PHOTON EMISSION IN MUON-PAIR PRODUCTION BY ELECTRON-POSITRON COLLISION

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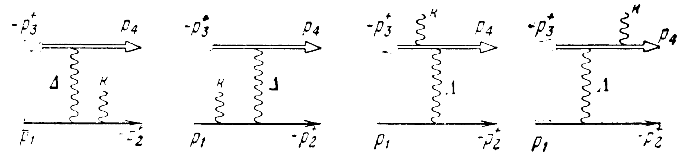
A simple method based essentially on gauge, charge, and Lorentz invariance is used to calculate the exact total cross section for emission of a photon in the production of a muon pair by electron-positron collision.

1. INTRODUCTION

ONE of the main types of process which will be observed in the near future in experiments with colliding electron beams and in colliding beams of electrons and positrons consists of processes of scattering and pair production accompanied by emission of hard photons. Whereas there is a simple and effective theoretical apparatus for describing the production of soft photons ($\omega/\epsilon \ll 1$), which makes use of the classical character of the emission of such photons, in the case of the emission of hard photons the calculations are exceptionally cumbersome, the resulting differential cross sections are extremely untransparent, and it is practically impossible to calculate the total cross sections by the standard method of integrating the differential cross sections. Since there have now already been measurements of the cross section for single photon emission in electron-positron collisions in colliding beams,[1] the problem of calculating the exact total cross sections has become important not only from the theoretical, but also from the practical point of view.

In the present paper we consider the process of emission of a photon in the production of a muon pair by electron-positron collision, $e^+ + e^- \to \mu^+ + \mu^- + \gamma$. A method is proposed which enables us to calculate in a simple way the total cross section of this process, integrated over the final muon states. The idea of the method is to integrate the separate parts of the diagrams by using the properties of relativistic, gauge, and charge invariance. It is then not actually necessary to make the extremely cumbersome calculation of the differential cross section, since the traces of the electron and muon parts of the diagrams are integrated directly. This is a rather universal method, and can be used for the calculation of the cross sections for various processes of this same type.



In the second section we give the derivation of the exact formula for the total cross section for emission of a photon in the production of a muon pair; in the third section we carry out an analysis of this formula.

2. CALCULATION OF THE TOTAL CROSS SECTION

The process of emission of a photon in the production of a muon pair is represented by four diagrams (see figure). Here we have used the notations: the momenta of the initial particles are $p_1(E_1, \mathbf{p}_1)$ and $p_2^+(E_2, \mathbf{p}_2)$; those of the final particles are $p_3^+(E_3, \mathbf{p}_3)$, $p_4(E_4, \mathbf{p}_4)$, and $k(\omega, \mathbf{k})$. We shall use a metric in which $(ab) = a_0b_0 - \mathbf{ab}$. The matrix element for the process is of the form

$$M = B\{\Delta^{-2}(\bar{v}(p_2^+)L_1^\nu u(p_1))(\bar{u}(p_4)\gamma_\nu v(p_3^+)) + \Lambda^{-2}(\bar{u}(p_4)L_2^\nu v(p_3^+))(\bar{v}(p_2^+)\gamma_\nu u(p_1))\}, \tag{2.1}$$

where

$$B = \frac{ie^3}{(2\pi)^{7/2}}\frac{m\mu}{(2\omega E_1E_2E_3E_4)^{1/2}}, \tag{2.2}$$

$$L_1^\nu = \gamma^\nu\frac{\hat{p}_1 - \hat{k} + m}{-2\varkappa}\hat{e} + \hat{e}\frac{-\hat{p}_2^+ + \hat{k} + m}{2\varkappa'}\gamma^\nu,$$

$$L_2^\nu = \gamma^\nu\frac{-\hat{p}_3^+ - \hat{k} + \mu}{-2\eta}\hat{e} + \hat{e}\frac{\hat{p}_4 + \hat{k} + \mu}{2\eta'}\gamma^\nu, \tag{2.3}$$

$$\Delta = p_3^+ + p_4, \quad \Lambda = p_1 + p_2^+, \quad \varkappa = (p_1k),$$

$$\varkappa' = -(p_2^+k), \quad \eta = -(p_3^+k), \quad \eta' = (p_4k). \tag{2.4}$$

