SCATTERING OF SLOW NEUTRONS IN He NEAR THE LAMBDA CURVE

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Using the previously developed theory [1] of phase transitions we find the cross section for the scattering of slow neutrons in He near the λ-curve. The results obtained make it possible to check experimentally the main assertion of the theory about the form of the fluctuation spectrum, $\epsilon = Aq^{3/2}$. We obtain some general relations between thermodynamic quantities and the neutron scattering cross section.

THE average occupation number of low-momentum states increases near the curve of the second-order phase-transition in liquid He I, and as a consequence there occurs a correlation between coordinates and velocities of particles over large distances. Such long-wavelength fluctuations are microscopic regions of a superfluid phase. Although these fluctuations are damped rapidly, one may speak of their momentum q and characteristic energy $\epsilon(q)$. The spectrum $\epsilon(q)$ is the most important characteristic of the fluctuations for knowledge of the spectrum enables us to construct the thermodynamics and kinetics of the λ-transition. The occurrence near the λ-curve of long-wavelength fluctuations is the cause of the anomalous behavior of the heat capacity and of other thermodynamic quantities. On the other hand, one should also expect near the λ-curve anomalous scattering of light and of particles by the fluctuations.[1)] Since the physical cause of these phenomena is the same, there must exist a relation between the thermodynamic quantities and the particle (neutron) scattering cross section. The present paper is devoted to the elucidation of this connection. We shall also explain the connection between the neutron scattering cross section and the fluctuation spectrum.

In a paper by Patashinskii and one of the authors [1] we constructed a theory of second-order phase transitions in a Bose liquid, an essential feature of which was the characteristic form of the fluctuation spectrum, $\epsilon(q) = Aq^{3/2}$. An experimental check of this statement would be an important argument for or against the theory.

1. GENERAL RELATIONS

We consider a macroscopic volume of a Bose liquid at temperature T, which is a target for the scattering of cold neutrons. The differential cross-section $\sigma(q, \epsilon)$ for the scattering of neutrons is in the well-known way connected with the Fourier transform $K(q, \epsilon)$ of the two-particle Green function $K(\mathbf{r}-\mathbf{r}', t-t')$ (see, for instance, [3]):

$$\sigma(q, \varepsilon) = -\left(\frac{m_n + m_{\rm He}}{m_{\rm He}}\right)^2 \frac{2\pi^2 q}{p^2} Va^2 \frac{\operatorname{Im} K(q, \varepsilon)}{1 + e^{-\varepsilon/T}}. \quad (1.1)$$

By $\sigma(q, \epsilon)$ we understand here the twofold differential cross section $d^2\sigma/dq\,d\epsilon$ involving a momentum transfer q and an energy transfer ϵ; m_n and m_{He} are the masses of a neutron and of a helium atom; V is the volume of the system, a the scattering amplitude for the scattering of a neutron by a separate helium nucleus, and p the momentum of the incident neutron (we put $\hbar = 1$).

The Green function $K(\mathbf{r}-\mathbf{r}', t-t')$ is connected with the functions for density correlations in different space-time points:

$$iK(\mathbf{r}-\mathbf{r}', t-t') = \begin{cases} \langle n(\mathbf{r}, t)\, n(\mathbf{r}, t')\rangle - n^2, & t > t' \\ \langle n(\mathbf{r}', t')\, n(\mathbf{r}, t)\rangle - n^2, & t < t' \end{cases}. \quad (1.2)$$

The symbol ⟨ ⟩ indicates averaging over an ensemble. Landau's well-known dispersion relation holds between the real and imaginary parts of $K(q, \epsilon)$

$$\operatorname{Re} K(q, \varepsilon) = \frac{1}{\pi} \oint \frac{\operatorname{Im} K(q, \omega)}{\omega - \varepsilon} \operatorname{th} \frac{\omega}{2T}\, d\omega. \quad (1.3)^*$$

Using Eq. (1.1) we find a relation between $\sigma(q, \epsilon)$ and $\operatorname{Re} K(q, \epsilon)$:

1)The anomalous scattering of neutrons by density fluctuations near a critical point was considered by Van Hove.[2] Some of the ideas in Secs. 2 and 4 of the present paper are close to the ideas given in his paper.

*th = tanh.

$$\oint \frac{\sigma(q,\omega)(1-e^{-\omega/T})}{\omega-\varepsilon}d\omega$$
$$= -\left(\frac{m_n+m_{He}}{m_{He}}\right)^2 \frac{2\pi^3 q}{p^2} Va^2 \operatorname{Re} K(q,\varepsilon). \qquad (1.4)$$

From the definition (1.2) it follows that

$$K(\mathbf{r}-\mathbf{r}', t-t') = K(\mathbf{r}'-\mathbf{r}, t'-t),$$
$$K(\mathbf{r}-\mathbf{r}', t-t') = K(\mathbf{r}-\mathbf{r}', t'-t). \qquad (1.5)$$

Hence it follows that the Fourier transform $K(q,\epsilon)$ satisfies the relation

$$K(q,\varepsilon) = K(q,-\varepsilon). \qquad (1.6)$$

From (1.1) and (1.6) we find

$$\sigma(q,-\varepsilon)/\sigma(q,\varepsilon) = e^{-\varepsilon/T}. \qquad (1.7)$$

Equation (1.7) expresses the principle of detailed balancing.

It is useful to introduce general relations connecting $K(q,\epsilon)$ with the thermodynamic quantity $(\partial n/\partial\mu)_{V,T}$ (n is the density and μ the chemical potential). By definition

$$K(0,\varepsilon) = -i\int_0^\infty dt e^{-i\varepsilon t}\int dV\,(\langle n(\mathbf{r},t)n(0,0)\rangle - n^2)$$
$$-i\int_{-\infty}^0 dt e^{-i\varepsilon t}\int dV\,(\langle n(0,0)n(\mathbf{r},t)\rangle - n^2). \qquad (1.8)$$

The quantity $\int dV\, n(\mathbf{r},t) = N(t)$, the number of particles in the system, is conserved, i.e., does not depend on time. Therefore

$$\langle N(t), n(0,0)\rangle = \langle N(0), n(0,0)\rangle = \langle n(0,0), N(0)\rangle,$$

and we get from (1.8)

$$K(0,\varepsilon) = -2\pi i\delta(\varepsilon)V^{-1}(\langle N^2\rangle - \langle N\rangle^2). \qquad (1.9)$$

