is caused by a change in the source of the field, leads to the appearance of true singularities of space-time on the Schwarzschild surface $g_{00} = 0$. On the other hand, in the co-moving system of a contracting body with small initial deviations from sphericity in the density distribution, the instant when the surface of the body crosses the Schwarzschild surface is in no way specially distinguished, and is not accompanied by the appearance of true singularities either in the metric or in the density. A comparison of these results leads to the conclusion that the quadrupole and higher multipole moments of the external gravitational field attenuate during the relativistic stages of the collapse of an asymmetrical body.

Deviations in the stationary metric from sphericity, connected with the components g_0^{α} , i.e., with the rotation of the local inertial system relative to a far inertial system, and the fields induced in the source by the "rotational" motions, do not lead to singularities when $g_{00} = 0$. During the process of collapse these deviations do not vanish. We note that the "rotational motions" are not necessarily connected with the rotation of

the body as a whole, and arise, for example, as a result of tangential velocities when an asymmetrical body is compressed. We present below a rigorous proof of the advanced considerations.

2. STATIONARY SOLUTIONS

In the spherically symmetrical case, the field is given by the known Schwarzschild solution and is static independently of the "spherically symmetrical" motion of the central mass which produces the field.^[4] The solution contains a critical surface—the Schwarzschild sphere S_S , characterized by the condition $g_{00} \equiv 1 - R_g/R = 0$. Near this surface, the red shift of the radiation line, emitted by a source at rest and received by a remote observer, is given by the expression

$$\omega_{\text{abs}} / \omega_{\text{rad}} = \sqrt{g_{00}} \sim l,$$

where l is a small distance from S_S ,

$$dl = \sqrt{g_{11}} dR \equiv \left(\frac{R}{R - R_g} \right)^{1/2} dR, \quad l = 2[R_g(R - R_g)]^{1/2}.$$

The observed frequency tends to zero as $l \rightarrow 0$. For a stationary external observer, a light ray and a trial particle can approach S_S only asymptotically, after an infinite time. For both the light ray and for the freely falling trial particle, this time is logarithmically infinite

$$t \sim \frac{R_g}{c} \ln \frac{R_g}{R - R_g}.$$

The four-dimensional space-time has no singularity on S_S , and in particular, when $R = R_g$ the curvature scalar $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, where $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, has a fully defined finite value $K = 12/R_g^4$. If the field source has dimensions smaller than S_S , then the Schwarzschild solution in the vacuum can be continued inside S_S into the so-called T-region.^[5,6]

a) Static field with axial symmetry. Regge and Wheeler^[3] considered the nonspherical problem in vacuum by the method of small perturbations superimposed on the Schwarzschild solution. From the solution of the equations for small perturbations, given in^[3], we see that in the stationary case any perturbation that decreases at infinity increases without limit on approaching the Schwarzschild sphere of the unperturbed problem. It follows therefore that no matter how small the deviations from spherical symmetry at a finite distance from S_S , the method of small perturbations used by Regge and Wheeler^[3] cannot give a correct answer up to S_S itself.

The static problem for some form of an axially symmetrical field of the quadrupole and higher

multipoles was solved by Erez and Rosen^[7] with the aid of Weyl's method.^[8] The corresponding expression for the interval for the quadrupole field, with the error contained in^[7] corrected by us, can be found in Appendix I. In this field, the surfaces of constant g_{00} , i.e., of constant gravitational potential, are singly-connected, closed, and imbedded in one another so that they do not differ topologically from the spherically symmetrical case, where they were concentric spheres. However, as $g_{00} \rightarrow 0$ the metric of the surfaces $g_{00} = \text{const}$ differs radically from the metric of a sphere. In particular, for a positive quadrupole moment (the body is elongated along the axis like a cucumber), the length of the equator tends to zero, and the length of the meridians to infinity as $g_{00} \rightarrow 0$. The area of the surface $g_{00} = \text{const}$ tends to infinity (but each surface with larger area lies completely inside the preceding surface with smaller area). The light and the freely-falling particle reach the surface $g_{00} = 0$ within a finite time of the external observer (see Appendix I). Finally, the invariant $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$, which characterizes the total curvature of space-time, becomes infinite for $q \neq 0$ as $g_{00} \rightarrow 0$ like q^2/g_{00} .

