PARTIAL SCATTERING AMPLITUDES IN A REPRESENTATION WITH PRESCRIBED COMPLEX ORBITAL ANGULAR MOMENTA

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The author considers relativistic partial amplitudes with prescribed orbital momenta and realizes their analytic continuation in the complex angular momentum plane. As a result of an analysis of the many-particle unitarity condition, expressed in terms of these partial amplitudes, it is shown in explicit fashion that both negative and positive integer values of the orbital angular momentum, which satisfy, however, definite inequalities, participate in the formation of the branch points of the scattering amplitude of scalar particles, corresponding to exchange of N reggeons.

THE present paper is aimed at explaining the role played by the usual orbital angular momentum in the investigation of analytic properties in many-point diagrams; this momentum is not involved directly in investigations of asymptotic values of amplitudes of elastic and inelastic processes,[1-4] in view of the use of the helicity-amplitude formalism.[5]

We consider the simplest of the inelastic processes:

$$a+b\to c+d+e, \tag{1}$$

$$a+b\to c+d+e+f. \tag{2}$$

For simplicity, all particles are assumed identical, spinless, and scalar. Let p_a, p_b, p_c, p_d, p_e, and p_f be respectively the 4-momenta of particles a, b, c, d, e, and f. We denote by $\mathbf{p}$ the 3-momentum of the complex of particles c and d in the c.m.s. of the colliding particles; by $\mathbf{p}_1$ we denote the 3-momentum of particle c in the c.m.s. of particles c and d; by $\mathbf{p}_2$ we denote the 3-momentum of particle e in the c.m.s. of particles e and f; by $\mathbf{p}_a$ we denote the 3-momentum of particle a in the total c.m.s., and by $\mathbf{p}'_a$ the 3-momentum of particle a in the c.m.s. of particles c and d; we also put

$$t=(p_a+p_b)^2,\quad t_1=(p_c+p_d)^2,\quad t_2=(p_e+p_f)^2.$$

We denote by l_1 (l—for a 5-point diagram) and l_2 the relative orbital angular momenta of particles c and d and particles e and f. Let L be the orbital angular momentum of the complex of particles c and d with respect to particle e for process (1), and with respect to the complex of particles e and f for process (2). Then the total angular momentum $\mathbf{j}+\mathbf{L}+\mathbf{l}$, in the case of a 5-point diagram, and $\mathbf{j}=\mathbf{L}+\mathbf{l}=\mathbf{L}+\mathbf{l}_1+\mathbf{l}_2$—for a 6-point diagram. The addition of orbital momenta must be taken in the sense in which it was introduced in the papers of Yu. Shirokov[6,7] and MacFarlane.[8]

The total amplitude $A_{2\to3}$ of the 5-point diagram in question is expanded in relativistic partial amplitudes in the following fashion:

$$A_{2\to3}=\sum_{j,L,l}\sum_{j_z,M,m}C_{L,M;l,m}^{j,j_z}Y_{jj_z}^*(\mathbf{n}_a)Y_{LM}(\mathbf{n})Y_{lm}(\mathbf{n}_1)f_{jLl}(t,t_1), \tag{3}$$

where $\mathbf{n}_a$, $\mathbf{n}$, and $\mathbf{n}_1$ are unit vectors along the momenta $\mathbf{p}_a$, $\mathbf{p}$, and $\mathbf{p}_1$, respectively, and $C_{L,M;l,m}^{\gamma,\gamma_z}$ are Clebsch-Gordan coefficients,

$$Y_{lm}(\mathbf{n}(\theta,\varphi))=(-1)^m\sqrt{2l+1}\left[\frac{\Gamma(l-m+1)}{\Gamma(l+m+1)}\right]^{1/2}P_{lm}(z)e^{im\varphi},$$

$$P_{lm}(z)=(1-z^2)^{m/2}\frac{d^mP_l(z)}{dz^m}.$$

Analogously, for the 6-point diagram

$$A_{2\to4}=\sum_{j,L,l,l_1,l_2}\sum_{M,m_1,m_2}C_{L,M;l,m}^{j,j_z}C_{l_1,m_1;l_2,m_2}^{l,m}\times Y_{jj_z}^*(\mathbf{n}_a)Y_{LM}(\mathbf{n})Y_{l_1m_1}(\mathbf{n}_1)Y_{l_2m_2}(\mathbf{n}_2)f_{jLll_1l_2}(t,t_1,t_2), \tag{4}$$

where $\mathbf{n}_2$ is a unit vector along the momentum $\mathbf{p}_2$.