It is, however, well known that

$$\langle N^2\rangle - \langle N\rangle^2 = T(\partial N/\partial\mu)_{V,T}, \qquad (1.10)$$

so that

$$\operatorname{Im} K(0,\varepsilon) = -2\pi\delta(\varepsilon)T(\partial n/\partial\mu)_{V,T}. \qquad (1.11)$$

Now substituting (1.11) into the dispersion relation (1.3) we get

$$\operatorname{Re} K(0,\varepsilon) = \begin{cases} 0, & \varepsilon \neq 0 \\ -(\partial n/\partial\mu)_{V,T}, & \varepsilon = 0 \end{cases}. \qquad (1.12)$$

At first sight it seems physically meaningless to isolate a finite value of a function in an isolated point. In fact, the value of $\operatorname{Re} K(0,\epsilon)$ given by Eq. (1.12) is the limit to which $\operatorname{Re} K(q,0)$ tends as $q \to 0$. Indeed, for small non-vanishing q instead of an infinite δ-peak there occurs in Eq. (1.9) a sharp peak with a finite, albeit narrow width. In order that we may substitute in Eq. (1.3)

$$\frac{\operatorname{th}(\omega/2T)}{\omega-\varepsilon} \approx \frac{1}{2T},$$

the width of the peak must be much larger than ϵ which is clearly satisfied when $\epsilon = 0$ and $q \neq 0$. We get thus

$$\lim_{q\to 0} \operatorname{Re} K(q,0) = -(\partial n/\partial\mu)_{V,T}, \qquad (1.13)$$
$$\operatorname{Re} K(0,\varepsilon) = 0 \quad (\varepsilon \neq 0). \qquad (1.14)$$

Far from the point where $(\partial n/\partial\mu)_{V,T}$ tends to infinity (the phase transition point), the quantity $\operatorname{Re} K(q,0)$ can apparently be expanded in a power series in q^2. However, near the phase transition curve $[(\partial n/\partial\mu)_{V,T} \to \infty]$ the q-dependence of $\operatorname{Re} K(q,0)$ changes: a singularity occurs in $\operatorname{Re} K(q,0)$ as $q \to 0$.

On the other hand, for fixed non-vanishing ϵ, the function $\operatorname{Re} K(q,\epsilon)$ does not have a singularity as $q \to 0$. We show in Appendix A that for a relatively large class of systems of interacting particles the following formula holds:

$$\operatorname{Re} K(q,\varepsilon) = -nq^2/\varepsilon^2, \qquad \varepsilon \gg q\sqrt{T/m}. \qquad (1.14')$$

We not consider the behavior of $\operatorname{Im} K(q,\epsilon)$ for small q. In that region $\operatorname{Im} K(q,\epsilon)$ is a δ-function-like function of ϵ with a wide peak depending on q. When $\epsilon \gg q\sqrt{T/m}$ the function $\operatorname{Im} K(q,\epsilon)$ decreases exponentially. Indeed, the maximum energy transfer to a particle in the liquid with momentum p for a given momentum transfer q is equal to pq/m. The average momentum of a particle in the liquid $p \sim \sqrt{Tm}$ and the number of particles in the liquid with momenta $p \gg \sqrt{Tm}$ is exponentially small. Of course, these considerations are not true for very low temperatures when quasi-particles with a dispersion law different from $p^2/2m$ play the main role.

2. ANOMALOUS SCATTERING BY FLUCTUATIONS

Near the phase transition curve the quantity

$$\operatorname{Re} K(q,0)_{q\to 0} = -(\partial n/\partial\mu)_{V,T}$$

has a singularity connected with the growth of the long-wavelength fluctuations. The energies comparable with the fluctuation energies $\epsilon(\eta, q)$ give the main contribution to $(\partial n/\partial\mu)_{V,T}$ in the integral (1.4). Here, q is the momentum of the fluctuation and

$$\eta = a(\mu-\mu_\lambda) + b(T-T_\lambda)$$

is the distance from the λ-curve. On approaching the λ-curve ($\eta \to 0$) the characteristic dimension

$\bar{r}$ of the fluctuations increases and the characteristic momentum $\bar{q}$ and characteristic energy $\epsilon(\eta, \bar{q})$ decrease. A neutron moving with speed v passes through the fluctuation region $\bar{r}$ in a time $\tau = 1/\bar{q}v$. If the fluctuation spectrum $\epsilon(\eta, q)$ for $q \gtrsim \bar{q}$ is proportional to q^β, the characteristic fluctuation energy $\epsilon(\eta, \bar{q})$ will be proportional to $\bar{q}^\beta$ and the average life time of the fluctuation

$$\tau_{fl} \gtrsim 1/\varepsilon(\eta, \bar{q}) \approx 1/\bar{q}^\beta.$$

If $\beta > 1$, in the fairly immediate vicinity of the λ-curve the time of passage τ of a neutron may become arbitrarily small compared to the life time τ_{fl} of the fluctuations, i.e., during the time of passage of the neutron the fluctuation does not manage to undergo essential changes and the scattering becomes elastic. The scattering takes place as if by static potential fields.

We consider the scattering kinematics in detail. Let a neutron with momentum p be scattered at a given angle θ with a momentum transfer equal to q. The energy transfer $\Delta\epsilon$ is then determined in terms of p, q, and θ by the equation

$$\Delta\varepsilon = \frac{p^2 - (\mathbf{p} - \mathbf{q})^2}{2m_n} = \frac{p^2}{2m_n}\left\{2\left(\sin^2\theta \pm \cos\theta \times \left[\left(\frac{q}{p}\right)^2 - \sin^2\theta\right]^{1/2}\right) - \left(\frac{q}{p}\right)^2\right\}, \tag{2.1}$$

where m_n is the neutron mass. For small angles θ Eq. (2.1) becomes

$$\frac{\Delta\varepsilon}{E(p)} = 2\left(\theta^2 \pm \left[\left(\frac{q}{p}\right)^2 - \theta^2\right]^{1/2}\right) - \left(\frac{q}{p}\right)^2,$$

$$E(p) = \frac{p^2}{2m_n}. \tag{2.2}$$

For elastic scattering $q_0/p = 2\sin(\theta/2) \approx \theta$. The energy transfer of the neutron when scattered by fluctuations has a characteristic value $\Delta\epsilon \approx q\,\partial\epsilon(\eta, Q)/\partial Q$. The value of the derivative is taken in the point $\max(q, \bar{q})$. Indeed, if $q \gg \bar{q}$ the scattering of the neutron is accompanied by the creation (or absorption) of a fluctuation with momentum of the order q. If, however, $q \ll \bar{q}$ the main role in the scattering is played by fluctuations with momentum of the order $\bar{q}$. In both these cases the fluctuation group velocity

$$v_{fl} = \left.\frac{\partial\varepsilon(\eta, Q)}{\partial Q}\right|_{\max(q, \bar{q})}$$

is small compared to the neutron speed v. It follows from Eq. (2.2) that the difference between the momentum transfer $\Delta q = |q - q_0|$ from the value of the transfer q_0 for elastic scattering is given by the quantity

$$\frac{\Delta q}{q_0} \approx \left(\frac{p\Delta\varepsilon}{q_0 E(p)}\right)^2 \approx \left(\frac{v_{fl}}{v}\right)^2 \ll 1. \tag{2.3}$$

In other words, we may assume that for all energy transfers which are important in the present problem the angle of scattering of the neutron remains unchanged for a fixed momentum transfer q. We assume here that $v_{fl}(q)$ decreases as $q \to 0$.