These results are not limited to a quadrupole only, and are, as shown in Appendix II, general for any static axially-symmetrical solution.

b) External field of a rotating body. We now consider the deviations from spherical symmetry connected not with the change in the distribution of the masses in the field source, but with rotation. Kerr^[9] obtained an exact solution of Einstein's equations in vacuum. This solution describes the field of a body of mass m with total momentum $M = amc$, where a is a constant with the dimension of length. For a body whose particles possess only rotational motion about a symmetry axis, the only nonvanishing-diagonal component of the metric, in a suitable coordinate system and in an external field, is g_{03} . This follows immediately from symmetry considerations and from the equivalence of the past and of the future. Kerr's solution contains non-removable off-diagonal components $g_{\mu\nu}$, in addition to g_{03} . Consequently, if this solution is realized as an external field of some stationary body, then the particles of the material of the body should execute not only rotational motion about the symmetry axis, but also some other motions (for example, such as rising at the poles and dropping at the equator), leading to non-equivalence of the past and the future. An analysis of Kerr's solution^[15] leads to the following conclusions.

1) For arbitrarily small but non-vanishing a ,

the lengths of the "parallel" L on the surface $g_{00} = \text{const}$ [these lengths are proportional to $(-g_{33} + g_{03}^2/g_{00})^{1/2}$ at $\theta = \text{const}$ and $g_{00} = \text{const}$] tend to infinity as $g_{00} \rightarrow 0$. The asymptotic value of L is of the form

$$L = 2\pi a \sin^2 \theta / \sqrt{g_{00}}.$$

2) The precession of a gyroscope away from the body is determined by the known expression^[4]:

$$\Omega^2 = c^2 a^2 R_g^2 R^{-6} (1 + 3 \cos^2 \theta).$$

Near S_g the precession in local time tends to infinity.

3) The scalar K , unlike the preceding type of deviations from spherical symmetry, does not have singularities on S_g and, in particular, we have on the equator, as in the Schwarzschild solution on S_g ,

$$K = 12/R_g^4, \quad R_g = 2Gm/c^2.$$

In this solution the field in vacuum can be continued inside S_g into the T-region. Kerr's solution has a space-time singularity (like Schwarzschild's solution) at $R = 0$.

4) A light ray traveling towards S_g in the direction of the pole, and light rays traveling in the plane of the "equator," reach S_g after a logarithmically infinite time of the external observer. (The clocks are synchronized here against the trajectories of the rays.)

In Appendix III we give the field of a slowly rotating sphere with $a \ll R_g$. This solution is valid not only far away, where $R \gg R_g$, but also near S_g . In this solution of the equations of small perturbations superimposed on the Schwarzschild field, only the terms linear in a and the higher mechanical moments are retained in the corrections to the components $g_{\mu\nu}$, and the terms with a^2 and higher order have been discarded. Those of the effects of Kerr's solution on $g_{00} = 0$ which depend on the linear corrections to $g_{\mu\nu}$ are retained in this solution, too. In particular, we have here

$$K|_{g_{00}=0} = 12/R_g^4 < \infty,$$

and the rotation does not give terms of first order in a .

c) Schwarzschild sphere in an external quadrupole field. There exist solutions of Einstein's equations in which there is a surface S_g which does not qualitatively differ at all from the Schwarzschild surface for the spherical case. In this case, however, the deviations from spherical symmetry should be brought about by the external field. For example, if we can consider a spheri-

cal mass in an external quadrupole field (which increases with increasing distance from the mass m) then the exact solution of Einstein's equations in vacuum is of the form given in Appendix IV. In this field the surface S_g is a Schwarzschild sphere deformed by the external fields, with all its properties.

3. COLLAPSE OF A PERTURBED SPHERICAL DUST CLOUD

Let us consider the motion of the dust in a co-moving reference frame.¹⁾ It is known (see^[4]) that in spherically symmetrical motion in this reference frame the transition to S_g occurs within a finite time, and in this system S_g is no singularity whatever. The density of matter in this case is finite, and its order of magnitude is $\bar{\rho}_{\text{crit}} = 2 \times 10^{16} (M_{\odot}/M)^2 \text{ g/cm}^3$. The spherically symmetrical motion of dust with small perturbations has likewise no singularities at this average density.^[10] The invariant K here is finite. From a comparison with the invariant K of the stationary solution follows the conclusion, mentioned in the introduction, that the multipole moments of the external field attenuate during the course of the collapse.

It is shown in Appendix III that during the collapse of a rotating body, the "rotational-type" deviations from sphericity are conserved.

