SU_n SYMMETRY IN THE THEORY OF CLEBSCH-GORDAN COEFFICIENTS

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Their transformation properties under SU_n are used as the basis of the theory of the Clebsch-Gordan coefficients. It is shown that the Regge relations for the Clebsch-Gordan coefficients are manifestations of SU_n symmetry. Various relations are found between Clebsch-Gordan coefficients and combinations of them. Some new types of combinations are introduced. The relation of the results to field theory is discussed.

RECENTLY a great deal of attention has been given to the unitary unimodular groups SU_n in the theory of elementary particles. This is particularly true for SU_3, whose representations have been used for classifying elementary particles for strong interactions.[1] There have also been some attempts to use SU_4 for classification.[2] This situation does not appear to be accidental; it may well be that the unitary unimodular transformations must be the basis of a future theory. Until recently, however, the use of the SU_n groups in physics has been extremely limited. We may mention the use of SU_n in the analysis of fractional parentage coefficients[3] and the use of SU_3 in the treatment of the three-dimensional oscillator.[4]

The present paper demonstrates that unitary unimodular symmetry plays an important role in the theory of the ordinary Clebsch-Gordan coefficients and their combinations. This is the case first for the new symmetry relations of the Clebsch-Gordan coefficients[5] that were discovered by Regge in 1958. Regge showed that the 3jm Wigner symbol can be written in the form

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left\| \begin{matrix} -j_1+j_2+j_3 & j_1-j_2+j_3 & j_1+j_2-j_3 \\ j_1-m_1 & j_2-m_2 & j_3-m_3 \\ j_1+m_1 & j_2+m_2 & j_3+m_3 \end{matrix} \right\| = \left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| = \| R_{ik} \|, \qquad (1)$$

where the 3×3 quadratic symbol $\| R_{ik} \|$ is the coefficient in the expansion of the Jth power of a determinant: [1)]

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix}^J = \sqrt{(J!)^3(J+1)} \sum_{\sum_k R_{ik}=J} \left\| \begin{matrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{matrix} \right\| \times \frac{u_1^{R_{11}} u_2^{R_{12}} u_3^{R_{13}} v_1^{R_{21}} v_2^{R_{22}} v_3^{R_{23}} w_1^{R_{31}} w_2^{R_{32}} w_3^{R_{33}}}{(R_{11}!\, R_{12}!\, R_{13}!\, R_{21}!\, R_{22}!\, R_{23}!\, R_{31}!\, R_{32}!\, R_{33}!)^{1/2}}. \qquad (2)$$

In accordance with the symmetry of the determinant Δ, the symbol $\| R_{ik} \|$ is invariant under the transposition operation and under even permutations of rows and columns, and is multiplied by $(-1)^J$ for odd permutations. There are all together $3! \times 3! \times 2 = 72$ symmetry relations. These new symmetry properties of the Clebsch-Gordan coefficients were later discussed in several papers.[6-9] Regge[6] found new symmetry relations for the Racah coefficients; it was proposed[9] that one should write all relations in the theory of angular momenta in terms of the new variables R_{ik} in place of the usual jm. But there were several aspects of the properties found by Regge, and especially the question of their physical origin, which were not discussed sufficiently in these papers.

An important point is that the determinant on the left of (2) is invariant under the unitary unimodular group SU_3.[10] The products on the right side of the equation also transform according to representations of SU_3; this is obviously also the case for the projection matrix $\| R_{ik} \|$. Thus the Regge symbol transforms according to some representation of SU_3.

In the same way as for the 3×3 symbol $\| R_{ik} \|$, one can construct symbols corresponding to any group SU_n. Thus for SU_4 the expansion (2) takes the form

[1)] In accordance with (1), the normalization factor $\sqrt{(J!)^3(J+1)}$ has been introduced in the expansion (2). This differs from the factor used by Regge.[1]

$$\begin{vmatrix} u_1 & v_1 & w_1 & t_1 \\ u_2 & v_2 & w_2 & t_2 \\ u_3 & v_3 & w_3 & t_3 \\ u_4 & v_4 & w_4 & t_4 \end{vmatrix}^J = \sqrt{(J!)^3(J+1)}\sum \left\| \begin{matrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{matrix} \right\| \times \frac{\prod_i u_i^{R_{1i}} v_i^{R_{2i}} w_i^{R_{3i}} t_i^{R_{4i}}}{\left(\prod_{ik} R_{ik}!\right)^{1/2}}. \quad (3)$$