It now turns out to be possible to connect in the vicinity of the phase transition curve the differential scattering cross section for a given angle $d\sigma/d\Omega = \sigma(\theta)$ with Re K(q, 0). Indeed, for small q and $\epsilon = 0$ the integration region $\omega \ll T$ contributes to the integral (1.4). After replacing $1 - e^{-\omega/T}$ by ω/T we get thus

$$\sigma(q) = \frac{d\sigma}{dq} = \int_{-\infty}^{\infty} \sigma(q, \omega)\, d\omega = -T\left(\frac{m_n + m_{He}}{m_{He}}\right)^2 \frac{2\pi^3 q}{p^2} Va^2 \operatorname{Re} K(q, 0). \tag{2.4}$$

Experimentally, however, one measures the scattering $\sigma(\theta)$ for fixed angle θ rather than the scattering for fixed momentum transfer $\sigma(q)$. As we have shown, on approaching the λ-curve the scattering by the fluctuations becomes elastic, i.e., to each fixed scattering angle there corresponds a well-defined momentum transfer $q_0 = 2p\sin(\theta/2)$. For the cross section for the anomalous scattering by fluctuations the following relation will therefore be valid

$$\sigma(\theta) = \frac{d\sigma}{d\Omega} = \frac{pp'}{2\pi q}\frac{d\sigma}{dq} = -T\left(\frac{m_n + m_{He}}{m_{He}}\right)^2 \frac{\pi^2 p'}{p} Va^2 \operatorname{Re} K(q, 0). \tag{2.5}$$

($p' = |\mathbf{p} - \mathbf{q}|$ is the momentum of the scattered neutron).

Substituting from Eq. (1.13) for $\operatorname{Re}_{q\to 0} K(q, 0)$ into (2.5) we find for the forward scattering cross section

$$\sigma(0) = T\left(\frac{m_n + m_{He}}{m_{He}}\right)^2 \pi^2 Va^2 \left(\frac{\partial n}{\partial\mu}\right)_{V,T}. \tag{2.6}$$

Near the λ-curve $(\partial n/\partial\mu)_{V,T} \to \infty$ and the forward scattering cross section also tends to infinity. We note once again that Eqs. (2.5) and (2.6) refer not to the total cross section, but only to its singular part connected with the scattering by fluctuations. Equations (2.5) and (2.6) refer to the total cross section only when the neutron energy $p^2/2m_n$ is much larger than the average thermal energy of the target particles.

3. CORRELATION FUNCTION OF A PERFECT BOSE GAS NEAR THE EINSTEIN CONDENSATION POINT

The present section has an illustrative character. We show with a perfect gas as an example how the correlation function $K(q, \epsilon)$ possesses a singularity for small q and ϵ in the vicinity of the phase-transition point. The function $K(r, t)$ for a perfect Bose gas can be written as an integral

$$iK(\mathbf{r}, t) = \int\int \frac{d^3p d^3q}{(2\pi)^6} n_p (n_q + 1) \exp\{i(\varepsilon_p - \varepsilon_q)t - i(\mathbf{p}-\mathbf{q})\mathbf{r}\},$$

$$n_p = (e^{(\varepsilon_p - \mu)/T} - 1)^{-1}. \tag{3.1}$$

For $t = 0$ this function has the form (see [4], p. 385)

$$iK(\mathbf{r}, 0) = n\delta(\mathbf{r}) + \nu(\mathbf{r}), \qquad \nu(\mathbf{r}) = \left|\int \frac{d^2p}{(2\pi)^3} n_p e^{i\mathbf{pr}}\right|^2, \tag{3.2}$$

where $n = N/V$ is the density of the system. For Boltzmann statistics of distinguishable particles $(-\mu/T \gg 1)$ the first term in Eq. (3.2) describes the correlation between the positions of one particle, and the second the position correlation in the positions of two different particles.

Near the phase transition point the function $\nu(\mathbf{r})$ has for small μ $(-\mu/T \ll 1,\ r \gg \hbar/mT)$ the form

$$\nu(\mathbf{r}) \approx \left| 2mT \int \frac{d^3p}{(2\pi)^3} \frac{\exp(i\mathbf{pr})}{p^2 - 2m\mu} \right|^2$$

$$= \frac{m^2T^2}{(2\pi)^2} \frac{\exp(-\sqrt{-8m\mu}\, r)}{r^2} \tag{3.3}$$

We see that when we approach the phase transition point the characteristic dimension of the fluctuations $\bar{r}$ increases as $1/\sqrt{-\mu}$. The corresponding characteristic momentum $\bar{q} \sim 1/\bar{r}$ decreases as $\sqrt{-\mu}$. Hence it follows that the time, characteristic for the fluctuations

$$\tau_{fl} \approx 1/\varepsilon(\mu, \bar{q}) \approx 2m/q^2$$

increases as $-1/\mu$ when the phase transition curve is approached, i.e., much faster than the time of passage $\tau \approx 1/\sqrt{-\mu}$ of the neutron through the characteristic fluctuation region.