The foregoing considerations do not as yet exclude the possibility of the following situation.

The body contracts, and after a finite proper time it passes through S_g with small perturbations, and then, already in the T-region, after being compressed to a high density and strongly deformed, it gives rise to strong perturbations of the metric of the surrounding space, making it possible for radiation to be emitted and for the body itself to expand again beyond S_g . It would seem that for a remote external observer, the question of the possibility of such a situation should not arise: after all, if the body crosses after a finite proper time S_g then this process stretches out for the external observer into an infinitely long one, and what happens afterwards is immaterial to the observer. Actually, however, this very conclusion, to which we are so used, that the time of approach to S_g stretches to infinity, is obtained from the fact that the world line of the ray emitted from the surface of the body arbitrarily close to S_g , proceeds for an arbitrary long time (in the time of any system!) near the world line of the point S_g (see the figure). In our problem it

¹⁾The results are valid also for a gas.

is not at all obvious beforehand that the perturbations of the metric will not change after an infinitely long time the world line of the ray to such an extent that this ray and other rays, which have already been emitted after the surface of the body crossed S_S , could go to a remote observer. In other words, it must be proved that the going over to the asymptotic solution is a monotonic process and that any oscillations during the relativistic stage of collapse are already impossible for an external observer.

We shall prove the following statement: assume that at the instant when the surface of the sphere crosses the S_S of the unperturbed problem the perturbations of the metric in the body, and the perturbations of the density and the velocity of the matter are small. Then, for an external observer, the picture of the contraction will be the same as in the case of an exactly spherical collapse—he sees the approach of the surface of the sphere to S_S as a process that stretches out to infinity, and the possibility of rays emitted by the surface of the body after crossing S_S is actually eliminated.

The proof (the details of which are given in Appendix V) consists in the following. We prove first that if in a co-moving freely-falling system of reference at some instant of proper time (close to the instant when the surface of the body crosses S_S) the perturbations in all of space are small and if the perturbations at infinite space remain small in all the succeeding instants of time, then in all of space outside of S_S and (this is particularly important) also in the T-region of space-time near S_S , the perturbations will always remain small. Then, using the smallness of the perturbations of the metric inside S_S in the T-region, it is proved that a light ray can never leave this region and consequently an external observer will never find out what occurred after S_S was crossed, and the process of the approach of the surface of the body to S_S stretches out for him to infinity (see Appendix V).

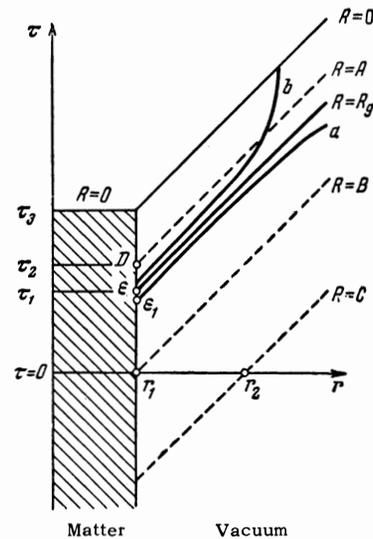
This completes the proof of the statement. This result of our paper is important for a description of the picture of the collapse from the point of view of the external observer.

We note that this result cannot be obtained by the method of Regge and Wheeler,^[3] since they work in the Schwarzschild reference frame, which cannot be used when $g_{00} = 0$ or in the T-region.

In the proper time, the star can be compressed after crossing the S_S surface to tremendous densities, and the perturbations become colossal. But no matter what takes place there, this will never

be manifest in the region of space-time to the right and below the dashed line $R = A$ in the figure, i.e., it will not be manifest in any way in the space outside S_S at any time t . This question is discussed in^[16]. The conclusions of^[16] contradict those of^[10].

The conclusions obtained are obviously important primarily in attempts to attribute phenomena occurring in quasars (and also in supernovas) to the relativistic effects due to the collapse of large masses.



Collapse of a dust sphere in a freely falling reference frame (for notation see Appendix VI), a and b — world lines of light rays. Ray a emitted from ϵ_1 near ϵ continues for a long time along $R = R_g$ (in a time measured in an arbitrary reference system).

