SCATTERING OF ELECTROMAGNETIC WAVES BY PLASMA OSCILLATIONS IN A PLANE PLASMA LAYER

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The scattering of electromagnetic waves by plasma waves in a plane plasma layer is investigated. The intensity of the scattered wave is estimated under the assumption that the layer thickness is small compared with the wavelengths of both the incident and scattered electromagnetic radiation. Possible applications of scattering for direct experimental verification of the existence of strong noise generation mechanisms in a turbulently heated plasma are discussed.

In connection with experiments on turbulence heating of plasmas [1-4] it would be of great interest to obtain direct experimental verification of the existence of intense plasma oscillations in a turbulent plasma. In particular, the existence of intense plasma oscillations can be verified by an experiment on scattering of electromagnetic waves by plasma oscillations. This problem has been treated by Kovrizhnykh and Tsytovich for the case of an infinite plasma.[5] Under laboratory conditions, however, a plasma always occupies a bounded volume, so that one should actually investigate scattering of electromagnetic waves on a plasma of finite extent. For this reason, in the present work we consider the scattering of electromagnetic waves in a plane plasma layer of thickness 2a.

In order to simplify the calculations we limit ourselves to the case in which the wavelengths of the incident and scattered electromagnetic radiation are much greater than the thickness of the layer; we assume further that $a \ll c/\omega_0$ where ω_0 is the electron plasma frequency while c is the velocity of light. This condition is satisfied in many experiments.[2,4] The analysis of the general case does not offer any fundamental difficulties but the calculations become extremely complicated.

As in the earlier work,[6] we neglect thermal motion of the electrons, this procedure being valid if the perturbation wavelength is appreciably greater than the Debye radius; we also assume that the ions are infinitely heavy and uniformly distributed throughout the layer. Assuming that in the unperturbed state the electric field, the magnetic field, and the directed motion of the electrons all vanish we obtain the following equation for the electric field perturbation $\mathbf{E}$:[6]

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{4\pi e^2 n_0}{mc^2}\mathbf{E} = \frac{4\pi e n_0}{c^2}(\mathbf{v}\nabla)\mathbf{v} + \frac{4\pi e^2 n_0}{mc^3}[\mathbf{vH}] + \frac{1}{c^2}(\mathbf{v}\operatorname{div}\mathbf{E}), \qquad (1)^*$$

where n_0 is the unperturbed plasma density while $\mathbf{v}$ and $\mathbf{H}$ are respectively the perturbations in velocity and magnetic field; the remaining notation is conventional. We note that $n_0 = 0$ outside the plasma.

The process we investigate arises as a result of the nonlinear interaction of the incident electromagnetic wave with the plasma oscillations. Hence, on the right side of Eq. (1) we retain only the nonlinear cross terms of the form $(\mathbf{v}_p\nabla)\mathbf{v}_e$, $(\mathbf{v}_e\nabla)\mathbf{v}_p$, etc., where the subscripts e and p refer to perturbed quantities associated with the incident electromagnetic wave and the plasma oscillations respectively.

Equation (1) is written in the form

$$\Delta\mathbf{E} - \operatorname{grad}\operatorname{div}\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} - \frac{4\pi e^2 n_0}{mc^2}\mathbf{E} = \frac{4\pi}{c^2}\frac{\partial\mathbf{j}_s}{\partial t}, \qquad (2)$$

where $4\pi c^{-2}\partial j_s/\partial t$ denotes the right side of Eq. (1). This manner of writing Eq. (1) shows that the problem of determining the amplitude of the scattered electromagnetic wave is essentially the classical electrodynamic problem of finding the fields due to a specified source current. Equation (2) for a plane plasma layer with arbitrary source currents j_s has been solved in an earlier work.[6]

As indicated above, the density of the source current j_s is expressed in terms of the perturbations of the electric field and the electron velocity. The perturbations of all quantities associated with

*$[\mathbf{vH}] = \mathbf{v}\times\mathbf{H}$.

the plasma oscillations are conveniently expressed in terms of the perturbation of the electrostatic potential φ which we expand in characteristic functions of the plasma oscillations:

$$\varphi = (2\pi)^{-2}\int d\mathbf{k}e^{i\mathbf{kr}}\left(\varphi_{\mathbf{k}}(x)e^{-i\omega_0 t} + \varphi^*_{-\mathbf{k}}(x)e^{i\omega_0 t}\right),$$

$$\varphi_{\mathbf{k}}(x) = \sum_{q=0}^{\infty}\left(\varphi^{(a)}_{\mathbf{k}q}\sin\frac{q\pi}{a}x + \varphi^{(s)}_{\mathbf{k}q}\cos\frac{(q+{}^1\!/_2)\pi}{a}x\right). \quad (3)$$

Here we have used a coordinate system with origin midway between the planes which define the layer (x axis perpendicular to the layer); $\mathbf{k}$ is a two-dimensional vector in the plane parallel to the layer. The subscripts s and a refer to Fourier coefficients for the even and odd characteristic functions respectively.