Here the 4×4 symbol $\| R_{ik} \|$ now contains $4! \cdot 4! \cdot 2 = 1152$ symmetry rules (permutations of rows and columns and transposition). For the group SU$_n$ the symbol $\| R_{ik} \|$ possesses $n! \cdot n! \cdot 2$ symmetries.

The intimate relation of SU$_2$ to the rotation group is well known.[11] Similarly we may say that Regge's work[5] established the connection between the properties of the Clebsch-Gordan coefficients and the group SU$_3$. From the definitions (2), (3), etc., one can now look at the connection between the Clebsch-Gordan coefficients and their combinations and the symbols $\| R_{ik} \|$ for the successive groups SU$_3$, SU$_4$, ... SU$_n$. Before considering this connection in general, we look at some important formulas which follow from (1) and (2).

If the determinant on the left of (2) has two identical rows $v_i = u_i$, then since the equality must hold for arbitrary u_i, w_i, we get the following relation:

$$\sum_{A_{1k}+A_{2k}=J-A_{3k}} \left\| \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix} \right\| \times (A_{11}!\, A_{12}!\, A_{13}!\, A_{21}!\, A_{22}!\, A_{23})^{-1/2} = 0. \quad (4)$$

The relation (4) is trivial for odd J, since it is satisfied automatically because of the Regge symmetry. For even J, in the standard notation of Wigner, we find

$$\sum_{m_i} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \left[\prod_i (j_i - m_i)!\,(j_i + m_i)! \right]^{-1/2} = 0. \quad (5)$$

We note that, in addition to (5), using the jm-notation we can get other relations from (4); these can all be gotten from (5) using the Regge symmetry and the technique described in an earlier paper.[9]

Writing the Jth power of the determinant as the product of determinants to the powers J_1 and J_2 (with $J_1 + J_2 = J$), expanding each of the determinants according to (2), and equating the factors of equal powers of $u_i v_i w_i$, we find

$$\|C_{ik}\| = \left[\frac{(J_1!)^3 (J_2!)^3 (J_1+1)(J_2+1)}{(J!)^3 (J+1)} \right]^{1/2} \times \sum_{A_{ik}+B_{ik}=C_{ik}} \|A_{ik}\| \, \|B_{ik}\| \frac{\left(\prod_{ik} C_{ik}!\right)^{1/2}}{\left(\prod_{ik} A_{ik}! B_{ik}!\right)^{1/2}}. \quad (6)$$

Here

$$\sum_i C_{ik} = J, \quad \sum_i A_{ik} = J_1, \quad \sum_i B_{ik} = J_2.$$

In the jm representation, one of the formulas corresponding to (6) has the form

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \left[\left(\frac{J_1! J_2!}{J!} \right)^3 \frac{(J_1+1)(J_2+1)}{J+1} \right]^{1/2} \times \sum_{\substack{j_i' + j_i'' = j_i \\ m_i' + m_i'' = m_i}} \begin{pmatrix} j_1' & j_2' & j_3' \\ m_1' & m_2' & m_3' \end{pmatrix} \begin{pmatrix} j_1'' & j_2'' & j_3'' \\ m_1'' & m_2'' & m_3'' \end{pmatrix} \left[\frac{\prod_{ik} A_{ik}!}{\prod A_{ik}'! A_{ik}''!} \right]^{1/2}. \quad (7)$$

Here $j_1' + j_2' + j_3' = J_1$, $j_1'' + j_2'' + j_3'' = J_2$, $j_1 + j_2 + j_3 = J$.