4. COLLAPSE OF AN ASYMMETRICAL BODY FROM THE POINT OF VIEW OF AN EXTERNAL OBSERVER

We have proved that a nonspherically symmetrical mass collapses for an external observer qualitatively in the same way as a spherical one. The change in the multipole moments during the course of contraction of the body should be accompanied by radiation of gravitational waves, but the energy carried by this radiation is small. The radiation of waves is a consequence of the change in the multipole moments, and cannot be regarded as the cause of their total damping. We note that in Newtonian theory, the moments also vary during the course of compression of the body, but for finite body dimensions they are finite. In Einstein's theory, a relativistic damping is superimposed on this change in the moments of the external field, due to the change in the dimensions of the contracting body.

Let us find the law governing the attenuation of q for an external observer during the course of the collapse. As shown in Appendix VI

$$q \sim \ln^{-1} [R_g / (R - R_g)],$$

but the approach to S_g proceeds like

$$t \sim \ln [R_g / (R - R_g)];$$

hence $q \sim t^{-1}$, i.e., the attenuation obeys a power law. The external observer "sees" (for example, with the aid of neutrino and antineutrino radiation) in the ultimate "cooled" state the finite nonsphericity of the distribution of the masses in the source of the field. However, this nonsphericity is not at all manifest in the external field.

The deviation of the limiting external field from the Schwarzschild field lies in the presence of components g_0^α which do not vanish during the process of contraction. These components cause quadratic deviations of other components of the metric from the Schwarzschild values. As was already noted in the introduction, the components g_0^α arise in the external field even in the absence of rotation of the body as a whole, for example as a result of tangential velocities arising when an asymmetrical body is compressed. When a sphere rotating like a rigid body collapses, the only component that differs from the Schwarzschild components is g_{03} , with $\partial g_{03} / \partial t = 0$.

Thus, g_{03} does not change in the external space during the course of the collapse of the body. For an external observer, the surface of the collapsing rotating sphere approaches asymptotically S_g after an infinite time. The sphere has time to execute only a finite number of revolutions. The external field in the terms linear in a remains constant all the time²⁾.

APPENDIX I

We present the Erez and Rosen^[7] solution of Einstein's equations for a static axially-symmetrical field in vacuum. The solution is presented after correction of the error that has crept into^{[7] 3)} which changes the final form of the formulas appreciably

$$ds^2 = e^{2\psi} dt^2 - m^2 e^{2\gamma - 2\psi} (\lambda^2 - \mu^2) \left(\frac{d\lambda^2}{\lambda^2 - 1} + \frac{d\mu^2}{1 - \mu^2} \right) - m^2 e^{-2\psi} (\lambda^2 - 1) (1 - \mu^2) d\varphi^2,$$

²⁾Of course, the theory of small perturbations gives only terms which are linear in a .

³⁾The expression for γ given in^[7] is in error.

$$\begin{aligned} \psi &= \frac{1}{2} \left\{ \left[1 + \frac{1}{4} q (3\lambda^2 - 1) (3\mu^2 - 1) \right] \ln \frac{\lambda - 1}{\lambda + 1} \right. \\ &\quad \left. + \frac{3}{2} q \lambda (3\mu^2 - 1) \right\}, \\ \gamma &= \frac{1}{2} (1 + q + q^2) \ln \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} - \frac{3}{2} q (1 - \mu^2) \\ &\quad \times \left[\lambda \ln \frac{\lambda - 1}{\lambda + 1} + 2 \right] + \frac{9}{4} q^2 (1 - \mu^2) \\ &\quad \times \left[(\lambda^2 + \mu^2 - 1 - 9\lambda^2 \mu^2) \frac{1}{16} (\lambda^2 - 1) \ln^2 \frac{\lambda - 1}{\lambda + 1} \right. \\ &\quad \left. + \frac{1}{4} (\lambda^2 + 7\mu^2 - \frac{5}{3} - 9\mu^2 \lambda^2) \lambda \ln \frac{\lambda - 1}{\lambda + 1} \right. \\ &\quad \left. + \frac{1}{4} \lambda^2 (1 - 9\mu^2) + \left(\mu^2 - \frac{1}{3} \right) \right]. \end{aligned} \quad (\text{I.1})$$

Here m is the mass of the body producing the field, q characterizes the quadrupole moment. The units used are chosen such that $c = 1$ and $G = 1$.

The scalar $K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ for the metric (I.1) has for small q and for $\mu = 0$ the following asymptotic form as $g_{00} \rightarrow 0$:

$$K = Bq^2 g_{00}^{-1} + 12/R_g^4, \quad B = \text{const.}$$

We have written out the principal diverging term and the term that remains when $q = 0$.

