MOTION OF SINGULARITIES OF PARTIAL AMPLITUDES IN THE COMPLEX j PLANE

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Submitted to JETP editor June 29, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 242-252 (January, 1965)

The explicit form of the many-particle unitarity relations proposed by Gribov, Pomeranchuk, and Ter-Martirosyan is used for a detailed study of the genesis, motion, and nature of the singularities of the partial scattering amplitudes in the j plane. In particular it is shown that singularities that depend on the masses of the particles go off from the first sheet in the j plane at threshold values of the energy. Assuming that the particles are related to definite Regge trajectories, we can include the three-particle unitarity relation in the four-particle relation. Then the three-particle singularities also pass out onto the second sheet over the four-particle branching in the j plane for $t_0 < (m + 2\mu)^2$. The same situation appears when the $(n - 1)$-particle and $(n - 2)$-particle unitarity relations are included in the n-particle relation. Brief consideration is given to the singularities in the j plane which appear in cases in which the expression $\alpha(t)$ for a Regge trajectory itself has a singularity or a zero of its derivative at some point t_0.

1. Branch points of the partial amplitude in the j plane were first reliably established by Mandelstam [1] for the example of a particular class of Feynman diagrams. For diagrams that contain three particles in the intermediate state in the t channel a branch point appears at $j = \alpha[(t^{1/2} - m)^2] - 1$, where $\alpha(t)$ is the Regge trajectory. Besides this, Wilkin has shown [2] that in the same class of diagrams a branch point $j = \alpha(0) - 1$ arises, and this one alone remains for $t < m^2$.

A different, and in our opinion a more general, approach has been used by V. N. Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan (private communication), who have proposed a method of analytic continuation of the n-particle unitarity relation with respect to the total angular momentum j and have established that the 2n-particle unitarity relation generates a singularity of the form $j_n = n\alpha(t/n^2) - n + 1$ and singularities which depend on the masses of the particles. For the four-particle unitarity relation these latter singularities are given by

$$j = \alpha((\sqrt{t} - 2\mu)^2) + \alpha(4\mu^2) - 1, \qquad j = 2\alpha(4\mu^2) - 1.$$

For the vacuum trajectory $\alpha(4\mu^2) > \alpha(0) = 1$, and singularities of this type would lead to an asymptotic behavior in conflict with Froissart's theorem. [3]

The singularity $j = \alpha(0) - 1$ found by Wilkin puts a limit on the rate of decrease of the amplitude, which in principle can be tested by experiment.

In the present paper we shall in all cases use the unitarity relation in the form proposed by V. N. Gribov and others. By means of the three-particle unitarity relation we shall obtain in Sec. 2 the Mandelstam and Wilkin singularities, and shall study their motions as functions of the energy. The case in which the trajectory $\alpha(t)$ has a singularity or its derivative has a zero is considered briefly.

In Sec. 3 we study the motion of singularities that arise in the four-particle unitarity relation. It is shown that singularities that depend on the masses pass off from the physical sheet for $t < 16\mu^2$, which leads to asymptotic behavior satisfying Froissart's theorem. [3] If in the three-particle unitarity relation the particles lie on the Regge trajectories, then, as will be shown, this relation can be included in the four-particle unitarity relation, and the three-particle singularities pass off the first sheet in the j plane for values $t < (m + 2\mu)^2$, so that no Wilkin singularity not depending on t remains in the j plane.

2. The term in the three-particle unitarity relation for the partial scattering amplitude $\varphi_j(t)$ which leads to singularities in the j plane is of the form [3]

$$\Delta_3\varphi_j(t) = \frac{1}{2\pi i}\int\limits_{C_1} dt_1 \frac{p(t, t_1, m^2)}{\sqrt{t}} \frac{N_{j\alpha\alpha}N^{(3)}_{j\alpha\alpha}}{j + 1 - \alpha(t_1)}. \qquad (1)$$

Here $N_{j\alpha\alpha}$ has the meaning of the amplitude for transition of two particles into a particle and a reggeon,

$$p = t^{-1/2}\{[t - (\sqrt{t_1} + m)^2][t - (\sqrt{t_1} - m)^2]\}^{1/2}$$

is the momentum of the particle with mass m, and the path C_1 is represented by the solid line in Fig. 1.