Formula (6) allows a simple generalization to an arbitrary number of factors:

$$\|C_{ik}\| = \left[\frac{(J_1!)^3 \ldots (J_n!)^3 (J_1+1) \ldots (J_n+1)}{(J!)^3 (J+1)} \right]^{1/2} \times \sum_{A_{ik}'+\ldots+A_{ik}^{(n)}=C_{ik}} \|A_{ik}'\| \ldots \|A_{ik}^{(n)}\| \left[\frac{\prod_{ik} C_{ik}!}{\prod A_{ik}'! \ldots A_{ik}^{(n)}!} \right]^{1/2}. \quad (8)$$

Finally all the symbols $\| A_{ik} \|$ can be reduced to symbols of the first degree $\varphi_{A_{ik}}/\sqrt{2}$, where $\varphi_{A_{ik}} = \pm 1$ and the sign φ depends on the distribution of ones in the symbol. Relation (8) then becomes

$$\|C_{ik}\| = \left[\frac{\prod_{ik} C_{ik}!}{(J+1)!} \right]^{1/2} \frac{1}{J!} \sum_{A_{ik}'+\ldots+A_{ik}^{(n)}=C_{ik}} \varphi_{A_{ik}'} \cdots \varphi_{A_{ik}^{(n)}} = \left[\frac{\prod_{ik} C_{ik}}{(J+1)!} \right]^{1/2} \frac{1}{J!} \Phi. \quad (9)$$

The coefficient Φ is thus determined by the number of ways of splitting $\| C_{ik} \|$ into factors, taking account of signs. Since the Regge symbol can be written as

$$\left\| \begin{matrix} q_1+p_1 & q_1+p_2 & q_1+p_3 \\ q_2+p_1 & q_2+p_2 & q_2+p_3 \\ q_3+p_1 & q_3+p_2 & q_3+p_3 \end{matrix} \right\|, \tag{10}$$

we easily find the general expression for Φ:

$$\Phi = \sum (-1)^{q_1+q_2+q_3} \frac{J!}{p_1!p_2!p_3!q_1!q_2!q_3!}. \tag{11}$$

Substituting Φ in formula (9), we get the familiar relation for the Regge symbol.[6] This method of derivation has the advantage that it can be immediately generalized to an arbitrary $n \times n$ symbol.

Considering the product of two determinants to the Jth power and using the expansion (2), we get

$$\|A_{ik}\|\,\|B_{ik}\| = \left[\frac{\prod\limits_{ik} A_{ik}!B_{ik}!}{(J!)^3(J+1)}\right]^{1/2} \sum_{\substack{l_{\sigma ik}=A_{ik} \\ l_{i\sigma k}=B_{ik}}} \|l_{ik\sigma}\| \frac{\left(\prod\limits_{ik} l_{ik\sigma}!\right)^{1/2}}{\prod\limits_{ikr} l_{ikr}!}. \tag{12}$$

For brevity we have introduced the notation $l_{\sigma ik} = \sum\limits_r l_{rik}$, and the powers of each of the Regge symbols is J.

Applying the invariant operator $u_2\partial/\partial u_1 + v_2\partial/\partial v_1 + w_2\partial/\partial w_1$ to the right and left sides of (2), using standard properties of the minors of a determinant, and collecting terms of the same degree, we get the recursion relations treated in [7,9].

We note that all the properties of the 3×3 symbol enumerated above arise from the properties of determinants and can be immediately generalized to symbols of arbitrary degree.

The transformation properties of the $n \times n$ symbols $\|R_{ik}\|$ also have a great deal in common. According to [10], the list of basic invariants of SU_n is exhausted by the contraction of covariant and contravariant quantities and by arbitrary powers J of the determinants formed from co- and contravariant vectors. For SU_3, the contraction is written as

$$(u_1U_1 + u_2U_2 + u_3U_3)^J = J! \sum_{\Sigma_i A_{ik}=J} \frac{u_1^{A_{11}} u_2^{A_{12}} u_3^{A_{13}} U_1^{A_{11}} U_2^{A_{12}} U_3^{A_{13}}}{A_{11}!\,A_{12}!\,A_{13}!} \tag{13}$$