By virtue of the symmetry, the light rays at $\mu = 0$ and $\mu^2 = 1$, which have initially a radial direction, will move all the time in this direction. Near $g_{00} = 0$, the time of propagation of light from a certain point with $\lambda = \lambda_0$ to $g_{00} = 0$ ($\lambda = 1$) will be

$$\begin{aligned} t &= \text{const} \cdot (\lambda_0 - 1)^{q/8} \text{ for } \mu = 0, \\ t &= \text{const} \cdot (\lambda_0 - 1)^{-q}, \quad q < 0 \text{ for } \mu^2 = 1. \end{aligned}$$

This time is finite⁴⁾, unlike in the case of the Schwarzschild field.

APPENDIX II

Weyl's equations^[8] for an axially-symmetrical Einstein field in vacuum can be written in the form

$$\begin{aligned} \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad \frac{\partial \gamma}{\partial \rho} = \rho \left[\left(\frac{\partial \psi}{\partial \rho} \right)^2 - \left(\frac{\partial \psi}{\partial z} \right)^2 \right], \\ \frac{\partial \gamma}{\partial z} = 2\rho \frac{\partial \psi}{\partial \rho} \frac{\partial \psi}{\partial z}. \end{aligned} \quad (\text{II.1})$$

The coordinates ρ and z are connected with the coordinates λ and μ of Appendix I by the expressions

$$\rho = m [(\lambda^2 - 1)(1 - \mu^2)]^{1/2}, \quad z = m\lambda\mu.$$

⁴⁾An only exception is the case $q > 0$, $\mu^2 = 1$.

For sources of the type⁵⁾ $\sigma = \sigma(z) \delta(\rho) = 0$, the solution of (II.1) is obviously the potential of a filament with linear density $\sigma = \sigma(z)$ in flat space. Near $g_{00} = 0$, ψ and γ are written in the following manner⁶⁾:

$$\psi = \sigma(z) \ln \rho, \quad \gamma = \sigma^2(z) \ln \rho,$$

where $\sigma(z)$ is arbitrary. The expression for the metric is of the form

$$ds^2 = \rho^{2\sigma} dt^2 - \rho^{2\sigma(1-\sigma)} (d\rho^2 + dz^2) - \rho^{2(1-\sigma)} d\varphi^2.$$

The properties of this metric are analogous to those discussed in Appendix I. In particular, from the point with coordinates ρ_0, z_0, φ_0 , moving along the line $z = z_0$ and $\varphi = \varphi_0$ with a velocity sufficiently close to that of light, we can reach $g_{00} = 0$ after a time

$$t = \rho_0^{[\sigma(z_0)-1]^2} [\sigma(z_0) - 1]^{-2}$$

measured with the clock of the external observer.

APPENDIX III

Let us consider the field of a rotating sphere in vacuum. The state of this sphere need not be static—the sphere can expand radially or contract. From symmetry considerations it is clear that in the case of weak rotation the only ones of the perturbations $h_{\mu\nu}$ of the components of the Schwarzschild solution in first approximation will be h_{03} , h_{13} , and h_{23} (the perturbations in the diagonal components are of second order of smallness). By means of a small coordinate transformation we can always cause one of these quantities to vanish: after a transformation $\varphi = \tilde{\varphi} + \xi$ the components h_{03} , h_{13} , and h_{23} receive increments

$$\Delta h_0^3 = \partial \xi / \partial t, \quad \Delta h_1^3 = \partial \xi / \partial R, \quad \Delta h_2^3 = \partial \xi / \partial \theta.$$

Let us cause h_{23} to vanish. We write out the non-trivial components

$$\begin{aligned} \delta R_{23} &= -\frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial t} \frac{g_{11} h_{03}}{\sin^2 \theta} - \frac{\partial}{\partial R} \frac{g_{00} h_{13}}{\sin^2 \theta} \right) = 0, \\ \delta R_{13} &= -\frac{1}{R^2} \left(\sin \theta \frac{\partial}{\partial \theta} \sin^{-1} \theta \frac{\partial h_{13}}{\partial \theta} + 2h_{13} \right) + g_{11} \frac{\partial^2 h_{13}}{\partial t^2} \\ &\quad - R^2 g_{11} \frac{\partial^2}{\partial R \partial t} \frac{h_{03}}{R^2} = 0, \\ \delta R_{03} &= -g_{00} \frac{\partial^2 h_{03}}{\partial R^2} - \frac{2}{R} h_{03} \frac{\partial g_{00}}{\partial R} - \frac{\sin \theta}{R^2} \frac{\partial}{\partial \theta} \sin^{-4} \theta \\ &\quad \times \frac{\partial h_{03}}{\partial \theta} + g_{00} \frac{\partial}{\partial t} \left(\frac{\partial h_{13}}{\partial R} + \frac{2}{R} h_{13} \right) = 0. \end{aligned} \quad (\text{III.1})$$