Imaging of the contour C_1 in the plane of $\alpha(t_1)$ gives the contour C_α, which is shown in Fig. 2. From Fig. 2 and Eq. (1) we can easily draw the following conclusions: a) for sufficiently large j the quantity $\Delta_3\varphi_j(t)$ has no singularities on the first sheet of the t plane; b) Eq. (1) defines $\Delta_3\varphi_j(t)$ as an analytic function of j in the entire plane except for a cut connecting the two branch points $j+1 = \alpha[(t^{1/2} - m)^2]$ and $j + 1 = \alpha^*[(t^{1/2} - m)^2]$; c) since near the upper limit $t_1 = (t^{1/2} - m)^2$ and the momentum $p(t, t_1, m^2) \sim [(t^{1/2} - m)^2 - t_1]^{1/2}$, the branchings are simple square roots: $\Delta_3\varphi_j(t) \sim [j - j(t)]^{1/2}$.

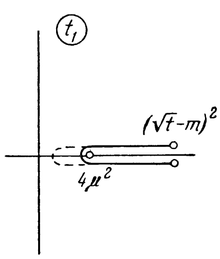

FIG. 1

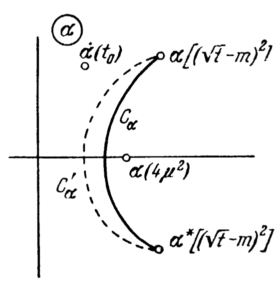

FIG. 2

The contour C_1 can be deformed to the left of the point $t_1 = 4\mu^2$, as shown by a dashed line in Fig. 1, but cannot be carried farther to the right than $t_1 = 4\mu^2$, since $\alpha(t_1)$ has a singularity at this point. Similarly, the contour C_α can be deformed to the position shown by the dashed line in Fig. 2, but cannot be drawn to the right of the point $\alpha(4\mu^2)$. This means that the function $\Delta_3\varphi_j(t)$ can be analytically continued from the side $j + 1 > \alpha(4\mu^2)$ into the region of smaller values of j, and in addition can be continued on the second sheet in the j plane according to the stated branchings; the point $j + 1 = \alpha(4\mu^2)$ is a singularity.[1)]

Let us now consider the case in which at some point t_0 we have $\alpha'(t_0) = 0$. If $\alpha(t_1)$ is the vacuum trajectory farthest to the right, then $\alpha'(t) > 0$ for $0 < t_1 < 4\mu^2$.[4] Therefore the point t_0 is either complex or negative. We shall see that the latter possibility is uninteresting; in the former case $\alpha(t_0)$ lies somewhere outside the contour C_α.

Going over to the variable α in (1), by writing $dt_1 = d\alpha/\alpha'(t_1)$, we get the singularity $j + 1 = \alpha(t_0)$ (and a corresponding singularity at the conjugate point). As can be seen from Fig. 2, however, this singularity is on the second sheet and is bared if the contour C_α is deformed to the left as far as the point $j + 1 = \alpha(t_0)$. As t is decreased the contour C_α is squeezed together and leaves the singularity $j + 1 = \alpha(t_0)$ on the second sheet. The same situation arises if $\alpha(t_1)$ has a singularity in the complex plane of t_1. The case in which the singularity of $\alpha(t_1)$ lies on the real axis will be considered below.

To examine the motion of the singularities for $t < (m + 2\mu)^2$ it is necessary to go over from $\Delta_3\varphi_j(t)$ to the amplitude $\varphi_j(t)$ itself. The function $\Delta_3\varphi_j(t)$ has a singularity at $j + 1 = \alpha[(t^{1/2} - m)^2]$, or for fixed j a singularity at $t = t(j) \equiv [[\alpha^{-1}(j+1)]^{1/2} + m]^2$, where $\alpha^{-1}(j+1)$ is the function inverse to α.

In the j plane there exists a region of values $D(j)$ for which $t(j)$ does not lie on the first sheet of the t plane with a cut from $t = (m + 2\mu)^2$—that is, a region which is external relative to the imaging of the first sheet of the t plane by means of the formula $j + 1 = \alpha[(t^{1/2} - m)^2]$. Hereinafter for simplicity we assume that the image of the first sheet of t on the α plane is of the shape shown in Fig. 3 by the solid or by the dashed line. Then for sufficiently large j the singularity $t = t(j)$ lies on the second sheet, and the dispersion relation for $\varphi_j(t)$ is of the usual form. When j is decreased the singularity $t(j)$ moves onto the first sheet and deforms the path of integration in the dispersion relation.