Averaging over the spins of the initial electrons and summing over the spins of the final muons and the polarizations of the photon, we get

$$\bar{S}_i S_f |M|^2 = -\frac{|B|^2}{4}\left[\frac{\mathscr{L}_1}{\Lambda^4}+\frac{\mathscr{L}_2}{\Lambda^4}+\frac{2\mathscr{L}_3}{\Lambda^2\Delta^2}\right] \quad (2.5)$$

where

$$\mathscr{L}_1 = \frac{1}{m^2\mu^2}[M_e^{\nu\nu'}(J_\mu)_{\nu\nu'}],$$

$$\mathscr{L}_2 = \frac{1}{m^2\mu^2}[M_\mu^{\nu\nu'}(J_e)_{\nu\nu'}], \quad \mathscr{L}_3 = \frac{1}{m^2\mu^2}[K_1^{\nu\nu'}K_{2\,\nu\nu'}]. \quad (2.6)$$

Here M_e and M_μ are Compton tensors proportional to the cross section for Compton scattering of a polarized heavy "photon" with mass Δ^2:

$$\frac{M_e^{\nu\nu'}}{m^2} = \mathrm{Sp}\,[L_1^\nu\Lambda_+(p_1)\bar{L}_1^{\nu'}\Lambda_-(p_2^+)], \quad \frac{M_\mu^{\nu\nu'}}{\mu^2} = R_1\left\{\frac{M_e^{\nu\nu'}}{m^2}\right\}; \quad (2.7)$$

J_e and J_μ are current tensors:

$$\frac{J_e^{\nu\nu'}}{m^2} = \mathrm{Sp}\,[\gamma^\nu\Lambda_+(p_1)\gamma^{\nu'}\Lambda_-(p_2^+)], \quad \frac{J_\mu^{\nu\nu'}}{\mu^2} = R_1\left\{\frac{J_e^{\nu\nu'}}{m^2}\right\}; \quad (2.8)$$

and K_1 and K_2 are interference tensors:

$$\frac{K_1^{\nu\nu'}}{m^2} = \mathrm{Sp}\,[L_1^\nu\Lambda_+(p_1)\gamma^{\nu'}\Lambda_-(p_2^+)], \quad \frac{K_2^{\nu\nu'}}{\mu^2} = R_1\left\{\frac{K_1^{\nu\nu'}}{m^2}\right\}. \quad (2.9)$$

The operation R_1 consists of the interchanges

$$p_1 \leftrightarrow -p_3^+, \quad p_2^+ \leftrightarrow -p_4, \quad m \leftrightarrow \mu. \quad (2.10)$$

We represent the tensor $M_e^{\nu\nu'}$ in the following form:

$$M_e^{\nu\nu'} = \frac{1}{4}[b_1^{\nu\nu'} + b_2^{\nu\nu'} + b_3^{\nu\nu'}], \quad (2.11)$$

$$b_2^{\nu\nu'} = \frac{4m^2}{\varkappa^2}\left\{g^{\nu\nu'}\left[m^2+\varkappa'-\varkappa+(p_2^+p_1)+\frac{\varkappa\varkappa'}{m^2}\right]-(p_1^\nu p_2^{+\nu'} + p_1^{\nu'}p_2^{+\nu})+\left(\frac{\varkappa}{m^2}+1\right)(k^\nu p_2^{+\nu'}+k^{\nu'}p_2^{+\nu})\right\}, \quad (2.12)$$

$$b_1^{\nu\nu'} = b_2^{\nu\nu'}(p_1 \leftrightarrow -p_2^+,\ k \leftrightarrow -k), \quad (2.13)$$