The choice of the quantization axis along the direction of the momentum $\mathbf{p}$ leads to the helicity-amplitude formalism.[2] We choose the angular momentum quantization axis along the momentum $\mathbf{p}_a$. Then the expansion (3) assumes the form

$$A_{2\to3}(t,t_1,z,z_1,\varphi)=\sum_{l=0}^{\infty}\sum_{L=0}^{\infty}\sum_{m=0}^{\infty}{}'(2l+1)$$
$$\times(2L+1)P_{Lm}(z)P_{lm}(z_1)\cos m\varphi\Psi_{Ll}{}^{m}(t,t_1),\qquad(5)$$

$$\Psi_{Ll}^{m}(t,t_1)=\sum_{j=|L-l|}^{L+l}\Big[\frac{2j+1}{(2l+1)(2L+1)}$$
$$\times\frac{\Gamma(L+m+1)\Gamma(l+m+1)}{\Gamma(L-m+1)\Gamma(l-m+1)}\Big]^{1/2}C^{j,0}_{l,m;L,-m}f_{jLl}(t,t_1).\qquad(6)$$

The prime denotes that when $m = 0$ a factor $\frac{1}{2}$ is introduced in the corresponding term of the sum[1]. We have introduced new amplitudes $\Psi^{m}_{Ll}(t, t_1)$ which are the analogs of the helicity amplitudes. In expansion (5) $z = \cos\theta_1$, where θ_1 is the angle between the momenta $\mathbf{p}_1$ and $\mathbf{p}'_a$, while φ is the azimuthal angle between the planes $(\mathbf{p}, \mathbf{p}'_a)$ and $(\mathbf{p}_1, \mathbf{p}'_a)$.

From (6) we have

$$f_{jL}{}^{l}(t,t_1)=\Big[\frac{(2L+1)(2l+1)}{2j+1}\Big]^{1/2}\sum_{m=0}{}'C^{j,0}_{l,m;L,-m}$$
$$\times\Big[\frac{\Gamma(L-m+1)\Gamma(l-m+1)}{\Gamma(L+m+1)\Gamma(l+m+1)}\Big]^{1/2}\Psi_{Ll}{}^{m}(t,t_1).\qquad(7)$$

Analogously, for a 6-point diagram

$$f_{jL}^{ll_1l_2}(t,t_1,t_2)=\Big[\frac{(2L+1)(2l_1+1)(2l_2+1)}{2j+1}\Big]^{1/2}$$
$$\times\sum_{m_1=0}^{\infty}{}'\sum_{m_2=0}^{\infty}{}'C^{j,0}_{l,m;L,-m}C^{l,m}_{l_1,m_1;l_2,m_2}$$
$$\times\Big[\frac{\Gamma(L-m+1)\Gamma(l_1-m_1+1)\Gamma(l_2-m_2+1)}{\Gamma(L+m+1)\Gamma(l_1+m_1+1)\Gamma(l_2+m_2+1)}\Big]^{1/2}$$
$$\times\Psi^{mm_1m_2}_{Ll_1l_2}(t,t_1,t_2).\qquad(8)$$

In order to avoid the appearance of root singularities in the continuation into the complex angular momentum plane, we introduce the amplitudes

$$F_{jL}{}^{l}(t,t_1)=\frac{f_{jL}{}^{l}(t,t_1)}{[(2L+1)(2l+1)]^{1/2}\Delta(j,L,l)},$$

$$\Delta(j,L,l)$$
$$=\Big[\frac{\Gamma(j+L-l+1)\Gamma(j+l-L+1)\Gamma(L+l-j+1)}{\Gamma(j+L+l+2)}\Big]^{1/2},$$

$$F_{jL}^{ll_1l_2}(t,t_1,t_2)$$
$$=\frac{f_{jL}^{ll_1l_2}(t,t_1,t_2)}{[(2L+1)(2l+1)(2l_1+1)(2l_2+1)]^{1/2}\Delta(j,L,l)\Delta(l,l_1,l_2)}.\qquad(9)$$

In the analytic continuation of the Clebsch-Gordan coefficients we shall make use of the Racah representation

$$C^{j,j_z}_{L,M;l,m}=\sqrt{2j+1}\,\Delta(j,L,l)[\Gamma(L+M+1)\Gamma(L-M+1)$$
$$\times\Gamma(l+m+1)\Gamma(l-m+1)\Gamma(j+j_z+1)$$
$$\times\Gamma(j-j_z+1)]^{1/2}G^{j,j_z}_{L,M;l,m},\qquad(10)$$

$$G^{j,j_z}_{L,M;l,m}=\sum_{k=0}^{\infty}(-1)^k[\Gamma(k+1)\Gamma(L+l-j-k+1)$$
$$\times\Gamma(L-M-k+1)\Gamma(l+m-k+1)$$
$$\times\Gamma(j-l+M+k+1)\Gamma(j-L-m+k+1)]^{-1}.\qquad(11)$$

We introduce the notation $n = j - L$. For integer l, the function $G^{j,j_z}_{j-n,M;l,m}$ is an analytic function of j in the region $\mathrm{Re}\, j > -1$.