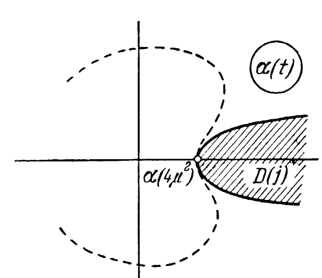

FIG. 3

In the more general case it is necessary to continue with respect to j from the region $D(j)$ which has been mentioned, and all of the conclusions of the present paper remain unchanged.

Thus for sufficiently large j the dispersion relation for $\varphi_j(t)$ is of the following form (the integral over the left-hand cut is unimportant):

1)There is no singularity at $j + 1 = \alpha(4\mu^2)$ if $\alpha(t) \sim (t - 4\mu^2)^{1/2}$ near $t = 4\mu^2$.

$$\varphi_j(t)=\frac{1}{\pi}\int\limits_{(m+2\mu)^2}^{\infty}\frac{dt'}{t'-t}\Delta_3\varphi_j(t'). \tag{2}$$

We substitute here the value of $\Delta_3\varphi_j(t)$ and interchange the integrations over t' and t_1. $\varphi_j(t)$ then takes the form

$$\varphi_j(t)=\frac{1}{2\pi i}\int\limits_{C_1'}\frac{dt_1 N_{j\alpha\alpha}N^{(3)}_{j\alpha\alpha}}{j+1-\alpha(t_1)}K(t,t_1,m^2), \tag{3}$$

where

$$K(t,t_1,m^2) = \int\limits_{(\sqrt{t_1}+m)^2}^{A}\frac{dt'}{t'-t}\frac{p(t',t_1,m^2)}{\sqrt{t'}}=\ln\frac{A}{\sqrt{t_1m^2}}-\frac{p(t,t_1,m^2)}{\sqrt{t}} \times\ln\frac{[(\sqrt{t_1}+m)^2-t]^{1/2}-[(\sqrt{t_1}-m)^2-t]^{1/2}}{[(\sqrt{t_1}+m)^2-t]^{1/2}+[(\sqrt{t_1}-m)^2-t]^{1/2}} +\frac{m^2-t_1}{t}\ln\sqrt{\frac{m^2}{t_1}}. \tag{4}$$

We have brought $N_{j\alpha\alpha}N^{(3)}_{j\alpha\alpha}$ out from under the sign of integration in (4), since we are interested in the analytic properties of $K(t, t_1, m^2)$ with respect to t_1 and are assuming that $N_{j\alpha\alpha}$ has no singularities with respect to t_1 that lie outside the contour C_1'.

Since t_1 has the meaning of the square of the mass of the reggeon, this assumption seems justified physically, at least for values of t smaller than the threshold of the two-particle state, that is, in the region of greatest interest from the point of view of the asymptotic behavior. We note that the analytic properties of K with respect to t_1 that are derived below from the explicit form of K can also be obtained directly from the integral (4) containing $N_{j\alpha\alpha}N^{(3)}_{j\alpha\alpha}$, but this is a less intuitive approach.

It follows from (4) that K has a branch point $t_1 = 0$ and behaves in the neighborhood of this point like $t_1 \ln t_1$. For $t > m^2$ another branch point appears at $t_1 = (t^{1/2} - m)^2$; for $\mathrm{Re}\, t < m^2$ this point passes off onto the second sheet via the branching at $t_1 = 0$. The locations of the singular points and the cuts in the t_1 plane, and also the path of integration C_1' in Eq. (3), are shown in Fig. 4

As was shown above, for large j the root $t_1(j)$ of the equation $j + 1 - \alpha(t_1) = 0$ lies on the second sheet in the t_1 plane. As j decreases the root $t_1(j)$ goes over onto the first sheet, deforming the contour C_1', and when $t_1(j)$ coincides with any singularity of $K(t, t_1, m^2)$ this gives rise to a singularity of the integral (3), that is, of $\varphi_j(t)$. The situation that leads to a singularity of $\varphi_j(t)$ is shown in Fig. 4, where the dots show the path of $t_1(j)$ from the second sheet.