$$b_3^{\nu\nu'} = -\frac{4}{\varkappa\varkappa'}\{2g^{\nu\nu'}(p_1p_2^+)[\varkappa-\varkappa'-(p_1p_2^+)-m^2] - 2p_1^\nu p_1^{\nu'}\varkappa' + 2p_2^{+\nu}p_2^{+\nu'}\varkappa - 2k^\nu k^{\nu'}m^2 + (p_1^\nu p_2^{+\nu'} + p_1^{\nu'}p_2^{+\nu})[2(p_1p_2^+)+\varkappa'-\varkappa] - (p_1^\nu k^{\nu'}+p_1^{\nu'}k^\nu)(p_1p_2^+) - (p_2^{+\nu}k^{\nu'}+p_2^{+\nu'}k^\nu)(p_1p_2^+)\}. \quad (2.14)$$

We write the cross section for the process in question in the form

$$d\sigma = d\sigma_e + d\sigma_\mu + d\sigma_{e\mu}, \quad (2.15)$$

where $d\sigma_e$ is the contribution in which the photon is emitted by the initial particles, $d\sigma_\mu$ is that in which it is emitted by the final muons, and $d\sigma_{e\mu}$ is the interference term. Let us consider the cross section

$$d\sigma_e = -\frac{\alpha^3}{(2\pi)^2|F|}\int\frac{d^3k}{\omega\Delta^4}M_e^{\nu\nu'}N_{\nu\nu'}, \quad (2.16)$$

where

$$N_{\nu\nu'} = \int\frac{d^3p_3}{F}\frac{d^3p_4}{E_4}(J_\mu)_{\nu\nu'}\delta(\Delta-p_3^+-p_4). \quad (2.17)$$

The Lorentz-invariant tensor $N_{\nu\nu'}$ can depend only on the four-vector Δ_ν, and consequently the most general form for this tensor is

$$N_{\nu\nu'} = c(\Delta^2,\mu^2)g_{\nu\nu'} + c_1(\Delta^2,\mu^2)\Delta_\nu\Delta_{\nu'}. \quad (2.18)$$

Because of the law of conservation of current

$$\Delta^\nu(J_\mu)_{\nu\nu'} = \Delta^{\nu'}(J_\mu)_{\nu\nu'} = 0 \quad (2.19)$$

the tensor $N_{\nu\nu'}$ is transverse:

$$N_{\nu\nu'} = c(\Delta^2,\mu^2)\left[g_{\nu\nu'}-\frac{\Delta_\nu\Delta_{\nu'}}{\Delta^2}\right]. \quad (2.20)$$

To calculate the coefficient $c(\Delta^2, \mu^2)$ we need only contract the tensor $N_{\nu\nu'}$ in Eq. (2.17) with $g^{\nu\nu'}$ and calculate the resulting integral. This integral is invariant and is most simply calculated in the c.m.s. of the two muons. We then get

$$c(\Delta^2,\mu^2) = \frac{2\pi}{3}[2\mu^2+\Delta^2]\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}; \quad (2.21)$$

the factor $(\Delta^2 - 4\mu^2)^{1/2}$ characterizes the behavior of the cross section at the threshold.

Owing to the gauge invariance of the Compton tensor,

$$\Delta_\nu M_e^{\nu\nu'} = \Delta_{\nu'}M_e^{\nu\nu'} = 0, \quad (2.22)$$

only the contraction $M_e^{\nu\nu'}g_{\nu\nu'}$ contributes to the cross section $d\sigma_e$. Calculating this contraction and carrying out a trivial integration over the azimuthal angle at which the photon emerges, we get the differential cross section with respect to the angle between the initial electron and the photon in the c.m.s. of the initial particles:

$$\frac{d^2\sigma_e}{d\cos\vartheta_k\,d\omega} = \frac{\alpha^3\omega}{6E^2\beta}\frac{(2\mu^2+\Delta^2)}{\Delta^4}\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2} \times\left\{m^2(\Delta^2+2m^2)\left(\frac{1}{\varkappa^2}+\frac{1}{\varkappa'^2}\right)+2\left[\frac{m^2}{\varkappa'}-\frac{m^2}{\varkappa}\right] + 2\left[\frac{\varkappa}{\varkappa'}+\frac{\varkappa'}{\varkappa}\right]+\frac{4}{\varkappa\varkappa'}[E^2\Delta^2+m^2(E\omega-m^2)]\right\}, \quad (2.23)$$

where in this system

$$\Delta^2 = 4E(E-\omega), \quad E_1 = E_2 = E. \quad (2.24)$$

Performing the elementary integration over the angle of emission of the photon, we get

$$d\sigma_e = \frac{2\alpha^3}{3E^2\beta}\frac{d\omega}{\omega}\frac{2\mu^2+\Delta^2}{\Delta^4}\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}\left\{(L-1)\times(\Delta^2+2\omega^2)+m^2\left[L\left(\frac{2\omega}{E}-\frac{m^2}{E^2}\right)-2\right]\right\}; \quad (2.25)$$

here

$$L=\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta},\quad \beta=\frac{(E^2-m^2)^{1/2}}{E}.$$

We proceed to the calculation of $d\sigma_\mu$. In analogy with (2.16) we can write

$$d\sigma_\mu=-\frac{\alpha^3}{(2\pi)^2|F|}\int\frac{d^3k}{\omega\Lambda^4}J_e^{\nu\nu'}Q_{\nu\nu'},\tag{2.26}$$

$$Q_{\nu\nu'}=\int\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\delta(\Lambda-p_4-p_3^+-k)(M_\mu)_{\nu\nu'}.\tag{2.27}$$

In the further work we use arguments analogous to those applied for $N_{\nu\nu'}$ in (2.17). Using the gauge invariance of $(M_\mu)_{\nu\nu'}$

$$\Lambda^\nu(M_\mu)_{\nu\nu'}=\Lambda^{\nu'}(M_\mu)_{\nu\nu'}=0\tag{2.28}$$

and the fact that $Q_{\nu\nu'}$ can depend only on the four-vectors Λ_ν, k_ν, we get the following expression[1)] for $Q_{\nu\nu'}$:

$$Q_{\nu\nu'}=a_1g_{\nu\nu'}+\frac{\Lambda^2(a_1+\Lambda^2a_2)}{(k\Lambda)^2}k_\nu k_{\nu'}$$
$$+a_2\Lambda_\nu\Lambda_{\nu'}-\frac{(a_1+\Lambda^2a_2)}{(k\Lambda)}(k_\nu\Lambda_{\nu'}+k_{\nu'}\Lambda_\nu);\tag{2.29}$$

the functions a_1 and a_2 depend on μ^2, Λ^2, and $(k\Lambda)$. It is clear that only the first two terms contribute to the cross section, since the current conservation law holds,

$$\Lambda^\nu(J_e)_{\nu\nu'}=\Lambda^{\nu'}(J_e)_{\nu\nu'}=0.\tag{2.30}$$

To calculate the functions a_1, a_2 we contract the tensor $Q_{\nu\nu'}$ with the tensors $g^{\nu\nu'}$ and $k^\nu k^{\nu'}$; we then get

$$Q_{\nu\nu'}g^{\nu\nu'}=2a_1-\Lambda^2a_2,\qquad Q_{\nu\nu'}k^\nu k^{\nu'}=a_2(k\Lambda)^2.\tag{2.31}$$