The following representations will be the analytic continuation of amplitudes (9) into the right half plane of the complex variable j for $\mathrm{Re}\, n \geq 0$:

$$F^{l(\pm)}_{j,j-n}(t,t_1)=\Gamma(j+1)\sum_{m=0}^{\infty}{}'G^{j,0}_{l,m;j-n,-m}\Gamma(j-n-m+1)$$
$$\times\Gamma(l-m+1)\Psi^{m(\pm)}_{j-n,l}(t,t_1),\qquad(12)$$

$$F^{ll_1l_2(\pm)}_{j,j-n}(t,t_1,t_2)=\Gamma(j+1)\sum_{m_1=0}^{\infty}{}'\sum_{m_2=0}^{\infty}{}'G^{j,0}_{l,m;L,-m}G^{l}_{l_1,m_1;l_2,m_2}$$
$$\times\Gamma(l-m+1)\Gamma(l+m+1)\Gamma(j-n-m+1)$$
$$\times\Gamma(l_1-m_1+1)\Gamma(l_2-m_2+1)\Psi^{mm_1m_2(\pm)}_{j-n;l_1l_2}(t,t_1,t_2).\qquad(13)$$

The amplitudes of $\Psi^{m(\pm)}_{Ll}(t, t_1)$ with specified signature L are the analytic continuation of the amplitudes $\Psi^{m}_{Ll}(t, t_1)$ in the complex L plane. The method of analytic continuation into the complex angular momentum plane of integrals of the type

$$\frac{1}{2}\int_{-1}^{+1}P_{Lm}(z)a_{lm}(t,t_1,z)\,dz=\Psi_{Ll}{}^{m}(t,t_1)$$

(with j in lieu of L) was given in the paper of Popova and Ter-Martirosyan.[2] We therefore present the final result

$$\Psi_{Ll}{}^{m(\pm)}=\Psi_{Ll}{}^{m(1)}\pm\Psi_{Ll}{}^{m(2)};$$

$$\Psi_{Ll}{}^{m(1)}(t,t_1)=\frac{1}{\pi}\int_{z_R}^{\infty}a_{lm}{}^{(1)}(t,t_1,z)e^{-i\pi m}Q_{Lm}(z)\,dz,$$

$$\Psi_{Ll}{}^{m(2)}(t,t_1)=-\frac{1}{\pi}\int_{z_L}^{\infty}a_{lm}{}^{(2)}(t,t_1,-z)e^{-i\pi m}Q_{Lm}(z)\,dz.\qquad(14)$$

for $a^{(1)}_{lm}$ and $a^{(2)}_{lm}$ are the absorption parts of the amplitude

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[1] In the derivation of (5) we used the symmetry property of the amplitude

$$A_{lm}(t,t_1,z)=\int d\Omega_{n'}A_{2\to3}(t,t_1,z,z_1,\varphi)Y_{lm}{}^{*}(\mathbf{n}')$$

with respect to the substitution $m \to -m$.

$$a_{lm}(t, t_1, z) = \frac{1}{4\pi}\int_{-1}^{+1} dz_1 \int_0^{2\pi} d\varphi P_{lm}(z_1)\cos m\varphi A_{2\to 3}(t, t_1, z, z_1, \varphi) \quad (15)$$

respectively on the right and left cuts with respect to z. We note that the amplitudes (14) are not a complete analytic continuation of the amplitudes $\Psi^m_{Ll}(t, t_1)$, but only of that part which plays an essential role in the determination of the singularities that determine the behavior of the amplitudes at large energies in the crossing channel.[2]

The amplitudes $\Psi^{m(\pm)}_{Ll}(t, t_1)$ with specified L 1) analytic functions of L in the region Re L > ν where ν determines the degree of growth of the amplitude $a_{lm}(t, t_1, z)$ as $z \to \infty$; 2) decrease exponentially as $L \to \infty$ along any straight line in the right half-plane L; 3) coincide for integer L with the partial amplitudes $\Psi^m_{Ll}(t, t_1)$ of the amplitudes $a_{lm}(t, t_1, z)$ from which the part connected with the complex singularities in z is separated; 4) vanish for integer L = n < m.[2] The amplitudes $\Psi^{Mm_1m_2}_{Ll_1l_2}(t, t_1, t_2)$ are continued in the complex L plane in similar fashion.