Let us now investigate the function $K(q, \epsilon)$. Taking the Fourier transform of both sides of Eq. (3.1) we get

$$K(q, \varepsilon) = \int \frac{d^3p}{(2\pi)^3} \frac{n_p - n_{\mathbf{p+q}}}{\varepsilon_p - \varepsilon_{\mathbf{p+q}} + \varepsilon} - i\pi \int \frac{d^3p}{(2\pi)^3} n_{\mathbf{p+q}}(1 + n_p)$$

$$\times [\delta(\varepsilon_p - \varepsilon_{\mathbf{p+q}} + \varepsilon) + \delta(\varepsilon_p - \varepsilon_{\mathbf{p+q}} - \varepsilon)]. \tag{3.4}$$

We can write the imaginary part of $K(q, \epsilon)$ in the following form

$$\operatorname{Im} K(q, \varepsilon) = -\frac{m^2T}{4\pi q}(1 + e^{-\varepsilon/T}) F\left(\frac{\varepsilon}{T}, \frac{q^2}{2mT}, -\frac{\mu}{T}\right), \tag{3.5}$$

$$F(x, y, z) = \int_{(x-y)^2/4y}^{\infty} d\xi \frac{e^{z+\xi}}{(e^{z+\xi} - 1)(e^{x+z+\xi} - 1)}. \tag{3.6}$$

The anomalous scattering near the λ-curve is connected with small values of q and ϵ. We are interested in the case of small values of the arguments x, y, and z. The value of the lower limit in (3.6) then remains undetermined.

We consider two limiting cases:

1. $(x-y)^2/4y \gg 1$ $(\epsilon \gg q\sqrt{T/2m})$. The asymptotic form of Eq. (3.6) is then

$$F(x, y, z) \approx e^{-(x-y)^2/4y} \approx e^{-x^2/4y} = e^{-m\varepsilon^2/2q^2T}. \tag{3.7}$$

2. $(x-y)^2/4y \ll 1$ $(\epsilon \ll q\sqrt{T/2m})$. Equation (3.6) can be approximately written as

$$F(x, y, z) = \int_{(x-y)^2/4y}^{\infty} \frac{d\xi}{(z+\xi)(x+z+\xi)}$$

$$= \frac{1}{x} \ln \frac{(x+y)^2 + 4yz}{(x-y)^2 + 4yz}. \tag{3.8}$$

The argument of the logarithm in (3.8) is a homogeneous function of degree zero in x, y, z and has no well-defined limit as $x, y, z \to 0$. For fixed y and z the function $F(x,y,z)$ as a function of x has a maximum, equal to $4/(y+4z)$, at $x = 0$. Its graph has the form of a peak, the width of which is determined by the largest of the quantities y and $\sqrt{yz}$.

Changing from the quantities x, y, z to ϵ, q, μ and from the function $F(x,y,z)$ to $\operatorname{Im} K(q, \epsilon)$ we can formulate our conclusions as follows. For small $|\mu| \ll T$ and $q^2 \ll 2mT$ the quantity $\operatorname{Im} K(q, \epsilon)$ as a function of ϵ has two characteristic dimensions

$$\varepsilon_1 = q\sqrt{\frac{T}{2m}}, \qquad \varepsilon_2 = \max\left(\frac{q^2}{2m}, q\sqrt{\frac{-\mu}{2m}}\right).$$

The physical meaning of ϵ_1 is that it is the characteristic energy transfer to a neutron when scattered by a separate atom. When $\epsilon \gg \epsilon_1$ the quantity $\operatorname{Im} K(q, \epsilon)$ decreases exponentially, in accordance with the general considerations given in Sec. 1. The smaller dimension ϵ_2 determines the width of the peak, in the center of which $K(q, \epsilon)$ has the value $-2m^3T^2/\pi q(q^2 - 8m\mu)$, increasing when $q, -\mu \to 0$. One easily verifies that ϵ_2 is the same as the fluctuation energy $\epsilon(q) = q^2/2m$ when $q \gg \bar{q} = \sqrt{-2m\mu}$ and the same as the quantity $q\, \partial\epsilon(Q)/\partial Q|_{Q=\bar{q}}$ when $q \ll \bar{q}$ (cf. Sec. 2). Here $\epsilon(\eta, q) = -\mu + q^2/2m$. The physical meaning of ϵ_2 is that it is the characteristic energy transfer of the neutron when scattered by the fluctuations.

It follows from Eq. (3.4) that when $\mu = 0$ the

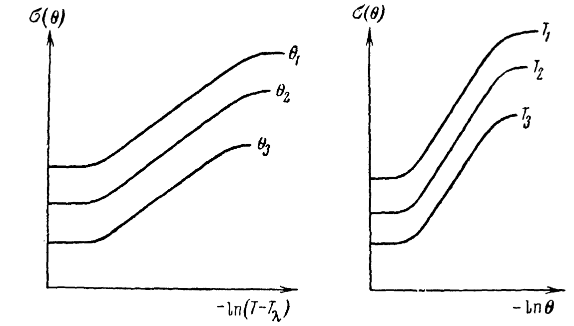

FIG. 1.

quantity Re K(q, 0) has a singularity of the form const/q in accordance with the general conclusions of Sec. 1. When $\mu \neq 0$, the function Re K(q, 0) can be expanded in a power series in $-q^2/2m\mu$.

4. THE CORRELATION FUNCTION AND SCATTERING CROSS SECTION IN He I

The most complete information about the fluctuation spectrum can be obtained by studying the inelastic scattering cross section $\sigma(q, \epsilon)$ for small transfers q and ϵ. As shown in Sec. 2, the characteristic energy transfer ϵ_2 when a neutron is scattered by fluctuations $\approx qv_{fl}$ is much smaller than the average energy transfer when scattered by He atoms. This transfer ϵ_2 is so small that at the present time it is impossible to measure it.

Let us now study the scattering cross section for a given angle [see Eq. (2.5)]. The theory constructed in [1] predicts the following behavior near the λ-curve:[2)]

$$\operatorname{Re} K(q, 0) = B \ln \frac{\max(\eta, Aq^{3/2})}{T_\lambda} + \operatorname{Re} K_{\text{reg}}(q, 0), \quad (4.1)$$

where A and B depend solely on the position of the transition point on the λ-curve, Re $K_{reg}(q, 0)$ is a regular function of T, μ, q. From Eqs. (2.5) and (4.1) we find the angular differential cross section:

$$\sigma(\theta) = -T\left(\frac{m_n + m_{\text{He}}}{m_{\text{He}}}\right)^2 \pi^2 V a^2 B \ln \frac{\max(\eta, Aq^{3/2})}{T_\lambda} + \sigma_{\text{reg}}(\theta). \quad (4.2)$$

Equation (4.2) can be checked experimentally. In the limiting cases of small and large angles, (4.2) becomes

$$\sigma(\theta) = -c \ln (T - T_\lambda) + c' \quad (\theta \ll \eta^{2/3} / A^{2/3} p), \quad (4.3)$$

$$\sigma(\theta) = -{}^3\!/_2 c \ln \theta + c'' \quad (\theta \gg \eta^{2/3} / A^{2/3} p). \quad (4.4)$$

2)In[1] the value of Re K(q,0) was given only for q = 0 (see Eq.(5.4) in[1]).