Accordingly we get singularities of $\varphi_j(t)$:

a) Owing to coincidence of $t_1(j)$ with the singularity of K at $t_1 = 0$, which gives $j = \alpha(0) - 1$.

b) Owing to coincidence of $t_1(j)$ with $(t^{1/2} - m)^2$, which gives $j + 1 = \alpha[(t^{1/2} - m)^2]$.

Since the singularity $(t^{1/2} - m)^2$ exists only for $t > m^2$, the singularity b) goes off the first sheet of the j plane for $t < m^2$.

The nature of the singularity a) is determined by the nature of the behavior of K at $t_1 = 0$; therefore near $j = \alpha(0) - 1$ the function $\varphi_j(t)$ behaves like $[j + 1 - \alpha(0)] \ln [j + 1 - \alpha(0)]$.

Since a singularity of the integral (3) occurs when a pole $t_1 = t_1(j)$ coincides with a singularity of $K(t, t_1, m^2)$, we can separate out the contribution of the pole by taking the residue of the integrand in (3) at $t_1 = t_1(j)$; this gives

$$t_1'(j)\{N_{j\alpha\alpha}N^{(3)}_{j\alpha\alpha}K(t,t_1,m^2)\}_{t_1=t_1(j)}. \tag{5}$$

It can be seen from this that additional singularities of $\varphi_j(t)$ appear in cases in which $t_1(j)$ has a singularity as a function of j. This is possible either when $\alpha'(t_0) = 0$, or when $\alpha(t_1)$ has a singularity that leads to a singularity of $t_1(j)$.

In the special case when α has a square-root singularity $\alpha(t) = \alpha(t_0) + c(t - t_0)^{1/2}$ the root $t_1(j)$ is an analytic function of j and there is no new singularity of $\varphi_j(t)$.

The motion of the singularity $j = \alpha[(t^{1/2} - m)^2] - 1$ is shown by the dashed line in Fig. 5; the solid line indicates the cut running out from the point $j = \alpha(0) - 1$.

3. The term in the four-particle unitarity relation that leads to singularities in the j plane is of the form [1]

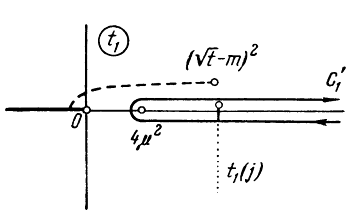

FIG. 4

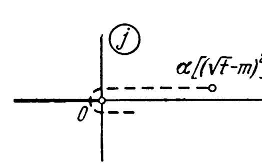

FIG. 5

$$\Delta_4\varphi_j(t)=\frac{1}{(2i)^2}\int\limits_{C_1}\int\limits_{C_2}\frac{N^{j}_{\alpha_1\alpha_2}N^{j(4)}_{\alpha_1\alpha_2}dt_1\,dt_2}{j+1-\alpha(t_1)-\alpha(t_2)}\,\frac{p(t,t_1,t_2)}{\sqrt{t}}. \quad (6)$$

Here $N^{j}_{\alpha_1\alpha_2}$ denotes the amplitude for transition of two particles into two reggeons, and $p(t, t_1, t_2)$ is the momentum of the reggeons with masses $t_1^{1/2}$ and $t_2^{1/2}$ in their center-of-mass system. The contours C_1 and C_2 go around the upper and lower sides of the cuts that run from $t_1 = 4\mu^2$ and $t_2 = 4\mu^2$. The upper boundary of the region of integration is fixed by the conservation laws and is of the form $t^{1/2} = t_1^{1/2}$. The region of integration is shown in Fig. 6; here t_1 and t_2 run through values on the upper and lower sides of the cut.

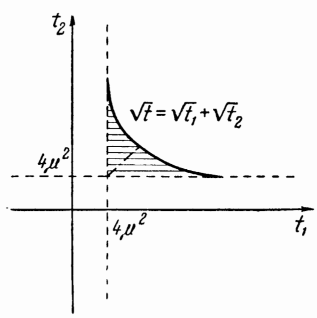

FIG. 6

Let us introduce the complex quantity $z = \alpha(t_1) + \alpha(t_2) - 1$. Then the region of integration in the plane of t_1 and t_2 is imaged into a certain region in the z plane. If we denote by M_{++} the region of integration over t_1, t_2 (shown in Fig. 6), when t_1 and t_2 are both on the upper side of the cut, and by M_{+-} the same region but with t_1 on the upper side and t_2 on the lower side, and similarly for the regions M_{-+} and M_{--}, then in general these regions are all imaged into different parts of the z plane. In the special case when the reggeons $\alpha(t_1)$ and $\alpha(t_2)$ are identical the images of M_{-+} and M_{+-} coincide.