It is easy to show that the representations (12) and (13) with Re n ≥ 0 are analytic functions on the right of some value Re j = j_0 with an asymptotic value $\sim e^{-\alpha j}$ (α > 0). The amplitudes $F^{l(\pm)}_{jL}(t, t_1)$ of the proper signature L for integer j, L, and l satisfy the triangle relation

$$|j-l| \leqslant L \leqslant j+l \quad (16)$$

by virtue of the properties of the amplitudes $\Psi^{m(\pm)}_{Ll}(t, t_1)$ of the proper signature L, which vanish for integer L < m. The amplitudes $F^{l(\pm)}_{jL}(t, t_1)$ of the improper signature L do not satisfy relation (16)[2]. This is precisely the reason for the difficulties that arise in the analytic continuation of the many-particle unitarity condition.

Let us examine the 3-particle unitarity condition of the elastic scattering amplitude $f_j(t)$, corresponding to the production of three spinless particles in the intermediate state. The corresponding jump through the 3-particle cut will be of the form

$$\Delta_3 f_j(t) = \frac{1}{3!}\frac{1}{2i}\int_{C_1} dt_1 \sum_{l=0}^{\infty}\sum_{L=|j-l|}^{j+l} f_{jLl}(t, t_1) \times f^{(3)}_{jLl}(t, t_{1-})\frac{2p(t,t_1)}{\sqrt{t}}\frac{2p_1^{2l+1}}{\sqrt{t_1}}$$

$$= \frac{1}{3!}\frac{1}{2i}\int_{C_1} dt_1 \sum_{l=0}^{\infty}\sum_{n=-l}^{+l}(2j-2n+1)(2l+1) \times \frac{\Gamma(2j-n-l+1)\Gamma(l+n+1)\Gamma(l-n+1)}{\Gamma(2j-n+l+2)} \times F^l_{j,\,j-n}(t, t_1)F^{l(3)}_{j,\,j-n}(t, t_{1-})\frac{2p(t,t_1)}{\sqrt{t}}\frac{2p_1^{2l+1}}{\sqrt{t_1}}, \quad (17)$$

where the contour C_1 is chosen the same as in[3,4]. Analytic continuation in the plane of complex j of a function of the form

$$\sum_{l=0}^{\infty}\sum_{L=|j-l|}^{j+l}\Phi_{jL}{}^l(t),$$

which is contained in the right side of (17) is generally speaking not unique. However, the only possible analytic continuation of (17), for which the function

$$\sum_{L=|j-l|}^{j+l}\Phi_{jL}{}^l(t)$$

is an analytic function in some region in the right half-plane of j for fixed l, will be

$$\sum_{n=-l}^{+l}\Phi^l_{j,\,j-n}(t) = \sum_{n=l}^{-\infty}\Phi^l_{j,\,j-n}(t) = \sum_{s=0}^{\infty}\Phi^l_{j,\,j+s-l}(t).$$

We then have

$$\Delta_3 f_j(t) = \frac{1}{3!}\frac{1}{2i}\int_{C_1} dt_1 \sum_{l=0}^{\infty}\sum_{s=0}^{\infty}(2j-2l+2s+1)(2l+1) \times \frac{\Gamma(2j-2l+s+1)}{\Gamma(2j+s+2)}\Gamma(2l-s+1)\Gamma(s+1)F^l_{j,\,j+s-l}(t, t_1) \times F^{l(3)}_{j,\,j+s-l}(t, t_{1-})\frac{2p(t,t_1)}{\sqrt{t}}\frac{2p_1^{2l+1}}{\sqrt{t_1}}. \quad (18)$$

However, the presence of poles of the function $\Gamma(2j - 2l + s + 1)$, contained in (18), leads to a difficulty in the analytic continuation of the unitarity condition in the form (18).[3] Mainly, for an amplitude of foreign signature j in the complex j plane there arises an infinite number of poles, located at integral positive points. Physically this is not admissible (see the paper of Gribov et al.[3]).