When, on the other hand, we calculate these contractions in the integral (2.27) and take the resulting invariant integrals, which is most simply done in the c.m.s. of the final muons, we can easily find the functions a_1 and a_2. We present these functions in the c.m.s. of the initial particles:

$$a_1=2\pi\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}\frac{1}{\omega^2}\left\{\Delta^2\left(2+\frac{\mu^2}{2E^2}\right)+2\omega^2\right.$$
$$\left.-L_1\left[\Delta^2\left(1+\frac{\mu^2}{2E^2}\right)+2\omega^2-\frac{\mu^4}{E^2}\right]\right\},\tag{2.32}$$

$$a_2\Lambda^2=4\pi\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}\frac{1}{\omega^2}[\Delta^2-2\mu^2L_1],\tag{2.33}$$

where

$$L_1=\frac{1}{\beta_0}\ln\frac{1+\beta_0}{1-\beta_0},\qquad \beta_0=\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}.\tag{2.34}$$

Substituting the quantities so obtained in Eq. (2.26), we easily get the differential cross section with respect to the angle between the initial electron and the photon, as contributed by the radiation from the muons:

$$\frac{d^2\sigma_\mu}{d(\cos\vartheta_k)\,d\omega}=\frac{\alpha^3\omega}{(2\pi)8E^4\beta}\left\{\left(1+\frac{m^2}{2E^2}\right)a_1\right.$$
$$\left.+\frac{\varkappa\varkappa'}{2E^2\omega^2}(a_1+\Lambda^2a_2)\right\}.\tag{2.35}$$

Carrying out the integration over the angle of emission of the photon, we get

$$d\sigma_\mu=\frac{\alpha^3}{6E^4\beta}\frac{d\omega}{\omega}\left(1+\frac{m^2}{2E^2}\right)\left(\frac{\Delta^2-4\mu^2}{\Delta^2}\right)^{1/2}\left\{(L_1-1)\right.$$
$$\left.\times(\Delta^2+2\omega^2)-\mu^2\left[L_1\left(\frac{2\omega}{E}+\frac{\mu^2}{E^2}\right)+2\left(1-\frac{\omega}{E}\right)\right]\right\}.\tag{2.36}$$

We have still to calculate the contribution of the interference terms. First we note that it can easily be shown from the explicit form of $K_{2\nu\nu'}$, Eq. (2.9), that the interchange $p_3^+\rightleftarrows p_4$ changes the sign of $K_{2\nu\nu'}$:

$$K_{2\nu\nu'}(p_4,p_3^+,\mu)=-K_{2\nu\nu'}(p_3^+,p_4,-\mu)$$
$$=-K_{2\nu\nu'}(p_3^+,p_4,\mu).\tag{2.37}$$

In the calculation we encounter an integral of the type

$$O_{\nu\nu'}=\int\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}\delta(\Lambda-p_3^+-p_4-k)K_{2\nu\nu'}.\tag{2.38}$$

It is easy to see that the interchange $p_3\rightleftarrows p_4$ changes the sign of the integrand, and so

$$O_{\nu\nu'}=0.\tag{2.39}$$

Thus when integrated over the final muon states the interference cross section $d\sigma_{e\mu}$ is zero. Therefore the total differential cross section is

$$\frac{d^2\sigma}{d(\cos\vartheta_k)\,d\omega}=\frac{d^2\sigma_e}{d(\cos\vartheta_k)\,d\omega}+\frac{d^2\sigma_\mu}{d(\cos\vartheta_k)\,d\omega}\tag{2.40}$$

and is given by Eqs. (2.23) and (2.35), and the integrated total cross section is

$$d\sigma=d\sigma_e+d\sigma_\mu\tag{2.41}$$

and is given by Eqs. (2.25) and (2.36).