Figure 7 shows the images z_{++} of the region M_{++} and z_{--} of the region M_{--}. Owing to the fact that z is symmetric in t_1 and t_2 the regions above and below the axis of symmetry in Fig. 6 give the same image in the z plane; therefore in the z plane a triangle z_{++} is formed with the vertices

$$z=\alpha((\sqrt{t}-2\mu)^2)+\alpha(4\mu^2)-1,$$
$$z=2\alpha(t/4)-1, \qquad z=2\alpha(4\mu^2)-1.$$

The regions M_{+-} and M_{-+} are uniquely imaged in the z plane, into a curvilinear triangle with the vertices

$$2\alpha(4\mu^2)-1, \quad \alpha((\sqrt{t}-2\mu)^2)+\alpha(4\mu^2)-1,$$
$$\alpha^*((\sqrt{t}-2\mu)^2)+\alpha(4\mu^2)-1.$$

It is now easy to find the singularities of $\Delta_4\varphi_j(t)$ as a function of j. In the j plane $\Delta_4\varphi_j(t)$ is an analytic function everywhere outside z_{++}, z_{--}, z_{+-}; the boundaries of the region of analyticity, shown in Fig. (7) (together with the boundaries of z_{+-}), are cuts connecting the singular points:

1) $j=2\alpha(t/4)-1, \qquad j=2\alpha^*(t/4)-1,$

2) $j=\alpha((\sqrt{t}-2\mu)^2)+\alpha(4\mu^2)-1,$

$j=\alpha^*((\sqrt{t}-2\mu)^2)+\alpha(4\mu^2)-1,$

3) $j=2\alpha(4\mu^2)-1.$

These points are singular points of the boundary, unlike the point $j = \alpha(t/4) + \alpha^*(t/4) - 1$, in whose neighborhood the boundary is analytic.

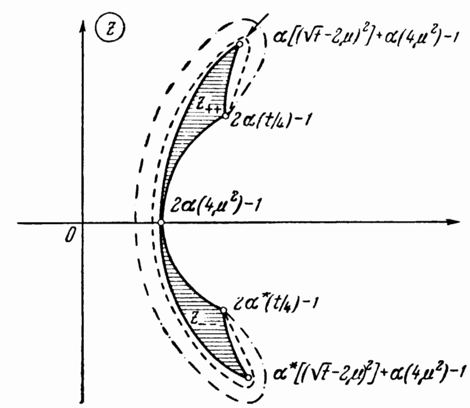

FIG. 7

The function $\Delta_4\varphi_j(t)$ can be analytically continued across any of the cuts connecting the singular points, but in general all of these continuations will lead to different results. It is convenient to draw a cut as shown by dashed lines in Fig. 7, from the point $j = 2\alpha(t/4) - 1$ to the left of $j = 2\alpha(4\mu^2) - 1$ and then to the point $j = 2\alpha^*(t/4) - 1$. This means analytic continuation of $\Delta_4\varphi_j(t)$ across the cuts connecting the points $j = 2\alpha(t/4) - 1$ and $j = 2\alpha^*(t/4) - 1$ with the point $2\alpha(4\mu^2) - 1$.

The contours C_1 and C_2 can then be deformed to the left of $4\mu^2$, as shown in Fig. 1. This causes creeping of the singularities 2), deforming the cut to the position shown by the dashed-and-dotted line in Fig. 7. We can conclude from this that singularities of types 2) and 3) exist when these points are approached from outside the cut (in the direction indicated by the arrow in Fig. 7), but do not exist when these points are approached from inside.

Consequently, with this way of drawing the cut these singularities arise on the second sheet and are laid bare if we want to draw the cut along the lines which in Fig. 7 connect the points $j = 2\alpha(t/4) - 1$ and $j = 2\alpha^*(t/4) - 1$ with the point $j = 2\alpha(4\mu^2) - 1$.