An attempt to circumvent this difficulty consists in realizing the analytic continuation of the right side of (18) in the complex j plane not in the form of a series in l, but in the form of a contour integral with respect to the variable l with suitable choice of the contour:

$$\Delta_3 f_j(t) = \frac{1}{3!}\frac{1}{(2i)^2}\int_{C_1} dt_1 \int_L \frac{dl}{\mathrm{tg}\,\pi l}\sum_{s=0}^{\infty}(2j-2l+2s+1)(2l+1) \times \frac{\Gamma(2j-2l+s+1)}{\Gamma(2j+s+2)}\Gamma(2l-s+1)\Gamma(s+1) \times F^l_{j,\,j+s-l}(t, t_1)F^{l(3)}_{j,\,j+s-l}(t, t_{1-})\frac{2p(t,t_1)}{\sqrt{t}}\frac{2p_1^{2l+1}}{\sqrt{t_1}}, \quad (19)*$$

2) We shall henceforth, for simplicity, not indicate the signature.

*tg = tan.

where the contour L encircles not only the entire real l axis, but also the singularities of the integrand connected with presence of the function $\Gamma(2j - 2l + s + 1)$.

Such a continuation, as in [3], is not unique. The unitarity condition (19), continued analytically in the complex j plane, contains amplitudes $F^{l}_{j,j+s-l}(t, t_1)$, which are analytically continued in the complex j and l plane simultaneously. Such amplitudes are given as before by formulas (12) and (14), the only difference being that in formula (14) the $a^{(1,2)}_{lm}(t, t_1, z)$ are the absorption parts of the amplitude $a_{lm}(t, t_1, z)$ which is analytically continued in the plane of complex l, having in lieu of (15) the integral form

$$a^{(\pm)}_{lm}(t, t_1, z) = \frac{1}{\pi}\int\limits_{z_{1R}}^{\infty} A_m^{(1)}(t, t_1, z, z_1)\, e^{-i\pi m} Q_{lm}(z_1)\, dz_1$$

$$\pm\left(-\frac{1}{\pi}\right)\int\limits_{z_{1L}}^{\infty} A_m^{(2)}(t, t_1, z, -z_1)\, e^{-i\pi m} Q_{lm}(z_1)\, dz_1. \qquad (20)$$

Then the analytic continuation of the unitarity condition in the form (19) is possible for definite values of t, for which $\cosh^{-1} z_{1R} - \cosh^{-1} z_R > 0$. However, the result of the investigations can be extended to arbitrary values of t.

Let us assume that for $l = \alpha(t_1) = \alpha$ there is a pole of the amplitude of the 5-point diagram $F_{jLl}(t, t_1)$:

$$F_{jLl}(t, t_1) = \frac{\Gamma(j+1) R_{jLl}(t, t_1)}{l - \alpha(t_1)}. \qquad (21)$$

Then the singular part $\Delta_3' f_j(t)$ of the integral (19), due to the pole at $l = \alpha(t_1)$, will take the form [3]

$$\Delta_3' f_j(t) = \frac{2}{3!}\frac{1}{(2i)}\int\limits_{C_1} dt_1 \frac{1}{\operatorname{tg}\pi\alpha(t_1)} \sum_{s=0}^{\infty}(2j - 2\alpha + 2s + 1)$$
$$\times (2\alpha+1)\frac{\Gamma(2j - 2\alpha + s + 1)}{\Gamma(2j + s + 2)}\Gamma(2\alpha - s + 1)$$
$$\times \Gamma(s+1)\Gamma^2(s+1)\, R^{\alpha}_{j,\, j-\alpha+s}(t, t_1)$$
$$\times R^{\alpha(3)}_{j,\, j-\alpha+s}(t, t_{1-})\,\frac{2p(t, t_1)}{\sqrt{t}}. \qquad (22)$$

The amplitude $R^{\alpha}_{jL}(t, t_1)$ can be regarded as a partial amplitude of the transition of two particles into a particle and a reggeon. It is determined by formulas (12) and (21)

$$R^{\alpha}_{j;\, j-\alpha+s}(t, t_1) = \sum_{m=0}^{\infty}{}' G^{j,0}_{\alpha, m;\, j-\alpha+s, -m}\,\Gamma(j - \alpha + s - m + 1)$$
$$\times \Gamma(\alpha - m + 1)\; r^{\alpha m}_{j-\alpha+s}(t, t_1). \qquad (23)$$