To find the stationary solution we put

$$\partial h_{13} / \partial t = \partial h_{03} / \partial t = 0.$$

Then the solution of (III.1) takes the form

$$h_{13} = \psi(R) R^2 \sin^2 \theta, \quad h_{03} = \frac{R_g}{R} \sum_n a_n f_n \left(\frac{R}{R_g} \right) P_n^1(\cos \theta) \sin \theta. \quad (\text{III.2})$$

Here $c = 1$, $G = 1$, $\psi(R)$ is arbitrary, $R_g = 2m$, $a_n = \text{const}$,

$$f_n(x) = x^2 u_n(x) \int \frac{dx}{x^4 u_n^2(x)},$$

$$u_n(x) = F(2+n; 1-n; 4; x),$$

F is the Gauss hypergeometric function (see^[12]); P_n^1 is the first associated Legendre polynomial (see^[12]). Asymptotically we have

$$f_n(x) \sim x^{1-n}, \quad x \gg 1.$$

Making now a small transformation $\tilde{\varphi} = \varphi - \psi(R)$, we obtain $h_{13} = 0$, and the only non-vanishing component is h_{03} , for which (III.2) holds true. This is exactly the field to which the field of a contracting rotating sphere can asymptotically come over as $t \rightarrow \infty$ ($R_{\text{SUR}} \rightarrow R_g$).

The concrete form of the field in the vacuum is determined by the conditions for continuity of the internal solutions on the surface of the body. The continuity conditions which follow from the requirements that the field equations be satisfied on the boundary, necessitate that h_{03} be everywhere continuous.

For a sphere rotating like a rigid body (but not necessarily stationary—it can be deformed radially) this condition leads to $h_{03} \sim \sin^2 \theta$ and $h_{13} \sim \sin^2 \theta$ in vacuum. The first equation of (III.1) is then satisfied identically, while the solution of the two others, compatible with the boundary conditions, is transformed with the aid of a small coordinate transformation into

$$h_{03} = -\sin^2 \theta \frac{2M}{R}, \quad (\text{III.3})$$

where $M = -am$ is the total momentum.

Thus, the external field of such a contracting sphere is constant (with respect to the terms linear in a). Expression (III.3) coincides in form with that given in^[4] for a weak field. It is actually valid also in a strong field when $a \ll R_g$ (accurate to first order in a).

It is interesting to note that whereas the magnetic moment of a collapsing magnetic star attenuates,^[13] the field of the mechanical moment is conserved. This difference is explained in the following manner. The magnetic moment is connected with the current I , which tends to zero, for

⁵⁾A source of only this type gives at finite distances from a singular surface small deviations from the spherical solution.

⁶⁾An exception is the degenerate case of a "point singularity" [see^[11], p. 269, formula(8.30)].

a Schwarzschild observer, as the rate of collapse approaches c and $R_{\text{SUR}} \rightarrow R_g$. On the other hand, the mechanical moment remains unchanged, for although the velocity of rotation of the star v attenuates like I in the Schwarzschild system as $R_{\text{SUR}} \rightarrow R_g$, the mass of the volume element for a local Schwarzschild observer increases with increasing rate of collapse. As the result, the moment $M \sim mvR$ remains unchanged.

APPENDIX IV

The solution of Einstein's equation in vacuum for a spherical mass m in an external quadrupole field (which increases with increasing distance from the mass m) is of the form (the notation is the same as in Appendix I):

$$\begin{aligned} \psi &= \frac{1}{2} \ln \frac{\lambda - 1}{\lambda + 1} + \frac{1}{4} q (3\lambda^2 - 1) (3\mu^2 - 1), \\ \gamma &= \frac{1}{2} \ln \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} - 3q\lambda(1 - \mu^2) - \frac{9}{16} q^2 (\lambda^2 - 1) (1 - \mu^2) \\ &\quad \times [9\mu^2\lambda^2 - \lambda^2 - \mu^2 + 1]. \end{aligned}$$

The surface $g_{00} = 0$ is determined by the condition $\lambda = 1$. The Gaussian curvature of this two-dimensional surface is

$$R_G = \frac{1}{4m^2} e^q [1 + 3q - 12q\mu^2 - 9q^2\mu^2 + 9q^2\mu^4]$$

and is different for different μ , being everywhere finite. The constant external quadrupole field can be produced by remote masses which are secured to supports that prevent their displacement. Over a limited time interval, the same field can also be approximately produced by unfastened remote masses whose velocity of motion under the influence of the mutual gravitation is at first small, the field being almost static.

