# *CONTRIBUTION TO THE THEORY OF RUNAWAY ELECTRONS*

A. N. LEBEDEV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

Submitted to JETP editor November 23, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **48**, 1393-97 (May, 1965)

An analytical expression is derived for the electron distribution function in a neutral spatially-homogeneous plasma in the presence of a weak electric field. The dependence of the flux of the runaway electrons on the field strength is found.

1. The question of the velocity distribution of the electrons in a plasma with a weak electric field has been discussed in a number of papers[1-3] by both numerical and analytical methods. The main interest lies in the computation of the flux of the so called runaway electrons which have a sufficiently high velocity so that collisions with the other plasma particles have practically no influence on their acceleration. However, the expressions which have been obtained in the papers referred to above frequently give quite different results, since they contain very sensitive exponential factors. This is caused by the fact that, owing to the very complicated behavior of the distribution function, the flux of the runaway electrons was estimated from additional physical considerations, which were not always valid when the parameters of the problem were changed. So, for example, Gurevich[2] found the distribution function only for velocities below the runaway threshold. Evidently, this precludes the possibility of obtaining trustworthy expressions for the flux.

2. The kinetic equation for the distribution function of the velocities of the electrons, $f(\mathbf{v})$, can be written for a spatially uniform plasma in the stationary state in the form

$$f'' + f'(1-\alpha^2 y\mu) + (2y)^{-1}\{[(1-\mu^2)f_\mu]_\mu - \alpha^2 y(1-\mu^2)f_\mu\} = 0, \quad (1)$$

where these nondimensional variables are used:

$$y = v^2/2v_0^2; \quad \mu = \cos\vartheta; \quad \alpha^2 = ET/2\pi e^3 N\Lambda \ll 1, \quad (2)$$

and where the partial derivative with respect to y is denoted by a prime, and that with respect to $\mu$ is denoted by a subscript $\mu$. In (2), $v_0 = (T/m)^{1/2}$, T is the electron temperature, $\vartheta$ is the angle between the velocity $\mathbf{v}$ and the electric field $\mathbf{E}$, N is the electron density which is equal to the ion density, and $\Lambda$ is the Coulomb logarithm.

The structure of Eq. (1) has been sufficiently fully discussed by Gurevich.[2] We remark here only that (1) is, generally speaking, applicable when $y \gg 1$; at the lower limit of the range, i.e. when $y \gtrsim 1$, the distribution must be Maxwellian since in that case the collisions and the associated dynamic losses must outweigh the influence of the electric field. Evidently sources are needed in order to maintain the stationary state in the presence of the electric field. We shall assume in the following that they are concentrated at $y \ll 1$ and have no influence on the distribution function in our region of interest.

It is important to note that at very large velocities, larger than, in particular, the runaway threshold velocity which is of the order of $y_c = \alpha^{-2}$, the distribution must have a sharp angular dependence and must be concentrated in the region of small angles $\vartheta(\mu \approx 1)$. To compute the flux of the runaway electrons one needs to know, at the same time, the behavior of $f(y, \mu)$ as $y \to \infty$. Thus we shall seek a solution for the range $\mu \approx 1$ in analogy with the treatment in [2]:

$$f(y,\mu) = f_0 \exp\{\varphi_1(y) + \varphi_2(y)(\mu-1) + O[(\mu-1)^2]\}. \quad (3)$$

stipulating that when $\mu = 1$ the equation (1) hold as well as its derivative with respect to $\mu$. We then obtain a nonlinear system of equations for the functions $\varphi_1$ and $\varphi_2$:

$$\varphi_1'' = \varphi_2/y - \varphi_1'^2 - \varphi_1'(1-\alpha^2 y);$$

$$\varphi_2'' = \varphi_2^2/y + \alpha^2\varphi_1' y + \varphi_2 y^{-1}(1-\alpha^2 y) - \varphi_2'(2\varphi_1' + 1 - \alpha^2 y). \quad (4)$$

An analogous system of equations was solved by Gurevich[2] by the method of successive approximations, where the first approximation was obtained with $\varphi_2 = 0$. As a result of this the obtained solution has a meaning only for $y < \alpha^{-2}$, and it becomes complex for large velocities, although the physical meaning of the problem requires that a stationary solution should exist for arbitrary y. Such a solution is obtained below as an asymptotic expansion in terms of the small parameter $\alpha$.