Using (11), we obtain

$$R^{\alpha}_{j,\, j-\alpha+s}(t, t_1) = \sum_{m=0}^{\infty}{}' \sum_{k=0}^{s}(-1)^k \frac{\Gamma(j-\alpha+s-m+1)\Gamma(\alpha-m+1)}{\Gamma(k+1)\Gamma(s-k+1)\Gamma(j-\alpha+s-m-k+1)\Gamma(\alpha-m-k+1)}$$
$$\times \frac{r^{\alpha m}_{j-\alpha+s}(t, t_1)}{\Gamma(j-\alpha+m+k+1)\Gamma(\alpha-s+m+k+1)}, \qquad (24)$$

where the function $r^{\alpha m}_{j-\alpha+s}$ is the residue of the amplitude Ψ^{lm}_{j-l+s} at the pole $l = \alpha(t_1)$, and has a representation of the form (14) with suitable replacement of the drops $a^{(1)}_{lm}$ and $a^{(2)}_{lm}$ of the amplitude of the 4-point diagram (20) by the drops of the amplitude of the transition of two particles into a particle and a reggeon, that is, the function $r^{\alpha m}_{j-\alpha+s}(t, t_1)$ is represented by an integral of the form

$$\int dz \int dz_1 Q^m_{j-\alpha+s}(z)\, Q_{\alpha}{}^m(z_1)\, B_m(t, t_1, z, z_1).$$

It is easy to see that the amplitudes $R^{\alpha}_{j,j-\alpha+s}(t, t_1)$ have no "kinematic" singularities with respect to j.

The integrand in the right side of (22) which is a series $\sum\limits_{s=0}^{\infty}$, containing the function $(2j - 2\alpha + 2s + 1)\Gamma(2j - 2\alpha + s + 1)$, has poles at the points

$$j(t_1) = \alpha(t_1) - 1 - k/2, \quad k = 0, 1, 2, \ldots.$$

As shown in [3] these poles are the cause of the branch points of the amplitude $f_j(t)$ at the following values of j:

$$j(t) = \alpha((\sqrt{t} - \mu)^2) - 1 - k/2 \; (k = 0, 1, 2, \ldots). \qquad (25)$$

The branch points (25) correspond to values of the orbital angular momentum

$$L = j - \alpha + s = -1 - k/2 + s$$
$$(s = 0, 1, \ldots, k+1; \quad k = 0, 1, \ldots).$$

These values of L can be either positive or negative. Expression (22) shows, however, that only those states for which the values of L satisfy the inequalities

$$j - l \leqslant L \leqslant l - j - 1$$

participate in the formation of the branch points, or, taking into account the parity conservation requirement for scalar particles,

$$j-l \leqslant L \leqslant l-j-2. \quad (26)$$

Let us consider the 4-particle term of the unitarity condition:

$$\Delta_4 f_j(t)=\frac{1}{2}\sum_{l_1, l_2=0}^{\infty}\sum_{l=|l_1-l_2|}^{l_1+l_2}\sum_{n=-l}^{+l}\frac{1}{(2i)^2}\int_{C_1}dt_1\int_{C_2}dt_2\, f_{j,\,j-n}^{ll_1l_2}(t,t_1,t_2)$$

$$\times f_{j,\,j-n}^{(4)ll_1l_2}(t,t_{1-},t_{2-})\frac{2p(t,t_1,t_2)}{\sqrt{t}}\frac{2p_1^{2l_1+1}}{\sqrt{t_1}}\frac{2p_2^{2l_2+1}}{\sqrt{t_2}}. \quad (27)$$

The analytic continuation of the 4-particle condition of unitarity (27) is carried out in a form analogous to (19):

$$\Delta_4 f_j(t)=\frac{1}{2(2i)^4}\int_{C_1}dt_1\int_{C_2}dt_2\int_{L_1}\frac{dl_1}{\operatorname{tg}(\pi l_1/2)}\int_{L_2}\frac{dl_2}{\operatorname{tg}(\pi l_2/2)}\sum_{s_1, s_2=0}^{\infty}(2j-2l_1-2l_2+2s_1+2s_2+1)\,\Gamma(2j-2l_1-2l_2+2s_2+s_1+1)$$

$$\times\Delta'(j,l_1,l_2,s_1,s_2)F_{j,\,j-l_1-l_2+s_1+s_2}^{l_1+l_2-s_2,\,l_1,\,l_2}(t,t_1,t_2)\,F_{j,\,j-l_1-l_2+s_1+s_2}^{(4)l_1+l_2-s_2,\,l_1l_2}(t,t_{1-},t_{2-})\frac{2p(t,t_1,t_2)}{\sqrt{t}}\frac{2p_1^{2l_1+1}}{\sqrt{t_1}}\frac{2p_2^{2l_2+1}}{\sqrt{t_2}}.$$

$$\Delta'(j,l_1,l_2,s_1,s_2)=\frac{\Delta^2(j,j-l_1-l_2+s_1+s_2,l_1+l_2-s_2)\,\Delta^2(l_1+l_2-s_2,l_1,l_2)}{(2j-2l_1-2l_2+2s_1+2s_2+1)\,\Gamma(2j-2l_1-2l_2+2s_2+s_1+1)}. \quad (28)$$

The contours L_1 and L_2 circuit respectively not only the real positive axes of the complex variables l_1 and l_2, but also the singularities of the integrand which are brought about by the presence of the function $\Gamma(2j-2l_1-2l_2+2s_2+s_1+1)$.