One should construct from the experimental data the curves giving the dependence $\sigma(\theta)$ for fixed θ and $\sigma(\theta)$ for fixed T where along the abscissa axes we must plot, respectively, the logarithms of $T - T_\lambda$ and of θ. The curves thus constructed will have linear sections (as is schematically indicated in Fig. 1). The slopes of those sections must, according to the theory, differ by a factor $^3/_2$.

We now give some quantitative estimates starting from the existing experimental data on the thermodynamics of the λ-transition.[5-7] We elucidate how closely we must approach the λ-curve in temperature or pressure in order that the logarithmic singularity in $\sigma(\theta)$ becomes perceptible. Unfortunately, experiments show that the singular part of the compressibility $k_T = -V^{-1}(\partial V/\partial p)_T$ is a small quantity compared with the regular part of the compressibility. In the vicinity of the point $T_\lambda = 2.023$°K, $V_\lambda = 24.2$ cm^3/mole, and $p_\lambda = 13.04$ atm, Lounasmar's experiment gives in the interval $(p - p_\lambda)/p_\lambda \sim 10^{-3}$ to 10^{-4} a relative change in the compressibility of 3 to 4% in He I and 10 to 12% in He II.[6] Chase, Maxwell, and Millett's data on the relative change in the compressibility near the λ-point give 1 to 2% in He I and 2 to 3% in He II for a change $(T - T_\lambda)/T_\lambda \sim 10^{-3}$ to 10^{-4}.[7]

As the compressibility k_T is connected with $(\partial n/\partial \mu)_{V,T}$ by the simple relation

$$k_T = -n^2 \Omega_{\mu\mu} = n^{-2} (\partial n / \partial \mu)_{V, T}, \quad (4.5)$$

one must thus expect on the basis of Eq. (1.12) and (4.3) the same relative change in the differential cross section $\sigma(\theta)$. The at present attainable accuracy in measuring the differential cross section of neutron scattering makes it possible to detect such a change in the scattering cross section.

There remains the consideration of the problem whether the ranges of angles shown in Eqs. (4.3) and (4.4) are experimentally accessible; to see this we must give a numerical estimate of the quantities A and $\eta = a(\mu - \mu_\lambda) + b(T - T_\lambda)$. Unfortunately, we can only give a most tentative estimate. Experimentally only the ratio of the coefficients b/a $= -(\partial \mu/\partial T)_\lambda$ is known; it changes in the range 20 to 45 when one moves along the λ-curve. It is reasonable to assume that the coefficient b is of the order of unity. The coefficient a will then be of the order $^1/_{30}$. The basis for such a statement is that the quantity a is proportional to the universal numerical factor $1 - P(\infty)$ (see [1]). One must think that this quantity is small through accidental circumstances. In the opposite case one cannot explain why the ratio a/b is small along the whole of the λ-curve.

Taking the magnitude of b of the order of unity and determining A from the experimental data on the specific heat ($A \approx 5 \times 10^{14}$ cgs esu) we can find a connection between the range of angles $\Delta\theta$ and the range of temperatures ΔT inside of which one can trace the logarithmic dependence of the differential scattering cross section $\sigma(\theta)$. To do this, we use the relation

$$A(p\Delta\theta)^{3/2} \sim b\Delta T \tag{4.6}$$

(p is the momentum of the incident neutron). Taking the wavelength of the neutron to be ~1 Å and $\Delta T \sim 10^{-2}$ °K, we get $\Delta\theta$ of the order of tenths of a degree. If it is possible to measure at angles of the order of tens of seconds (corresponding to approaching the transition point at temperatures $T - T_\lambda \sim 10^{-5}$ to 10^{-6} °K) then the change in the logarithmic part of the cross section $\sigma(\theta)$ is a quantity of the order of 5% of the total cross section. To elucidate the character of the spectrum $\epsilon(q)$ we need thus a very delicate experiment with a very accurate thermostatic arrangement. It seems to us, though, that the requirements of experimental accuracy may be made easier if one approaches the λ-curve from the He II side.

5. NEUTRON SCATTERING NEAR THE λ-CURVE IN He II

As before, we start from the basic formulas (1.1) to (1.4). Following Belyaev[8] we write the wave function operator $\psi(\mathbf{r}, t)$ as a sum

$$\psi(\mathbf{r}, t) = \sqrt{n_0} + \tilde{\psi}(\mathbf{r}, t), \tag{5.1}$$

where n_0 is the density of the condensate, and $\tilde{\psi}(\mathbf{r}, t)$ the wave function of the uncondensed particles. Substituting (5.1) into (1.2) we get ($x \equiv (\mathbf{r}, t)$)

$$\begin{aligned} iK_{\rm II}(x-x') &= i\tilde{K}(x-x') + n_0[\langle T(\tilde{\psi}^+(x)\tilde{\psi}(x'))\rangle \\ &+ \langle T(\tilde{\psi}(x)\tilde{\psi}^+(x'))\rangle + \langle T(\tilde{\psi}^+(x)\tilde{\psi}^+(x'))\rangle \\ &+ \langle T(\tilde{\psi}(x)\tilde{\psi}(x'))\rangle] + \sqrt{n_0}[\langle T(\tilde{\psi}^+(x)\tilde{\psi}(x)\tilde{\psi}^+(x'))\rangle \\ &+ \langle T(\tilde{\psi}^+(x)\tilde{\psi}(x)\tilde{\psi}(x'))\rangle + \langle T(\tilde{\psi}^+(x)\tilde{\psi}^+(x')\tilde{\psi}(x'))\rangle \\ &+ \langle T(\tilde{\psi}(x)\tilde{\psi}^+(x')\tilde{\psi}(x'))\rangle], \\ i\tilde{K}(x-x') &= \langle T(\tilde{\psi}^+(x)\tilde{\psi}(x)\tilde{\psi}^+(x')\tilde{\psi}(x'))\rangle - \tilde{n}^2. \end{aligned} \tag{5.2}$$