3. For the case of small velocities it is natural to utilize as an initial condition the requirement that the distribution function become Maxwellian for $y \gtrsim 1$. Keeping in the expressions for $\varphi_{1,2}$ the terms of order unity, which is important because of the exponential character of the solution of (3), and introducing a new independent variable

$$w = 3^{-1} 2^{2/3} \alpha^{-2/3} (\alpha^2 y - 1), \qquad (5)$$

we find the following expressions:

$$\varphi_1 \approx \alpha^{-2/3} \psi (1 + \psi^3 / 2) + \ln \psi (1 + 2\psi^3)^{-5/6} + O(\alpha^{2/3});$$
$$\varphi_2 / y \approx \alpha^{4/3} \psi + \alpha^2 (1 + \psi^6)(1 + 2\psi^3)^{-2} + O(\alpha^{8/3}), \qquad (6)$$

which are applicable for negative w, and also for $0 \le w \ll \alpha^{-2/3}$. The function $\psi(w)$ satisfies the cubic equation

$$\psi^3 + 2^{-2/3} \cdot 3 w \psi - 1 = 0 \qquad (7)$$

and has the form

$$\psi = 2^{2/3} (-w)^{1/2} \cos\left[{}^1/_3 \arccos(-w)^{-3}\right], \qquad w < -1,$$
$$\psi = \left[\sqrt{1 + w^3} + 1\right]^{1/3} / 2^{1/3} - \left[\sqrt{1 + w^3} - 1\right]^{1/3} / 2^{1/3},$$
$$w > -1. \qquad (8)$$

A graph of the function $\psi(w)$ is given in Fig. 1.

The condition $w \ll \alpha^{-2/3}$ restricts the applicability of these formulae to the region below the threshold velocity. For the region of large y we introduce again a scale change of the independent variable

$$u = \alpha^2 y - 1 = 2^{-2/3} \cdot 3 \alpha^{2/3} w. \qquad (9)$$

Here we use as an initial condition the requirement that the solution join with the solution (6) for small u. Then we obtain for $\alpha^{2/3} \ll u < \infty$

$$\varphi_1 \approx -\ln \mathrm{Ei}_1(u^{-1}) + O(\alpha^2);$$
$$\varphi_2 / y \approx \alpha^2 \mathrm{Ei}_1^{-1}(u^{-1}) \exp(-u^{-1}) + O(\alpha^4) \qquad (10)$$

[the additive constant in the expression for $\varphi_1$ can be omitted since it is contained in the normalization constant $f_0$ of (3)]. Using the asymptotic expression of the exponential integral $\mathrm{Ei}_1(u^{-1})$ for $u \ll 1$ it is easy to show that the solutions of (6) and (10) can be joined together with sufficient accuracy in the whole region of applicability

$$\alpha^{2/3} \ll u \ll 1 \quad \text{or} \quad 1 \ll w \ll \alpha^{-2/3}. \qquad (11)$$

Only the normalization constant $f_0$ remains to be found for the complete determination of the distribution function, since for $y \sim 1$ the distribution has to become Maxwellian: $f \to N v_0^{-3} (2\pi)^{-3/2} e^{-y}$. We find from (6) and (8), with an accuracy given by the omission of certain small terms of the order $\sim \alpha$,

$$f_0 = 2^{5/6} (2\pi)^{-3/2} v_0^{-3} \alpha^{-1/2} \exp\left[-1 / 2\alpha^2 - 2 / \alpha - {}^1/_2\right]. \qquad (12)$$

Using the expansion of (6) and (8) in inverse powers of w it is easy to obtain expressions for the distribution function which are valid in the range where $|w| \gg 1$. The first terms of this expansion indeed coincide with those obtained in [2]. However, in the vicinity of and above the threshold the difference becomes quite substantial as one can see, e.g., in Fig. 2 where the mean square angular spread of the distribution is shown.

The applicability of the obtained expressions can be easily checked by substituting

$$f = G(y, \mu) \exp\left[\varphi_1 + \varphi_2 (\mu - 1)\right]$$

in the basic Eq. (1). It then becomes evident that the function $G(y, \mu)$ differs from unity by an amount of the order $(1 - \mu)^2$ uniformly in the whole range of y. We note, however, that for $\varphi_2 \gg 1$ the function $f(y, \mu)$ itself depends very sensitively on $\mu$ even for $1 - \mu^2 \ll 1$. This way the distribution indeed turns out to be concentrated at small $\vartheta$ for sufficiently large y.