Representation (28) is apparently unique, satisfying the following two requirements: a) that the double series $\sum_{s_1,s_2=0}^{\infty}$ be an analytic function of j in some region in the right half-plane of j at fixed values of l_1 and l_2 for definite values of t; b) that the amplitude of $f_j(t)$ have no infinite number of poles for positive integer values of j.

Let us assume that the amplitude of the 6-point diagram $F_{jL}^{ll_1l_2}(t,t_1,t_2)$ has poles of positive signature at $l_1=\alpha(t_1)=\alpha_1$ and $l_2=\alpha(t_2)=\alpha_2$:

$$F_{jL}^{ll_1l_2}(t,t_1,t_2)=\frac{\Gamma(j+1)R_{jL}^{ll_1l_2}(t,t_1,t_2)}{(l_1-\alpha(t_1))(l_2-\alpha(t_2))}. \quad (29)$$

Then the calculation of the singular part of the jump $\Delta_4' f_j(t)$ yields:

$$\Delta_4' f_j(t)=\frac{1}{2(2i)^2}\int_{C_1}dt_1\int_{C_2}dt_2\left[\operatorname{tg}\pi\frac{\alpha(t_1)}{2}\cdot\operatorname{tg}\pi\frac{\alpha(t_2)}{2}\right]^{-1}$$

$$\times\sum_{s_1, s_2=0}^{\infty}(2j-2\alpha_1-2\alpha_2+2s_1+2s_2+1)$$

$$\times\Gamma(2j-2\alpha_1-2\alpha_2+s_1+2s_2+1)\,\boldsymbol{\Delta'(j,\alpha_1,\alpha_2,s_1,s_2)}$$

$$\times R_{j,\,j-\alpha_1-\alpha_2+s_1+s_2}^{\alpha_1+\alpha_2-s_2;\,\alpha_1\alpha_2}(t,t_1,t_2)\,R_{j,\,j-\alpha_1-\alpha_2+s_1+s_2}^{(4)\alpha_1+\alpha_2-s_2,\,\alpha_1\alpha_2}(t,t_1,t_2)$$

$$\times\frac{2p(t,t_1,t_2)}{\sqrt{t}} \quad (30)$$

The partial amplitude $R(t,t_1,t_2)$ of the transition of two particles into two reggeons on the basis of formulas (13) and (29) is represented in the following fashion:

$$R_{j,\,j-\alpha_1-\alpha_2+s_1+s_2}^{\alpha_1+\alpha_2-s_2,\,\alpha_1\alpha_2}(t,t_1,t_2)=\sum_{m_1=0}^{\infty}{}'\sum_{m_2=0}^{\infty}{}'\,G_{\alpha_1+\alpha_2-s_2,\,m;\,j-\alpha_1-\alpha_2+s_1+s_2,\,-m}^{j,\,0}$$

$$\times G_{\alpha_1,\,m_1;\,\alpha_2,\,m_2}^{\alpha_1+\alpha_2-s_2,\,m}\Gamma(\alpha_1+\alpha_2-s_2-m+1)$$

$$\times\Gamma(\alpha_1+\alpha_2-s_2+m+1)$$

$$\times\Gamma(j-\alpha_1-\alpha_2+s_1+s_2-m+1)\Gamma(\alpha_1-m_1+1)$$

$$\times\Gamma(\alpha_2-m_2+1)\,r_{j-\alpha_1-\alpha_2+s_1+s_2,\,\alpha_1\alpha_2}^{mm_1m_2}(t,t_1,t_2), \quad (31)$$

where $m=m_1+m_2$, $r(t,t_1,t_2)$ is a double residue of the amplitude $\Psi_{Ll_1l_2}^{mm_1m_2}(t,t_1,t_2)$ at the poles $l_1=\alpha(t_1)=\alpha_1$ and l_1 and $l_2=\alpha(t_2)=\alpha_2$ for $L=j-\alpha_1-\alpha_2+s_1+s_2$, having an integral representation of the form

$$\int dz\int dz_1\int dz_2 Q_{L,m}(z)\,Q_{\alpha_1m_1}(z_1)\,Q_{\alpha_2m_2}(z_2)\,B_m(t,t_1,z,z_1,z_2).$$

The amplitudes (31), like the amplitudes (23) have no "kinematic" singularities in j.