APPENDIX V

Let us consider the collapse of a spherical dust mass. We introduce in the dust a co-moving system. We continue this freely falling system beyond the boundary of the dust, using the known solution of Tolman (see [4]). For concreteness we shall assume that a point on the boundary of the dust falls with parabolic velocity, and that the density of matter inside the dust is uniform.⁷⁾ The metric

⁷⁾If the collapse commenced far from R_g , then near R_g the velocity of the boundary is always close to parabolic. The extension of the proof to the case of motion of the boundary of the dust with elliptical or hyperbolic velocity and with a gradient of the dust density along the radius presents no difficulties.

inside the dust is the metric of Friedman's cosmological model (see [4]) with pressure equal to zero, while the metric outside the dust is the Lemaitre metric^[14] with ds^2 in the form

$$\begin{aligned} ds^2 &= dt^2 - [^{3/2}(r - \tau + \tau_0)]^{-2/3} dr^2 - [^{3/2}(r - \tau + \tau_0)]^{1/3} \\ &\quad \times (d\theta^2 + \sin^2 \theta d\varphi^2). \end{aligned} \quad (\text{V.1})$$

Here τ is the proper time, $\tau_0 = \text{const}$ and depends on the origin of the time, r is the co-moving coordinate, $c = 1$, $R_g = 1$.

The space-time of this model is shown in the figure. The dashed lines are the lines $R = \text{const}$, where $R = [^{3/2}(r - \tau + \tau_0)]^{2/3}$ is the Schwarzschild coordinate⁸⁾. Assume that at the instant $\tau = 0$ (close to the instant τ_1 when the boundary of the dust crosses the Schwarzschild surface $R = R_g$), the perturbations of the density, of the velocity of matter and of the metric h^β_α are small for all $0 \leq r < \infty$. Further, assume that at an arbitrarily large $R = \text{const}$ the perturbations are always small (the latter is obvious). Then, first, h^β_α will be always small in the system in question when

$$R = [^{3/2}(r - \tau + \tau_0)]^{2/3} > A,$$

i.e., to the right and below the dashed line $R = A$ in the figure; here A is some constant, $A < R_g$. Furthermore, the light ray leaving the dust after the instant τ_1 will never emerge outside the Schwarzschild surface $R = R_g$ (see the figure).

Let us prove the first statement. It is seen from (V.1) that in vacuum the components $g_{\alpha\beta}$ depend only on

$$R = [^{3/2}(r - \tau + \tau_0)]^{2/3}.$$

Therefore, if we now consider as independent variables not r and τ but R and τ , then small perturbations of the metric in vacuum can be written in the form $h = \exp(i\omega\tau)f(R)$ (we shall henceforth omit the indices α and β). The function $f(R)$ depends on θ and φ , but this dependence is now immaterial and will not be considered.

The idea of the proof consists in the fact that from the smallness of the perturbations on the lines (see the figure) $D - r_1 - r_2$ and further along $R = C$, and from the form of h , it follows that h is small everywhere inside the region bounded by $R = A$, $R = C$, and $D - r_1 - r_2$.

We present a formal proof. The boundary of the dust crosses R_g at a finite density $\rho_C \approx 2$

⁸⁾In the T-region (i.e., for $R < R_g$) R cannot be a space coordinate. See [6].

$\times 10^{16} (M_{\odot}/M)^{-2}$. The solution of the equations of small perturbations inside the dust^[13] shows that h increases without limit only when $\rho \rightarrow \infty$ and when $\rho = \rho_C$ it is finite. Thus, up to the instant τ_2 (which is still far from τ_3 when $\rho = \infty$), there will be $h < \epsilon_1$ in the dust if $r < r_1$.