Perturbations of all quantities in the incident electromagnetic wave can be expressed in terms of the wave amplitude in vacuum $\mathbf{E}_0$. To be definite we shall assume that the wave is incident on the layer from below, from the half-space $x < -a$:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}_i\mathbf{r}+\varkappa_i x-\omega_i t)} + \mathbf{E}_0{}^* e^{-i(\mathbf{k}_i\mathbf{r}+\varkappa_i x-\omega_i t)}, \qquad x < -a, \quad (4)$$

where $\mathbf{k}_i$ is the component of the wave vector of the incident wave parallel to the layer, $\kappa_i = (\omega_i^2/c^2 - k_i^2)^{1/2}$ is the projection of the wave vector on the x axis, and ω_i is the wave frequency.

Let us assume that a p-polarized wave is incident on the plasma layer. In general, the scattered radiation will contain waves with both p- and s-polarization. We first find the field $\mathbf{E}$ associated with the scattered p-polarized wave. For this purpose write $\mathbf{E}$ in the form

$$\mathbf{E} = (2\pi)^{-2}\int d\mathbf{k}e^{i\mathbf{kr}}\mathbf{E}_{\mathbf{k}}(x,t),$$

where $\mathbf{k}$ is a two-dimensional vector in the plane perpendicular to the x axis. If the frequency of the incident electromagnetic wave is ω_i the scattered-radiation spectrum consists of two lines at the combination frequencies $\Omega_{1,2} = \omega_i \pm \omega_0$:

$$\mathbf{E}_{\mathbf{k}}(x,t) = \mathbf{E}_{1\mathbf{k}}{}^{(+)}(x)e^{-i\Omega_1 t} + \mathbf{E}_{1\mathbf{k}}{}^{(-)}(x)e^{i\Omega_1 t} + \mathbf{E}_{2\mathbf{k}}{}^{(+)}(x)e^{-i\Omega_2 t} + \mathbf{E}_{2\mathbf{k}}{}^{(-)}(x)e^{i\Omega_2 t},$$

where $\mathbf{E}^{(-)}_{(1,2)\mathbf{k}} = \mathbf{E}^{(+)*}_{(1,2)-\mathbf{k}}$.

Using the results of [6] we find the projections of the vector $\mathbf{E}_{1\mathbf{k}}{}^{(+)}$ in the direction of $\mathbf{k}$ and along the x axis:

$$E_{1\mathbf{k}}{}^{(+)} = \frac{\mathbf{kE}_{1\mathbf{k}}{}^{(+)}}{k} = \frac{icee^{-i\varkappa_1(x+a)}}{m(\Omega_1^2-\omega_0^2)} \times\left(\frac{\Omega_1^2-\omega_0^2}{\Omega_1\omega_i^2}\varkappa_1\varkappa_i\frac{\mathbf{kk}_i}{kk_i} - \frac{\Omega_1+\omega_0}{\omega_i^2-\omega_0^2}kk_i\right)E_0R, \quad x < -a,$$

$$E_{1\mathbf{k}}{}^{(+)} = \frac{\mathbf{kE}_{1\mathbf{k}}{}^{(+)}}{k} = \frac{icee^{i\varkappa_1(x-a)}}{m(\Omega_1^2-\omega_0^2)} \times\left(\frac{\Omega_1^2-\omega_0^2}{\Omega_1\omega_i^2}\varkappa_1\varkappa_i\frac{\mathbf{kk}_i}{kk_i} + \frac{\Omega_1+\omega_0}{\omega_i^2-\omega_0^2}kk_i\right)E_0R, \quad x > a,$$

$$R = \sum_{q=0}^{\infty}(-1)^q\frac{(q+{}^1\!/_2)\pi}{a}\varphi^{(s)}_{\mathbf{k}-\mathbf{k}_i q},$$

$$E^{(+)}_{1\mathbf{k}x} = kE_{1\mathbf{k}}{}^{(+)}/\varkappa_1 \quad \text{for} \quad x < -a,$$

$$E^{(+)}_{1\mathbf{k}x} = -kE_{1\mathbf{k}}{}^{(+)}/\varkappa_1 \quad \text{for} \quad x > a. \quad (5)$$