The poles of the function $\Gamma(2j-2\alpha_1-2\alpha_2+2s_2+s_1+1)$, which is contained in the double series of the integrand of (30), lead to the occurrence of branch points of the elastic scattering amplitude $f_j(t)$ for the following values of $j(t)$:

$$j(t)=2\alpha(t/4)-1-k/2 \quad (k=0,1,2,\ldots). \quad (32)$$

These branch points correspond to values of the orbital angular momentum

$$L=j(t)-2\alpha(t/4)+s_1+s_2=-1-k/2+s_1+s_2$$
$$(k,s_1,s_2=0,1,2,\ldots;\ s_1+s_2\leqslant k+1). \quad (33)$$

According to (33), both positive and negative integer and half-integer values of L participate in the formation of the branch points (32), provided that they satisfy the inequalities

$$j(t)-2\alpha(t/4)\leqslant L\leqslant 2\alpha(t/4)-j(t)-2, \quad (34)$$

as can be readily seen from (30).

In the limiting case, when $t \to 4\mu^2$ (in this case $t_1, t_2 \to \mu^2$, $\alpha(t_1) = \alpha(t_2) \to \alpha(\mu^2) = 0$ and $L \to j$), the amplitude for the production of two reggeons $R(t, t_1, t_2)$ becomes equivalent to the amplitude of scattering of two identical scalar particles. It is to be expected that in this case the arguments presented by Gribov and Pomeranchuk[9] will be valid for the amplitudes $r(t, t_1, t_2)$. The symmetry property of the reggeon spectral function with respect to the substitution $s \rightleftarrows u$ (here s and u are the Mandelstam variables) will lead to the vanishing of the residues of the integrand in (30) for odd k and odd k/2. If these properties are conserved not only as $t \to 4\mu^2$, then the amplitude $f_j(t)$ has branch points

$$j(t) = 2\alpha(t/4) - k, \quad \text{where } k = 1, 3, 5, \ldots .$$

In the more general case of exchange of N reggeons, the many-particle terms of the unitarity condition lead to the occurrence of branch points

$$j_N(t) = N\alpha(t/N^2) - (N-1) - k/2 \quad (k = 0, 1, 2, \ldots), \tag{35}$$

which correspond to the addition of N reggeons with the N − 1 orbital angular momenta $L_1, L_2, \ldots, L_{N-1}$ which arise during the consideration of the problem, and assume integer and half integer positive and negative values. However, the symmetry of the spectral functions with respect to the substitution $s \rightleftarrows u$ causes the vanishing of residues for half-integer and odd k/2.

Thus, by turning to relativistic partial amplitudes with specified orbital angular momenta we were able to show directly the values of the orbital momentum corresponding to the branch points obtained in [3,4] (see also [10]). It is not excluded that the use of the amplitudes F^l_{jL} and Ψ^m_{Ll}, the analytic properties of which at complex values of the momenta were investigated in the present paper, may turn out to be convenient in the analysis of certain problems of "reggistics" of many-point diagrams. In particular, on the basis of formulas (12), (13), and (14) it is easy to see that the amplitudes $F^l_{jL}(t, t_1)$ and $F^{ll_1l_2}_{jL}(t, t_1, t_2)$, like the amplitudes $\Psi^m_{Ll}(t, t_1)$ and $\Psi^{mm_1m_2}_{Ll_1l_2}(t, t_1, t_2)$ have a threshold behavior of the type $\sim p^L$ for arbitrary complex L, which agrees with the result obtained in [11], if account is taken of the linear relation between the amplitude $F^l_{jL}(t, t_1)$ with the helicity amplitudes.[6] This circumstance can cause the amplitudes $r^{mm_1m_2}_{Ll_1l_2}(t, t_1, t_2)$ to have poles that condense as $p \to 0$ towards a certain value j in the complex plane (see the paper of Gribov and Pomeranchuk [12]). Inasmuch as in formulas (22) and (30) pinching of the contour, which leads to the appearance of branch points in the elastic-scattering amplitude, takes place at p = 0 (the reggeon production threshold), the aforementioned condensation of the poles can lead to the appearance of branch points of a new type in the elastic-scattering amplitude.

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