Taking the Fourier transform of (5.2) and using the usual definition of the Green function (see, e.g., [3], p. 280), we get

$$\begin{aligned} K_{\rm II}(q,\varepsilon) &= \tilde{K}(q,\varepsilon) + n_0(G(q,\varepsilon) + G(q,-\varepsilon) + F(q,\varepsilon) \\ &+ F^+(q,\varepsilon)) + 2\sqrt{n_0}(P(q,\varepsilon) + P(q,-\varepsilon)); \end{aligned} \tag{5.3}$$

$$\begin{aligned} iG(x-x') &= \langle T(\tilde{\psi}(x)\tilde{\psi}^+(x'))\rangle, \\ iF(x-x') &= \langle T(\tilde{\psi}(x)\tilde{\psi}(x'))\rangle, \\ iF^+(x-x') &= \langle T(\tilde{\psi}^+(x)\tilde{\psi}^+(x'))\rangle, \\ iP(x,x') &= \langle T(\tilde{\psi}^+(x)\tilde{\psi}^+(x')\tilde{\psi}(x'))\rangle. \end{aligned} \tag{5.4}$$

The first term on the right-hand side of (5.3) corresponds to the scattering of a neutron by the uncondensed particles, the second term to the scattering of a neutron by the particles in the condensate accompanied by a transition into an uncondensed state (or vice versa), while the third term is an interference term.

When we approach the λ-curve from below, the appearance of fluctuations, when a neutron is scattered by the condensate, becomes energetically much easier but, on the other hand, the number of particles in the condensate, n_0, diminishes, so that the total probability for the scattering of a neutron by particles in the condensate tends to a finite limit when the λ-curve is approached, as shown in Appendix B. We shall now determine the differential scattering cross section $\sigma(\theta)$.

Using Eq. (2.5), we get

$$\sigma_{\rm II}(\theta) = -T\left(\frac{m_n + m_{\rm He}}{m_{\rm He}}\right)^2 \pi^2 V a^2 \,{\rm Re}\, K_{\rm II}(q, 0). \tag{5.5}$$

Using the properties of $K_{II}(q, 0)$ which are given in Appendix B we find the relation

$$K_{\rm II}(q,0) = K_{\rm I}(q,0) - \Delta\left(\frac{\partial n}{\partial\mu}\right)_{V,T} \mathcal{G}\left(\frac{\eta}{Aq^{3/2}}\right); \tag{5.6}$$

$$\Delta\left(\frac{\partial n}{\partial\mu}\right)_{V,T} = \lim_{\eta\to 0}\left(\frac{\partial n(\eta)}{\partial\mu} - \frac{\partial n(-\eta)}{\partial\mu}\right), \tag{5.6'}$$

where $\mathcal{G}(x)$ is a function with well-known limiting values:

$$\mathcal{G}(0) = 0, \qquad \mathcal{G}(\infty) = 1. \tag{5.7}$$

We change now again to the cross section $\sigma(\theta)$. It then follows from (5.5) to (5.7) that, for instance, for fixed pressure the difference $\sigma_I(\theta) - \sigma_{II}(\theta)$ depends only on the ratio $(T - T_\lambda)/\theta^{3/2}$. Judging from the available experimental data on the compressibility, the jump $\Delta(\partial n/\partial\mu)_{V,T}$ is not large: 10 to 12% according to Lounasmaa's data[6] and 1 to 2% according to the data of Chase et al[7].[3)] However, the quantity $\sigma_I(\theta) - \sigma_{II}(\theta)$ changes appreciably faster when θ and $T - T_\lambda$ are changed than the logarithmic part of the cross sections $\sigma_I(\theta)$ and $\sigma_{II}(\theta)$ which makes the experimental verification of the theory easier.

3)We remind the reader that the data of[6] and[7] refer to different points on the λ-curve.

CONCLUSIONS

A study of neutron scattering near the λ-curve can give direct information about the fluctuation spectrum. Experimental data on the compressibility show that the scattering of neutrons by fluctuations will be rather weak. It is therefore necessary to measure in a very narrow range of angles (from tens of seconds to one degree) with very accurate thermostatic apparatus while the cross section $\sigma(\theta)$ itself must be measured with a very high accuracy (better than 1%). The experiments are greatly facilitated by the fact that it is not necessary to measure the twofold (in angles and energy) differential cross section $\sigma(\theta, \epsilon)$, but that it is sufficient to measure only the differential cross section $\sigma(\theta)$.

Most promising is a measurement of the cross section $\sigma(\theta)$ in He II, where a fast changing part due to the interaction of the neutrons occurs besides the slowly changing logarithmic part. The fast part of the cross section $\sigma(\theta)$ depends only on the ratio $(T - T_\lambda)/\theta^{3/2}$. The logarithmic part of the cross section varies like

$$\ln \frac{\max(Ap^{3/2}\theta^{3/2}, T - T_\lambda)}{T_\lambda}.$$

By comparing the slopes of the linear parts of the curves in the $\sigma(\theta)$, $\ln(T - T_\lambda)$ and the $\sigma(\theta)$, $\ln\theta$ planes we can judge whether indeed the fluctuation spectrum has the form $Aq^{3/2}$.

It was shown in [9] that when one approaches along the λ-curve from the He I–He II–gas triple point (λ-point) to the He I–He II–solid point the relative magnitude of the singular part of the compressibility k_T increases. The relative magnitude of the anomalous scattering increases, therefore, too. It would therefore be desirable to perform measurements in that part of the λ-curve which is close to the He I–He II–solid point. We obtained in the present paper a general relation [Eqs. (1.13) and (2.5)] which connects the cross section with the compressibility. It would be useful to check this relation experimentally.

We thank V. M. Galitskiĭ for discussions.

APPENDIX A

We consider $\text{Re}\, K(q, \epsilon)$ for large ϵ. We showed in Sec. 1 that $\text{Im}\, K(q, \epsilon)$ decreases exponentially when $\epsilon \gg q\sqrt{T/m}$. From Eqs. (1.3) and (1.6) it follows then that $\text{Re}\, K(q, \epsilon)$ for $\epsilon \gg q\sqrt{T/m}$ has the form

$$\text{Re}\, K(q, \varepsilon) = -B/\varepsilon^2, \qquad (A.1)$$

$$B = \frac{1}{\pi}\int_{-\infty}^{\infty} \omega \,\text{Im}\, K(q, \omega)\, \text{th}\frac{\omega}{2T}\, d\omega. \qquad (A.2)$$