For $t \to 16\mu^2$ all of the singular points come together at the value $j = 2\alpha(4\mu^2) - 1$, and in order to trace the further motion of the singularities it is necessary to go over to the partial amplitude $\varphi_j(t)$ itself by means of the dispersion relation

$$\varphi_j(t) = \frac{1}{\pi}\int_{16\mu^2}^{\infty} \frac{dt'}{t'-t}\Delta_4\varphi_j(t'). \tag{7}$$

By changing the order of integration over t' and t_1, t_2 in Eq. (7), we arrive at a formula analogous to (3) [here it is necessary to make the same stipulations as preceded the derivation of (3)]:

$$\varphi_j(t) = \frac{1}{(2i)^2\pi}\int_{C_1'}\int_{C_2'} \frac{dt_1 dt_2 N^j_{\alpha_1\alpha_2} N^{j(4)}_{\alpha_1\alpha_2}}{j+1-\alpha(t_1)-\alpha(t_2)} K(t, t_1, t_2). \tag{8}$$

$K(t, t_1, t_2)$ is obtained from (5) by the replacement $m^2 \to t_2$, and the paths C_1' and C_2', unlike C_1 and C_2, extend to infinity.

Let us consider the inner integral over dt_2 in (8)

$$I(t, t_1, j) = \frac{1}{2i}\int_{C_2'} \frac{dt_2 N^j_{\alpha_1\alpha_2} N^{j(4)}_{\alpha_1\alpha_2} K(t, t_1, t_2)}{j+1-\alpha(t_1)-\alpha(t_2)} \tag{9}$$

and determine its singularities.

In the t_2 plane the function K has a branch point at $t_2 = 0$, near which K is of the form $K \sim t_2 \ln t_2$, and a branch point at $t_2 = (t^{1/2} - t_1^{1/2})^2$. The latter arises on the first sheet for $t > t_1$, but when $\mathrm{Re}\, t_1^{1/2}$ becomes larger than $\mathrm{Re}\, t^{1/2}$ the point $(t^{1/2} - t_1^{1/2})^2$ goes off onto the second sheet by way of the branching at $t_2 = 0$. Figure 8 shows the singularities of the function K and the corresponding cuts, and also shows the path of integration C_2'. The arrangement corresponds to

$$t > 16\mu^2, \quad \mathrm{Im}\, t > 0, \quad t_1 \in C_1', \quad t_1 < t.$$

The denominator in (9) becomes zero for t_2

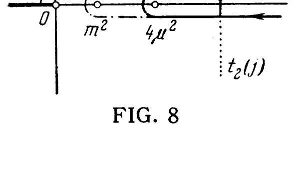

FIG. 8

$= t_2(j, t_1)$, where $t_2(j, t_1)$ is determined from the equation $j + 1 - \alpha(t_1) - \alpha(t_2) = 0$. For sufficiently large j, if the imaging of the first sheet of t by $\alpha(t)$ is as shown in Fig. 3 (in the general case, as was indicated in Sec. 2, for $j \in D_j$), the root $t_2(j)$ is on the second (or third, etc.) sheet in the t_2 plane at the branching $t_2 = 4\mu^2$. Then t_1 can be on either the first or the second sheet.

When j decreases the root $t_2(j, t_1)$ passes off from the second sheet, and deforms the contour C_2', pushing it either to the point $t_2 = (t^{1/2} - t_1^{1/2})^2$ or to $t_2 = 0$. Therefore the function $I(t, t_1, j)$ acquires a singularity when $t_2(j, t_1) = (t^{1/2} - t_1^{1/2})^2$ or $t_2(j, t_1) = 0$, that is, for values of t_1 satisfying the equations

$$j + 1 = \alpha(t_1) + \alpha((\sqrt{t} - \sqrt{t_1})^2), \tag{10}$$
$$j + 1 = \alpha(t_1) + \alpha(0). \tag{11}$$

Finally, as it deforms the contour C_2', $t_2(j, t_1)$ can pass around above the point $(t^{1/2} - t_1^{1/2})^2$ and push the contour to the point $t_2 = 4\mu^2$. This produces a singularity $t_2(j, t_1) = 4\mu^2$ or

$$j + 1 = \alpha(4\mu^2) + \alpha(t_1). \tag{12}$$

Thus for $t = t_0 + i\epsilon$, $t_0 > 16\mu^2$ the function $I(t, t_1, j)$ has singularities at the points t_1 given by Eqs. (10)—(12).