Comparing with (3), we have

$$\frac{U_1^{A_{11}}U_2^{A_{12}}U_3^{A_{13}}}{(A_{11}!A_{12}!A_{13}!)^{1/2}} = \sqrt{(J+1)!} \sum_{\sum\limits_i A_{2i}=\sum\limits_i A_{3i}=J} \left\| \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{33} \\ A_{31} & A_{32} & A_{33} \end{matrix} \right\|$$

$$\times \frac{v_1^{A_{21}}v_2^{A_{22}}v_3^{A_{23}}w_1^{A_{31}}w_2^{A_{32}}w_3^{A_{33}}}{(A_{21}!\,A_{22}!\,A_{23}!\,A_{31}!\,A_{32}!\,A_{33}!)^{1/2}}. \tag{14}$$

According to (14), the Regge symbol $\|A_{ik}\|$ changes a definite linear combination of covariant components into contravariant ones, i.e., it plays the role of a metric tensor in the corresponding tensor space.

Expanding the determinant in (2) in terms of the first column and collecting the appropriate factors, we have

$$\frac{1}{\sqrt{A_{11}!A_{12}!A_{13}!}} \begin{vmatrix} v_2 & w_2 \\ v_3 & w_3 \end{vmatrix}^{A_{11}} \begin{vmatrix} v_1 & w_1 \\ v_3 & w_3 \end{vmatrix}^{A_{12}} \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}^{A_{13}}$$

$$= \sqrt{(J+1)!} \sum \|A_{ik}\| \frac{\prod\limits_i v_i^{A_{2i}} w_i^{A_{3i}}}{\left(\prod\limits_i A_{2i}!A_{3i}!\right)^{1/2}}. \tag{15}$$

This formula is the starting point for the derivation of numerical values of the Clebsch-Gordan coefficients.[11] From (15) we see that the vectors of a basis contravariant to a given one are expressed in terms of the corresponding minors. For SU_2, the 2×2 symbol $\|A_{ik}\|$ in the expansion

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}^J = \sqrt{(J!)^3(J+1)} \sum \left\| \begin{matrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{matrix} \right\| \frac{u_1^{A_{11}}u_2^{A_{21}}v_1^{A_{12}}v_2^{A_{22}}}{(A_{11}!A_{21}!A_{12}!A_{22}!)^{1/2}} \tag{16}$$

is also a metric tensor; in the jm notation it has the form $(-1)^{j-m}\delta_{m,-m'}$. Comparing the contraction of vectors for SU_2,

$$(v_1V_1 + w_1W_1)^J = \sum_{A_1+A_2=J} \frac{J!}{A_1!A_2!} v_1^{A_1}w_1^{A_2}V_1^{A_1}W_1^{A_2} \tag{17}$$

with formula (2), we have

$$\frac{V_1^{A_{21}}W_1^{A_{31}}}{(A_{21}!A_{31}!)^{1/2}} = \sqrt{(J!)^3(J+1)} \sum_{A_{22}A_{23}A_{32}A_{33}} \left\| \begin{matrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{matrix} \right\|$$

$$\times \frac{v_2^{A_{22}}v_3^{A_{32}}w_2^{A_{23}}w_3^{A_{33}}}{(A_{22}!A_{32}!A_{23}!A_{33}!)^{1/2}} \tag{18}$$

Thus any two components appearing in the same column or same row of the $\|R_{ik}\|$ symbol transform according to a representation of SU_2. All the invariants of the rotation group (SU_2), including arbitrary j symbols and transformation matrices transform according to representations of SU_3. Using the analogous transformation properties and metric tensors for arbitrary groups SU_n, one can construct different combinations of Clebsch-Gordan coefficients (and any $\|R_{ik}\|$ symbols). It is of particular interest to consider new types of combinations.