On the other hand, the asymptotic behavior of $\text{Re}\, K(q, \epsilon)$ can be obtained directly from the definition (1.2). To do that we turn our attention to the fact that the function $K(\mathbf{r}, t)$ itself is continuous at $t = 0$ but its derivative $dK(\mathbf{r}, t)/dt$ has a discontinuity at $t = 0$. The same is also true of its Fourier transform

$$K(q, t) = \int K(\mathbf{r}, t)\, e^{i\mathbf{qr}}\, dV.$$

For large ϵ we have

$$K(q, \varepsilon) = \int_{-\infty}^{\infty} K(q, t)\, e^{-\varepsilon t}\, dt \approx -\frac{1}{\varepsilon^2}\left(\frac{dK}{dt}\bigg|_{t=+0} - \frac{dK}{dt}\bigg|_{t=-0}\right). \qquad (A.3)$$

According to the definition (1.2)

$$\frac{dK(\mathbf{r}, t)}{dt} = \begin{cases} \langle dn(\mathbf{r}, t)/dt,\, n(0, 0)\rangle, & t > 0 \\ \langle n(0, 0),\, dn(\mathbf{r}, t)/dt\rangle, & t < 0 \end{cases}.$$

Applying the well-known formula

$$dn(\mathbf{r}, t)/dt = i[H, n(\mathbf{r}, t)],$$

we get

$$K(q, \varepsilon) \approx -\frac{1}{\varepsilon^2}\int e^{i\mathbf{qr}} \langle [[H, n(\mathbf{r}, 0)], n(0, 0)]\rangle\, dV. \qquad (A.4)$$

Comparing (A.1) and (A.4) we find

$$\frac{1}{\pi}\int \omega\, \text{Im}\, K(q, \omega)\, th\frac{\omega}{2T}\, d\omega = \int e^{i\mathbf{qr}} \langle [[H, n(\mathbf{r}, 0)]\, n(0, 0)]\rangle\, dV. \qquad (A.5)$$

In the case of a potential interaction or in the more general case when the interaction Hamiltonian is a functional of the density $n(\mathbf{r})$ we can replace in the right-hand side of (A.5) the total Hamiltonian by the kinetic energy H_{kin}. After that all operations indicated in the right-hand side of (A.5) can be performed explicitly, using the well-known commutation relations

$$[\psi(\mathbf{r})\psi^+(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}').$$

The result of the calculations has the form

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \omega\, \text{Im}\, K(q, \omega)\, \text{th}\frac{\omega}{2T}\, d\omega = nq^2, \qquad (A.6)$$

$$\text{Re}\, K(q, \varepsilon) = -nq^2/\varepsilon^2, \qquad \varepsilon \gg q\sqrt{T/m}.$$

We note that in the case considered Eq. (A.6) is valid for arbitrary q.

APPENDIX B

The functions $G(q, \epsilon)$, $F(q, \epsilon)$, and $F^+(q, \epsilon)$ are real for $\epsilon = 0$ and can be expressed in terms

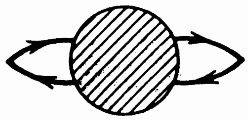

FIG. 2.

of the self-energy parts $\Sigma_1(q, 0)$ and $\Sigma_2(q, 0)$ in the well-known way (see [3]):

$$G(q, 0)=\frac{\mu-\Sigma_1(q, 0)}{(\mu-\Sigma_1(q, 0))^2-\Sigma_2^2(q, 0)}, \tag{B.1}$$

$$F(q, 0)=F^+(q, 0)=\frac{\Sigma_2(q, 0)}{(\mu-\Sigma_1(q, 0))^2-\Sigma_2^2(q, 0)}. \tag{B.2}$$

Because of the relation $\mu = \Sigma_1(0, 0) - \Sigma_2(0, 0)$ (see [3]) the functions $G(q, 0)$ and $F(q, 0)$ become infinite at $q = 0$, but their sum

$$G(q, 0)+F(q, 0)=\frac{1}{\mu-\Sigma_1(q, 0)-\Sigma_2(q, 0)} \tag{B.3}$$

is finite. Using the properties of $\Sigma_1(q, 0)$ and $\Sigma_2(q, 0)$ found in [1] we can write $G(q, 0) + F(q, 0)$ in the form

$$G(q, 0) + F(q, 0) = \eta^{-1} g(\eta / A q^{3/2}), \tag{B.4}$$

where $g(x)$ is a universal function with the following limits

$$g(x)=\begin{cases}1/2, & x\to\infty\\ 0, & x\to 0\end{cases}. \tag{B.5}$$

Using now for n_0 Eq. (6.10) from [1]:

$$n_0 = \eta X_0 / V_0, \tag{B.6}$$

where V_0 is a constant characterizing the interaction between He atoms and X_0 a universal numerical constant. The second term on the right-hand side of Eq. (4.3) can thus be written in the form

$$n_0(2G(q, 0)+F(q, 0)+F^+(q, 0))=\frac{2X_0}{V_0} g\left(\frac{\eta}{Aq^{3/2}}\right). \tag{B.7}$$

We now consider the quantity $\tilde{K}(q, 0)$. It can be represented as a sum of diagrams, the general form of which is depicted in Fig. 2. The circle indicates here a multipole with four uncondensed and an arbitrary number of incoming condensate lines.[4)] This multipole can be split into parts not containing condensate lines, those containing two, four, etc. condensate lines. The totality of diagrams not containing a single condensate line corresponds, as function of its arguments to the function $K_I(q, \epsilon)$. We now consider an arbitrary term of the series for $\tilde{K}(q, \epsilon)$ containing 2m condensate lines (the

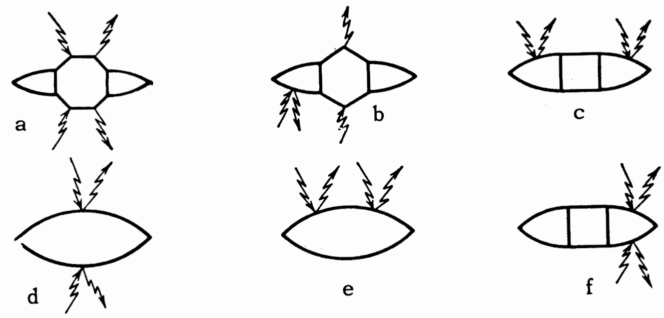

FIG. 3.

diagrams corresponding to the term with $m = 2$ are given in Fig. 3). As an example we give the analytical expression for the term of the type 3a:

$$n_0^m T^2 \int d^3p\, d^3p'\, G_I(\mathbf{p}) G_I(\mathbf{p}') G_I(\mathbf{p}-\mathbf{q}) G_I(\mathbf{p}'-\mathbf{q}) \times \Gamma^I_{m+2}(\mathbf{p}, \mathbf{p}', \mathbf{p}-\mathbf{q}, \mathbf{p}'-\mathbf{q}). \tag{B.8}$$

$\Gamma^I_N(p_1, \ldots, p_N)$ indicates here a 2N-pole without any condensate lines, i.e., being the same as a function of its arguments to a simular multipole for He I. In Eq. (B.8) we have omitted the sum over the frequencies $\omega_n \neq 0$ as it leads to small corrections.