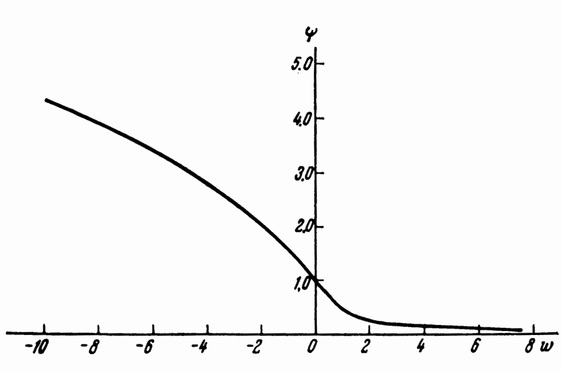


FIG. 1.

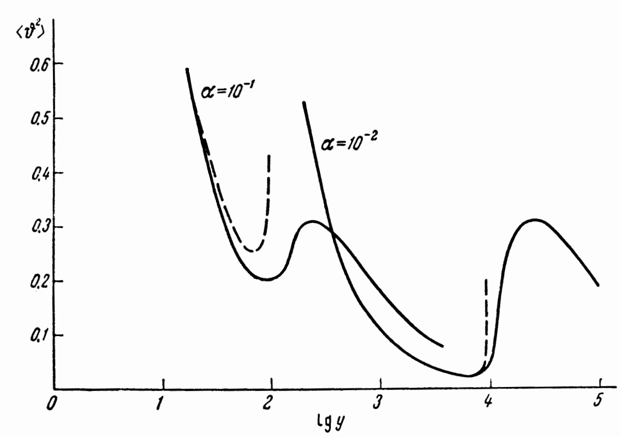


FIG. 2. Dependence of the angular dispersion on the velocity. The corresponding curves from[2] are shown dashed.

4. It is now easy to calculate the flux of the runaway electrons using the behavior of the distribution function at infinity. Since the fast electrons move practically in the direction of the electric field and the collisions are unimportant, we obtain for the flux, using (3), (10) and (12),

$$J = \lim_{y\to\infty} \frac{2\pi v^2 eE}{m} \int_{-1}^{+1} f(y, \mu)\, d\mu$$

$$= 2^{1/3}\pi^{-1/2} N\nu\alpha^{-1/6} \exp(-1/2\alpha^2 - 2/\alpha - 1/2), \qquad (13)$$

where J denotes the number of particles accelerated in a unit time from a unit volume, and

$$\nu = 4\pi e^4 N\Lambda / \sqrt{mT^3} \qquad (14)$$

is the electron-electron collision frequency for thermal velocities.

The analogous expressions obtained for the flux by Gurevich[2] in the same approximation differ from (13) by the pre-exponential factor. This, evidently, is the consequence of using the distribution function which is valid for $y < \alpha^{-2}$. A result more close to (13) ($J \sim \alpha^{-3/4}$ instead of $\alpha^{-1/2}$) has been obtained by Kruskal and Bernstein[3], but without the numerical coefficient.

We now describe some characteristic features of the distribution function. For small y, naturally, it is determined by the form of the source function. For $1 \ll y \ll \alpha^{-1}$ the distribution coincides with the Maxwell distribution, but it decreases somewhat slower as y increases. At the same time the dependence on $\mu$, i.e., the angular dependence, appears already below the runaway threshold. It becomes sharper as the threshold is approached, as has already been pointed out in[2]. We note that the maximum value of the mean square value $\langle\vartheta^2\rangle_{\min}$ turns out to be somewhat smaller than given by Gurevich[2] (see Fig. 2). Slightly above the runaway threshold,[1)] at $y \approx 2.59\,\alpha^{-2}$, the angular dispersion goes through a maximum which for arbitrary $\alpha$ has the value $\langle\vartheta^2\rangle_{\max} \approx 0.31$. It then begins slowly to decrease, approaching for large $y \gg \alpha^{-2}$ the behavior $y^{-1}\ln y$. This indicates that in this range the distribution of the perpendicular velocities is Gaussian with a slowly (logarithmically) growing dispersion. Along the line $\mu = 1$, i.e., along the direction of the electric field, the distribution function always decreases ($\varphi_1' < 0$). For large $y \gg \alpha^{-2}$ this decrease becomes very slow and is proportional $1/\ln y$.

It should be mentioned that the above method of asymptotic expansion can be easily used also for the case of a multiply ionized plasma, and also for the investigation of the process leading to the establishment of the stationary state.

[1] H. Dreicer, Phys. Rev. **117**, 329 (1960).

[2] A. V. Gurevich, JETP **39**, 1296 (1960), Soviet Phys. JETP **12**, 904 (1961).

[3] M. Kruskal and I. Bernstein, MATT-Q-20 (Semi-annual Report, 1962), p. 174.

Translated by M. Danos
204

[1)]Gurevich's results are not applicable in this range and formally give an infinite angular dispersion for $y = \alpha^{-2}$.