In obtaining these expressions we have retained the first nonvanishing term in the expansion in the parameters $a\omega_i/c$, $a\Omega_1/c$ and $a\omega_0/c$ in accordance with the assumption made above that the parameters $a\omega_i/c$ and $a\Omega_1/c$ are small compared with unity. The inequality $a\omega_0/c \ll 1$ follows from the relation $\omega_0 = \Omega_1 - \omega_i$ and $a\omega_i/c, a\Omega_1/c \ll 1$. The expressions for $\mathbf{E}_{2\mathbf{k}}{}^{(+)}$ are analogous in structure.

Averaging over time, we write the energy flux density of the scattered waves in the range $\mathbf{k}$, $\mathbf{k} + d\mathbf{k}$ in the form

$$\overline{\mathbf{S}}_{1\mathbf{k}}d\mathbf{k} = \mathbf{n}\frac{c}{4\pi L^2}\left[\mathbf{E}^{(+)}_{1\mathbf{k}}\mathbf{E}^{(+)*}_{1\mathbf{k}} + \mathbf{E}^{(-)}_{1\mathbf{k}}\mathbf{E}^{(-)*}_{1\mathbf{k}}\right]d\mathbf{k} = (\overline{\mathbf{S}}^{(+)}_{1\mathbf{k}} + \overline{\mathbf{S}}^{(-)}_{1\mathbf{k}})d\mathbf{k},$$

where L is the normalized length in the y and z directions and $\mathbf{n}$ is a unit vector in the direction of propagation of the scattered wave. Using the expressions found above for $\mathbf{E}^{(+)}_{1\mathbf{k}}$ and $\mathbf{E}^{(-)}_{1\mathbf{k}}$ we have

$$\overline{\mathbf{S}}^{(+)}_{1\mathbf{k}} = \mathbf{n}\frac{S_0}{2L^2}\frac{e^2}{m^2\varkappa_1^2}\frac{\Omega_1^2}{(\Omega_1^2-\omega_0^2)^2} \times\left[\frac{\Omega_1^2-\omega_0^2}{\Omega_1\omega_i^2}\varkappa_1\varkappa_i\frac{\mathbf{kk}_i}{kk_i} \mp \frac{\Omega_1+\omega_0}{\omega_i^2-\omega_0^2}kk_i\right]^2 \times\sum_{q',q''=0}^{\infty}(-1)^{q'+q''}\frac{(q'+{}^1\!/_2)\pi}{a}\frac{(q''+{}^1\!/_2)\pi}{a}\varphi^{(s)}_{\mathbf{k}-\mathbf{k}_i q'}\varphi^{(s)*}_{\mathbf{k}-\mathbf{k}_i q''},$$

$$\overline{\mathbf{S}}^{(-)}_{1\mathbf{k}} = \overline{\mathbf{S}}^{(+)}_{1-\mathbf{k}}, \quad S_0 = |E_0|^2/2\pi. \quad (6)$$

The upper sign is taken in front of the second term in the rectangular brackets for $x < -a$ and the lower sign for $x > a$; $\overline{S}_0$ is the time-averaged energy flux density of the incident electromagnetic wave.

We now average Eq. (6) over phases of the potential perturbations $\varphi_{\mathbf{k}p}{}^{(s)}$ using the relation

$$\langle\varphi^{(s)}_{\mathbf{k}q'}\varphi^{(s)*}_{\mathbf{k}q''}\rangle = |\varphi^{(s)}_{\mathbf{k}q'}|^2\delta_{q'q''}.$$

As a result we find

$$\overline{\mathbf{S}}^{(+)}_{1\mathbf{k}} = \mathbf{n}\frac{2\pi S_0}{\varkappa_1^2}\frac{e^2}{m^2}\frac{\Omega_1^2}{(\Omega_1^2-\omega_0^2)^2} \times\left[\frac{\Omega_1^2-\omega_0^2}{\Omega_1\omega_i^2}\varkappa_1\varkappa_i\frac{\mathbf{kk}_i}{kk_i} \mp \frac{\Omega_1+\omega_0}{\omega_i^2-\omega_0^2}kk_i\right]^2$$

$$\times \hbar\omega_0 \sum_{q=0}^{\infty} \frac{(q+{}^1/_2)^2\pi^2/a^2}{(\mathbf{k}-\mathbf{k}_i)^2+(q+{}^1/_2)^2\pi^2/a^2} N^{(s)}_{\mathbf{k}-\mathbf{k}_i q},$$

$$N^{(s)}_{\mathbf{k}q} = [\mathbf{k}^2+(q+{}^1/_2)^2\pi^2/a^2]\,|\varphi^{(s)}_{\mathbf{k}q}|^2/4\pi L^2\hbar\omega_0. \qquad (7)$$

The quantity $N^{(s)}_{\mathbf{k}q}$ has the meaning of the number of plasma oscillations characterized by **k**, q, (s) per unit volume.