According to [1], $\Gamma^I_N(p_1, \ldots p_N)$ is a homogeneous function of degree $\frac{3}{2}(N-2)$ of its arguments p_i and the quantities $(\eta/A)^{2/3}$. From this it follows that the integral (B.8) is a homogeneous function of degree $3m/2$ of the arguments q, $(\eta/A)^{2/3}$. Taking into account all dimensional factors in Γ^I_{m+2} and G_I and using Eq. (B.6) for n_0 we can write the integral (B.8) in the form

$$V_0^{-1} \mathcal{G}_m(\eta / A q^{3/2}), \tag{B.9}$$

where $\mathcal{G}_m(x)$ is a universal function which behaves as $\sim x^m$ when $x \to 0$ and has a finite limit as $x \to \infty$. One sees easily that any other diagram of order m in the condensate lines has similar properties. $\tilde{K}$ has thus the form

$$\tilde{K}=K_I+\frac{1}{V_0}\sum_{m=1}\mathcal{G}_m\left(\frac{\eta}{Aq^{3/2}}\right). \tag{B.10}$$

The interference term $2\sqrt{n_0}\,(P(q, \epsilon) + P(q, -\epsilon))$ is a sum of diagrams. The general form of these diagrams is given in Fig. 4a. Consider, for instance, the diagram of Fig. 4b which apart from a factor $\sqrt{n_0}$ is given by the expression

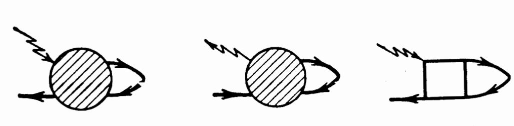

FIG. 4.

4)We note that we are considering here diagrams constructed from bare (and not complete) Green functions and bare vertices.

$$n_0 G_{\mathrm{I}}(q,0) T \sum_{\omega_n} \int d^3p \, \square(0,\mathbf{q},\mathbf{p},\mathbf{p}-\mathbf{q}) \times G_{\mathrm{I}}(\mathbf{p},\omega_n) G_{\mathrm{I}}(\mathbf{p}-\mathbf{q},\omega_n). \quad \text{(B.11)}$$

It was shown in [1] [Eqs. (4.14) to (4.16)] that the sum over the non-vanishing frequencies only leads to a renormalization. We can thus write Eq. (B.11) in the form

$$n_0 (G_{\mathrm{I}}(q,0)(c+(1-c)Q(\eta/Aq^{3/2}));$$

$$Q\left(\frac{\eta}{Aq^{3/2}}\right) = T\int \square(0,\mathbf{q},\mathbf{p},\mathbf{p}-\mathbf{q}) G_{\mathrm{I}}(\mathbf{p}-\mathbf{q},0)\, d^3p,$$

$$G_{\mathrm{I}}(q,0) = \frac{1}{\eta + Aq^{3/2}}, \quad \text{(B.12)}$$

where $Q(x)$ is a universal function of its arguments, c a renormalization constant. It is well-known that $Q(x) \to \text{const.}$ as $x \to 0$. The contribution from the diagram of Fig. 4b has thus, apart from a factor $\sqrt{n_0}$, the form

$$V_0^{-1}\varphi_1(\eta/Aq^{3/2}), \quad \text{(B.13)}$$

where $\varphi_1(x)$ is a function which tends to zero as x when $x \to 0$ and tends to a finite limit as $x \to \infty$.

One sees easily that one can also describe the remaining diagrams for the term $\sqrt{n_0}\, P(q,\epsilon)$ in an analogous fashion as $V_0^{-1}\varphi_n(\eta/Aq^{3/2})$ where all $\varphi_n(x)$ tend to zero as x^n when $x \to 0$ and to a finite limit as $x \to \infty$. The interference term can thus be described by the formula

$$\frac{1}{V_0}\sum_{n=1}^{\infty}\varphi_n(x) = \frac{1}{V}\varphi(x), \quad x = \frac{\eta}{Aq^{3/2}}, \quad \text{(B.14)}$$

where $\varphi(x) \to cx$ as $x \to 0$ and $\varphi(x) \to \text{const.}$ as $x \to \infty$.

Equation (4.3) can now be written in the form

$$K_{\mathrm{II}} = K_{\mathrm{I}} + V_0^{-1}\mathcal{G}(\eta/Aq^{3/2}), \quad \text{(B.15)}$$

$$\mathcal{G}(x) = \sum_{m=1}^{\infty}\mathcal{G}_m(x) + 2X_0 g(x) + 2\varphi(x). \quad \text{(B.16)}$$

The function $\mathcal{G}(x)$ is proportional to x for small x and tends to a constant as $x \to \infty$. One can normalize $\mathcal{G}(x)$ such that $\mathcal{G}(\infty) = 1$. The quantity $1/V_0$ attains then, according to Eq. (4.6), the meaning of the jump $\Delta(\partial n/\partial\mu)_{V,T}$.

[1] A. Z. Patashinskiĭ and V. L. Pokrovskiĭ, JETP 46, 994 (1964), Soviet Phys. JETP 19, 677 (1964).

[2] L. Van Hove, Phys. Rev. 95, 249 (1954).

[3] Abrikosov, Gor'kov, and Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics) Fizmatgiz, 1961; English translation published by Prentice Hall, 1963.

[4] L. D. Landau and E. M. Lifshitz, Statisticheskaya fizika (Statistical Physics) Gostekhizdat, 1951; English translation published by Pergamon Press, 1958.

[5] Fairbank, Buckingham, and Kellers, Proc. Fifth Internat. Conf. Low Temp. Phys., Madison, Wisc., 1957, p. 50; Bull. Amer. Phys. Soc. 2, 183 (1957).

[6] O. V. Lounasmaa, Phys. Rev. 130, 847 (1963).

[7] Chase, Maxwell, and Millett, Physica 27, 1129 (1961).

[8] S. T. Belyaev, JETP 34, 417 (1958), Soviet Phys. JETP 7, 289 (1958).

[9] Batyev, Patashinskiĭ, and Pokrovskiĭ, JETP 47, 598 (1964), Soviet Phys. JETP 20, 398 (1965).

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