As an example we consider SU_4. Expanding the determinant on the left of (3) in terms of the fourth column, we have

$$\begin{vmatrix} u_1 & v_1 & w_1 & t_1 \\ u_2 & v_2 & w_2 & t_2 \\ u_3 & v_3 & w_3 & t_3 \\ u_4 & v_4 & w_4 & t_4 \end{vmatrix}^J = \sum_{\sum\limits_k A_{ik}=J} \frac{J!}{A_{11}!\,A_{12}!\,A_{13}!\,A_{14}!} t_1^{A_{11}} t_2^{A_{12}}$$

$$\times\, t_3^{A_{13}} t_4^{A_{14}} \begin{vmatrix} w_2 & w_3 & w_4 \\ v_2 & v_3 & v_4 \\ u_2 & u_3 & u_4 \end{vmatrix}^{A_{11}} \begin{vmatrix} w_1 & w_3 & w_4 \\ v_1 & v_3 & v_4 \\ u_1 & u_3 & u_4 \end{vmatrix}^{A_{12}} \begin{vmatrix} w_1 & w_2 & w_4 \\ v_1 & v_2 & v_4 \\ u_1 & u_2 & u_4 \end{vmatrix}^{A_{13}} \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix}^{A_{14}}. \tag{19}$$

Using relations (2) and (3), and equating terms with the same powers of u_i, v_i, w_i and t_i, we get for the 4×4 symbol

$$\sqrt{(J+1)!} \left\| \begin{matrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{matrix} \right\| = \sqrt{\prod_i (A_{1i}+1)!} \sum_{\sum_{\beta\neq\alpha} B_{i\alpha}^{\beta} = A_{i\alpha}} \frac{\left(\prod_{ik} A_{ik}!\right)^{1/2}}{\left(\prod_{\substack{\beta=1,2,3,4 \\ \alpha\neq\beta}} B_{2\alpha}^{\beta}!\, B_{3\alpha}^{\beta}!\, B_{4\alpha}^{\beta}!\right)^{1/2}}$$

$$\times \left\| \begin{matrix} B_{22}^1 & B_{23}^1 & B_{24}^1 \\ B_{32}^1 & B_{33}^1 & B_{34}^1 \\ B_{42}^1 & B_{43}^1 & B_{44}^1 \end{matrix} \right\| \left\| \begin{matrix} B_{21}^2 & B_{23}^2 & B_{24}^2 \\ B_{31}^2 & B_{33}^2 & B_{34}^2 \\ B_{41}^2 & B_{43}^2 & B_{44}^2 \end{matrix} \right\| \left\| \begin{matrix} B_{21}^3 & B_{22}^3 & B_{24}^3 \\ B_{31}^3 & B_{32}^3 & B_{34}^3 \\ B_{41}^3 & B_{42}^3 & B_{44}^3 \end{matrix} \right\| \left\| \begin{matrix} B_{21}^4 & B_{22}^4 & B_{23}^4 \\ B_{31}^4 & B_{32}^4 & B_{33}^4 \\ B_{41}^4 & B_{42}^4 & B_{43}^4 \end{matrix} \right\|. \tag{20}$$

In this formula, in contrast to the usual summation over combinations of two symbols in the theory of Clebsch-Gordan coefficients, we have a summation over combinations of three symbols.

A similar expansion can be written for 5×5 symbols; then four symbols appear in the sum. In general the properties of arbitrary $n \times n$ $\| R_{ik} \|$ symbols are related to the specific features of SU_n. Using the expansion in terms of a row or column of the $n \times n$ symbol, we can express it in terms of a sum of products of $(n-1) \times (n-1)$ symbols. In turn, expanding the $(n-1) \times (n-1)$ symbols, etc., we express the $n \times n$ symbol successively in terms of symbols of lower and lower order, down to the Regge symbol (3×3 symbol).

It is of interest to compare the combinations of $\| R_{ik} \|$ symbols considered above with the usual combinations of Clebsch-Gordan coefficients (i.e., with invariants with respect to SU_2), where the summation contains only two coefficients. In the general case, by using the metric tensors we combine n symbols, in accordance with the degree of SU_n. For the specific analysis of different combinations of $\| R_{ik} \|$ symbols, it is most convenient to start from products of the corresponding determinants. In a certain sense the usual combinations of Clebsch-Gordan coefficients are subsymbols of the $n \times n$ symbols. As an example we consider the Racah coefficient. The 6j Wigner symbol was written in [9] as a 3×4 table