Assuming that $|\mathbf{k}-\mathbf{k}_i| \ll a^{-1}$ (since $k_i \sim \omega_i/c \ll a^{-1}$, $k \sim \Omega_1/c \ll a^{-1}$) we can simplify the sum in Eq. (7):

$$\sum_{q=0}^{\infty} \frac{(q+{}^1/_2)^2\pi^2/a^2}{(\mathbf{k}-\mathbf{k}_i)^2+(q+{}^1/_2)^2\pi^2/a^2} N^{(s)}_{\mathbf{k}-\mathbf{k}_i q} \approx \sum_{q=0}^{\infty} N^{(s)}_{\mathbf{k}-\mathbf{k}_i q}.$$

In order to find the energy ϵ radiated per unit time from a unit surface of the layer into the half space $x < -a$ we must integrate the expression for the normal component $\bar{S}_{1\mathbf{k}}$ with respect to **k** from $|\mathbf{k}| = 0$ to $|\mathbf{k}| = \Omega_1/c$. The integration can be carried out explicitly if the energy of the plasma oscillations $W^{(s,a)}_{k'q} = \hbar\omega_0 N^{(s,a)}_{k'q}$ is uniformly distributed over the spectrum from $q = 0$, $k' = 0$ to $q = q_0$, $k' = k_0 \gg k_i$, k and if the relation $N^{(a)}_{k'q} = N^{(s)}_{k'q} = N_{k'q}$ is satisfied. This is the case, for example, when nonlinear interactions between plasma oscillations establish a uniform distribution of energy over the degrees of freedom of oscillatory motion;[7] under these conditions all degrees of freedom of the plasma oscillations are excited from $q = 0$, $k' = 0$ to $q = q_0 \sim ak_D/\pi$, $k' = k_0 \sim k_D$ (where $k_D = \omega_0/v_{T_e}$, v_{T_e} is the electron thermal velocity).

With these assumptions the occupation numbers $N_{k'q}$ are determined from the relation

$$\hbar\omega_0 \sum_{q=0}^{q} \int dk'(N^{(a)}_{k'q}+N^{(s)}_{k'q}) = 2\pi k_0^2 q_0 \hbar\omega_0 N_{kp} = W,$$

where W is the energy density of the plasma oscillations. Thus

$$N_{k'q} = \frac{W}{2\pi k_0^2 q_0 \hbar\omega_0}, \qquad \sum_{q=0}^{q_0} N^{(s)}_{\mathbf{k}-\mathbf{k}_i q} = \frac{W}{2\pi k_0^2\hbar\omega_0}.$$

Carrying out the integration over **k** we find

$$\varepsilon_1 = \frac{\eta}{3}\frac{\Omega_1^4}{(\Omega_1^2-\omega_0^2)^2} \times\left[\frac{(\Omega_1^2-\omega_0^2)^2}{2\Omega_1^2\omega_i^2}\cos^2\theta_i + 2\frac{(\Omega_1+\omega_0)^2\omega_i^2}{(\omega_i^2-\omega_0^2)^2}\sin^2\theta_i\right]S_0,$$

$$\eta = (\omega_0/ck_0)^2 W/mnc^2, \quad \tan\theta_i = k_i/\varkappa_i. \qquad (8)$$

This same expression is obtained for the energy ϵ radiated from a unit surface of the layer into the half space $x > a$.

When $\omega_i \approx \omega_0$ it follows from Eq. (8) that the intensity of the scattered radiation becomes infinite at frequency $\Omega_1 = \omega_0 + \omega_i \approx 2\omega_0$; this situation corresponds to resonance between the incident wave and the characteristic oscillations of the layer. Actually, however, higher-order nonlinear effects tend to limit the intensity of the scattered wave to some finite level. Additionally, the thermal motion of the electrons tends to smear out the resonance.