3. ANALYSIS OF THE TOTAL CROSS SECTION

The expression (2.41) which we have obtained for the cross section $d\sigma$ is exact. Let us now

*A similar approach has been used in the paper by Gorgé and others[2] to calculate the contribution from the emission of a photon by the electron in a muon-electron collision. There, however, an integration over $d\omega$ was performed, which led to difficulties owing to the infrared divergence, and the integration was not done over the momentum of the final muon. An extensive bibliography is given in[2].

examine the behavior of the cross section in various limiting cases. First we study the behavior of the cross section near the threshold for production of muons; here it is obvious that $\mu^2/E^2 \sim 1$, $\omega/E \ll 1$. In this case the cross section, to terms of first order in ω/E, is of the form

$$d\sigma_e{}^{th} = \frac{2\alpha^3}{E^2}\frac{d\omega}{\omega}\left(1+\frac{\omega}{3E}\right)\beta_0\left(\ln\frac{2E}{m}-\frac{1}{2}\right), \quad (3.1)$$

$$d\sigma_\mu{}^{th} = \frac{4}{3}\frac{\alpha^3}{E^2}\frac{d\omega}{\omega}\beta_0{}^3, \quad (3.2)$$

where near threshold $\beta_0 \ll 1$ [cf. Eq. (2.34)]. It is seen from this that near threshold the cross section for emission of a photon by the muons is smaller by a factor $\beta_0^2/\ln(E/m)$ than the cross section for emission of a photon by the electrons and is extremely small.

For $E \gg \mu$ (far from threshold) and under the condition $\omega/E \ll 1$ we have, to terms of first order in μ^2/E^2 and ω/E,

$$d\sigma_e = \frac{4}{3}\frac{\alpha^3}{E^2}\frac{d\omega}{\omega}\left(\ln\frac{2E}{m}-\frac{1}{2}\right), \quad (3.3)$$

$$d\sigma_\mu = \frac{4}{3}\frac{\alpha^3}{E^2}\frac{d\omega}{\omega}\left(\ln\frac{2E}{\mu}-\frac{1}{2}\right)\left(1-\frac{\omega}{E}\right). \quad (3.4)$$

It is seen that in this case the ratio of the cross sections reduces to the ratio of the logarithmic factors, so that for E = 250 MeV $d\sigma_\mu$ is about 15 percent of $d\sigma_e$, and for E = 1 GeV it is about 30 percent of $d\sigma_e$. Consequently the contribution of photon emission by the muons is by no means small.

An important feature of these formulas, in which the case of photon emission in the production of a pair of particles differs decidedly from that of bremsstrahlung in electron scattering, is that the total cross section falls off with increase of the energy as $\ln(E/m)/E^2$, whereas the cross section for bremsstrahlung in scattering increases as $\ln(E/m)$. This is due to the fact that the main contribution to the bremsstrahlung cross section comes from small momentum transfers (cf. e.g., [3]). Indeed, in the case of bremsstrahlung in electron scattering by a Coulomb force center the minimum momentum transfer is $\omega m^2/2E^2$. In our present case, on the other hand, the momentum transfer cannot be small (it is larger than 2μ), and the result of this is that there is no compensation of the factor $1/E^2$. It is clear that this sort of situation is characteristic of all processes of emission of photons in the production of pairs of particles when there is annihilation of an electron-positron pair.

Let us also examine the hard part of the photon spectrum. The photon energy is a maximum in the case when the photon and the two final muons come out in opposite directions and the muons have equal momenta:

$$\omega_{max} = (E^2-\mu^2)/E. \quad (3.5)$$

It can be seen directly from the expressions (2.25) and (2.36) that for $\omega \to \omega_{max}$ the cross section $d\sigma$ goes to zero as $(\omega_{max}-\omega)^{1/2}$.

In conclusion the writers express their gratitude to V. M. Galitskiĭ for a discussion.

[1] Bernardini, Corrazza, Di Jiugno, Haissinski, Marin, Querzoli, and Touschek, Preprint LNF-64/33, 1964; Nuovo cimento (in press).

[2] Gorgé, Locher, and Rollnik, Nuovo cimento **27**, 928 (1963).

[3] V. Bayer and V. Galitsky, Physics Letters **13**, 355 (1964).

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