In a freely falling system in vacuum there are solutions which increase without limit on $R = R_g$. However, a correct formulation of the Cauchy problem excludes these solutions, and h is small near the surface of the sphere in vacuum up to $\tau = \tau_2$. We thus have in vacuum:

1) from the initial conditions: $h = f(R) < \epsilon_2$ when $\tau = 0$, $r > r_1$,

2) owing to the smallness of the perturbations on the boundary of the dust: $h < \epsilon_3$, when $0 \leq \tau \leq \tau_2$ when $r = r_1$.

It follows from 1) that $f(R) < \epsilon_2$ when $R \geq B = [(3/2)(r_1 + \tau_0)]^{2/3}$ (see the figure).

It follows from 2) that $f(R) < \epsilon_4$, where $\epsilon_4 = \epsilon_3 / |\exp(i\omega\tau)|_{\max}$ when $0 \leq \tau \leq \tau_2$ and $A < R \leq B$ (see the figure).

We thus always have

$$f(R) < \epsilon_5 \quad \text{for } R > A, \quad \epsilon_5 = \max(\epsilon_2\epsilon_4). \quad (\text{V.2})$$

Now, by definition, $h < \epsilon_6$ at sufficiently large $R = \text{const} = C$ and for arbitrary $\tau > 0$:

$$h_{R=C} = e^{i\omega\tau} f(B) < \epsilon_6, \quad \tau > 0.$$

Thus,

$$e^{i\omega\tau} < \epsilon_6 / f(B) = \epsilon_7, \quad \tau > 0. \quad (\text{V.3})$$

It follows from (V.2) and (V.3) that

$$h = e^{i\omega\tau} f(R) < \epsilon_5\epsilon_7 = \epsilon_8, \quad R > A, \quad \tau > 0.$$

This proves the first statement.

We now prove the second statement. In an unperturbed metric (V.1), for any light ray (not necessarily traveling along the radius) in the T-region, when $R < R_g - F$, where F is an arbitrary constant smaller than R_g , the following inequalities hold true⁹⁾:

$$d\tau/dr \geq (-g_{00}/g_{11})^{1/2} > 1 - N,$$

where $N = \text{const}$. This inequality denotes that the inclination of the ray is larger by a finite amount than the inclination of the line $R = R_g$ (see the figure). We have proved above that when $R > A$ the perturbations of the metric always remain small. It is clear that these perturbations change the value of $d\tau/dr$ of the ray little, and that the inequality

$$d\tau/dr > 1 - N$$

remains in force. Thus, the ray in the region $A < R < R_g$ never approaches $R = R_g$, and all the more, cannot cross it. Consequently, we have proved that in perturbed collapse the ray never emerges from the T-region.

APPENDIX VI

The axially symmetrical static quadrupole perturbations of the Schwarzschild metric as $g_{00} \rightarrow 0$ are written in the form ($c = 1$):

$$h_{00} \sim q \left(1 - \frac{R_g}{R}\right) \ln \left(1 - \frac{R_g}{R}\right),$$

$$h_{11} \sim q \left(1 - \frac{R_g}{R}\right)^{-1} \ln \left(1 - \frac{R_g}{R}\right),$$

$$h_{22} \sim h_{33} \sim q \ln \left(1 - \frac{R_g}{R}\right),$$

where q is the quadrupole parameter of the perturbation. In the collapse of a body with $q \neq 0$ in the co-moving system, all the quantities $h_{\mu\nu}$ are finite. Inasmuch as h_{22} and h_{33} are not transformed on going over from the co-moving system to the Schwarzschild system, it is obvious that as $R \rightarrow R_g$

$$q \sim \ln^{-1} \frac{R_g}{R_1 - R_g} \sim \frac{1}{t},$$

where R_1 is the position of the boundary of the collapsing body. Thus, in first order in q the perturbations in the diagonal terms vanish asymptotically. However, the density perturbations in the collapsing body are accompanied, in the general case, by the appearance of terms h_{13} , h_{23} , and h_{12} in the synchronous reference frame.^[10] This corresponds to the appearance of non-radial velocities, i.e., it is equivalent to some differential rotation with zero total momentum. Therefore in Schwarzschild coordinates there appear non-diagonal terms which depend on the time.

As was shown in Appendix III, h_0^α terms, describing nonspherically symmetrical motion of a central body, still remain asymptotically as $g_{00} \rightarrow 0$. Thus, when the body collapses with small deviations from spherical symmetry the external metric, in the limit as $g_{00} \rightarrow 0$, may differ in first order of perturbation theory from the Schwarzschild metric only in the terms $h_{0\alpha}$.

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