Carrying out the calculations for radiation at frequency $\Omega_2 = \omega_i - \omega_0$ we find

$$\varepsilon_2 = \frac{\eta}{3}\frac{\Omega_2^4}{(\Omega_2^2-\omega_0^2)^2} \times\left[\frac{(\Omega_2^2-\omega_0^2)^2}{2\Omega_2^2\omega_i^2}\cos^2\theta_i + 2\frac{(\Omega_2-\omega_0)^2\omega_i^2}{(\omega_i^2-\omega_0^2)^2}\sin^2\theta_i\right]S_0. \qquad (8')$$

As indicated above, when p-polarized waves are incident on the layer the scattered radiation will contain s-polarized waves. For these waves we have

$$\varepsilon_{1,2} = {}^1/_2\eta(\Omega_{1,2}/\omega_i)^2 S_0\cos^2\theta_i. \qquad (9)$$

We now present without derivation the results of calculations for the case in which the incident wave is s-polarized. For the scattered p-polarized waves we have

$$\varepsilon_{1,2} = {}^1/_6\eta(\Omega_{1,2}/\omega_i)^2 S_0, \qquad (10)$$

and for the scattered s-polarized waves

$$\varepsilon_{1,2} = {}^1/_2\eta(\Omega_{1,2}/\omega_i)^2 S_0. \qquad (11)$$

It is evident from (8)—(11) that when an electromagnetic wave of frequency ω_i is incident on a plasma layer the scattered radiation will, in general, contain waves with both combination frequencies $\Omega_{1,2} = \omega_i \pm \omega_0$. This is the essential difference between scattering from a bounded plasma and scattering from an infinite plasma. In an infinite plasma the scattered electromagnetic wave at frequency $\omega_i - \omega_0$ appears only when $\omega_i > 2\omega_0$ because electromagnetic waves with frequencies smaller than ω_0 cannot propagate in the infinite plasma.

Let us now estimate the intensity of the scattered radiation for experiments such as those described in [2] and [4]. At a plasma density $n \sim 10^{11}$ cm^{-3} the parameter $a\omega_0/c$ in these experiments is 0.3, so that Eqs. (8)—(11) apply, at least qualitatively. It is reasonable to expect that $W \sim nmv^2_{T_e}$, $k_0 \sim \omega_0/v_{T_e}$ in these experiments[2,4] (cf. [4]) so that

$$\eta = (\omega_0/ck_0)^2 W/nmc^2 \sim (T_e/mc^2)^2,$$

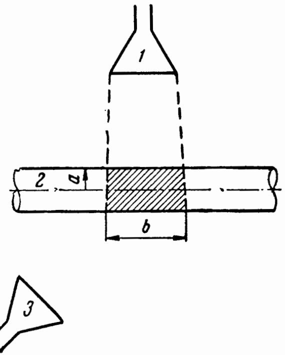

Diagram of an experiment to investigate scattering of electromagnetic waves by plasma oscillations: 1) transmitting horn, 2) plasma cylinder, 3) receiving horn. The region of interaction of the incident electromagnetic wave and the plasma oscillations is shown by cross-hatching.

where T_e is the electron temperature; the electron temperature can reach values of 10^3 eV. Thus η is of order 10^{-6}.

The scattered power incident on a receiving antenna P can be estimated from the formula (cf. the Figure),

$$P \approx {}^1\!/_2 ab\varepsilon o,$$

where o is the solid angle subtended by the receiving horn at the site of the nonlinear interaction. When $\Omega \sim \omega_0$ we find $\epsilon \sim \eta S_0$ from Eqs. (8)—(11) i.e.,

$$P \sim {}^1\!/_2 ab\eta S_0 o \sim {}^1\!/_2 ao\eta P_0 / b,$$

where P_0 is the power radiated by horn 1 (cf. the Figure). In the experiments [2,4] a ~ 1 cm. Assuming that b ~ 5 cm, o ~ 10^{-1} we find $P \sim 10^{-8} P_0$. For an easily achievable power in the 3-centimeter range $P_0 = 10^5$ we find $P \sim 10^{-3}$ W, a value which is completely adequate for reliable detection of the scattered radiation. A further increase in the power P of the scattered radiation can be achieved by choosing the frequency of the incident radiation ω_i so as to satisfy the resonance condition $\omega_i \approx \omega_0$. In this case the power of the scattered radiation at frequency $\Omega_1 = \omega_i + \omega_0$ increases sharply.

The proposed experiment is convenient in the sense that the frequency of the scattered radiation is very different from that of the incident wave. It is then possible to distinguish a weak signal at frequency $\Omega_1 = \omega_i + \omega_0$ against the strong background signal at the main frequency ω_i.

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