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} = \left\| \begin{matrix} j_1+j-j_{23} & j_2+j_3-j_{23} & j_1+j_2-j_{12} & j+j_3-j_{12} \\ j_1+j_{12}-j_2 & j_3+j_2-j & j_1+j_{23}-j & j_3+j_{23}-j_2 \\ j+j_{12}-j_3 & j_2+j_{12}-j_1 & j_2+j_{23}-j_3 & j+j_{23}-j_1 \end{matrix} \right\| = \left\| \begin{matrix} x_1+y_1 & x_2+y_1 & x_3+y_1 & x_4+y_1 \\ x_1+y_2 & x_2+y_2 & x_3+y_2 & x_4+y_2 \\ x_1+y_3 & x_2+y_3 & x_3+y_3 & x_4+y_3 \end{matrix} \right\|. \tag{21}$$

This symbol is part of the more general 4×4 table corresponding to the 4×4 symbol:

$$\left\| \begin{matrix} x_1+y_1 & x_4+y_2 & x_2+y_3 & x_3+y_4 \\ x_2+y_2 & x_3+y_1 & x_1+y_4 & x_4+y_3 \\ x_3+y_3 & x_2+y_4 & x_4+y_1 & x_1+y_2 \\ x_4+y_4 & x_1+y_3 & x_3+y_2 & x_2+y_1 \end{matrix} \right\| = \left\| \begin{matrix} x_1+y_1 & x_2+y_1 & x_3+y_1 & x_4+y_1 \\ x_1+y_2 & x_2+y_2 & x_3+y_2 & x_4+y_2 \\ x_1+y_3 & x_2+y_3 & x_3+y_3 & x_4+y_3 \\ x_1+y_4 & x_2+y_4 & x_3+y_4 & x_4+y_4 \end{matrix} \right\|. \tag{22}$$

In the j-notation, the symbol (22) has the form

$$\left\| \begin{matrix} j+j_3-j_{12} & j_1+j_{23}-j & j_1+j_{12}-j_3 & j_2+j_3+j_{12}+{}^1/_2J \\ j_1+j_{12}-j_2 & j_2+j_3-j_{23} & j_3+j+j_{23}+{}^1/_2J & j_2+j_{23}-j_3 \\ j_2+j_{12}-j & j_1+j_{12}+j+{}^1/_2J & j_1+j_2-j_{12} & j_3+j_{23}-j_2 \\ j_1+j_2+j_{23}+{}^1/_2J & j+j_{23}-j_1 & j_3+j_{12}-j & j_1+j-j_{23} \end{matrix} \right\|, \tag{23}$$

where ${}^1/_2J = j_1 + j_2 + j_3 + j + j_{12} + j_{23}$. The symbol (23) can be expressed in terms of a linear combination of Racah coefficients, which should now have not $3! \cdot 4!$, but $4! \cdot 4! \cdot 2$ symmetry properties, and to which the generalization of formulas (5)—(12) applies. A similar treatment can be given for the 9j symbols, and for arbitrary transformation matrices.

Altogether there is a considerable variety of types of invariant combinations of Clebsch-Gordan coefficients and relations between them; these relations may find an application in the theory of angular momentum for various calculations. It should be mentioned that in addition to its use for investigating the $\| R_{ik} \|$ symbols and relations between them, the approach used here is also useful for analyzing the groups SU_n themselves, and also for studying the fractional parentage coefficients. In the latter case one can enumerate a complete set of quantum numbers in terms of the successive SU_n groups (cf.[3]).

It appears that further investigation of the role of the unitary unimodular groups should essentially change the group theoretical basis of the present theory. Usually one takes the rotation group R_3 (or the Lorentz group) as a basis. For this reason up to now wide use has been made only of SU_2, which is holomorphic to R_3. As pointed out above, Regge's work[5] actually consisted in treating the group SU_3. One may expect that the groups SU_2, SU_3, ... SU_n have a no less fundamental significance for the theory than R_3. Thus, a term for interaction of wave functions which are invariant under SU_2, which is expressible in terms of Clebsch-Gordan coefficients, can already be written in invariant form with respect to SU_3.

The entire treatment given applied to R_3 (to the groups SU_n); for the Lorentz group one can use its connection with the unimodular group in two dimensions. Various properties of the symbols for the SU_n groups and relations between them remain valid for the unimodular groups.

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