Let us denote the two roots of (10) by t_1^+ and t_1^-, $t_1^+ > t_1^-$. They are connected by the equation $t_1^+ = (t^{1/2} - t_1^{(-)1/2})^2$. We denote the root of Eq. (11) by $t_1^{(0)}$ and that of Eq. (12) by $t_1^{(\mu)}$. For sufficiently large j the roots $t_1^{(0)}$ and $t_1^{(\mu)}$, and one of the roots $t_1^{(\pm)}$, are on the second sheet in the t_1 plane, as can be seen from the explicit forms of Eqs. (10)—(12) [for example, from (11), if we take $j + 1$ real and equal to $\alpha(4\mu^2) + \alpha(0) + \epsilon$ and use the fact that $\alpha(t_1) = \alpha(4\mu^2) + \epsilon$, then t_1 is on the second sheet]. When, as it decreases, j becomes equal to $\alpha(t) + \alpha(0) - 1$, the root t_1^- passes around the point $t_1 = 0$ and comes out onto the first sheet, and in the same way the root t_1^+ passes around the point $t_1^{(0)}$, which is on the second sheet, and comes out from under the cut connected with this point.

The function $I(t, t_1, j)$ has two other singularities with respect to t_1 which do not depend on j. The first arises when the singularity $(t^{1/2} - t_1^{1/2})^2$ in the t_2 plane coincides with $t_2 = 4\mu^2$, which gives $t_1 = (t^{1/2} - 2\mu)^2$. The second is the singularity at $t_1 = 0$, which is already known to us and is contained explicitly in $K(t, t_1, t_2)$.

Thus in the t_1 plane we have three singularities in the first sheet, on one side of the contour C_1': $t_1 = 0$, t_1^-, $(t^{1/2} - 2\mu)^2$. Four singularities arise on the second sheet, on the other side of C_1': $t_1^{(+)}$, $t_1^{(0)}$, and $t_1^{(\mu)}$, $4\mu^2$. The positions in the t_1 plane of all of these singularities for $j < \alpha(t) + \alpha(0) - 1$ are shown in Fig. 9.

The function $\varphi_j(t)$ can be written in the form

$$\varphi_j(t) = \frac{1}{2\pi i}\int_{C_1'} I(t, t_1, j)\,dt_1. \tag{13}$$

Coincidence of singularities that lie on different sides of C_1' leads to singularities of $\varphi_j(t)$.

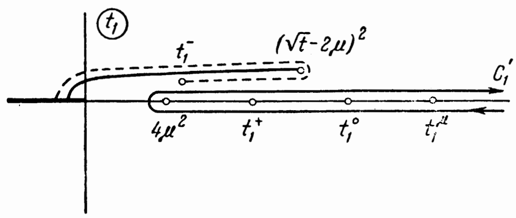

FIG. 9

By using the concrete form of K in (9) one can show that the discontinuities of I across the cuts connected with the singularities t_1^-, $t_1 = 0$ and $t_1 = (t^{1/2} - 2\mu)^2$ do not contain singularities for $t_1 = t_1^{(0)}$. Therefore the point $t_1 = t_1^{(0)}$ can be left out of consideration. Furthermore, the discontinuities of the function I at the cuts connected with the singularities t_1^+ and $t_1^{(\mu)}$ do not contain singularities for $t_1 = 0$. Therefore the point $t_1 = 0$ also need not be considered. Then we get the following singularities of $\varphi_j(t)$:

I) it can be seen from the condition $t_1^+ = (t^{1/2} - t_1^{(-)1/2})^2$ that coincidence of t_1^+ and t_1^- leads to a singularity $j = 2\alpha(t/4) - 1$;

II) coincidence of t_1^- and $t_1^{(\mu)}$, and also of t_1^+ and $(t^{1/2} - 2\mu)^2$ and of $t_1^{(\mu)}$ and $(t^{1/2} - 2\mu)^2$, gives $j + 1 = \alpha(4\mu^2) + \alpha[(t^{1/2} - 2\mu)^2]$;

III) passage of $t_1^{(\mu)}$ around $(t^{1/2} - 2\mu)^2$ with accompanying deformation of the contour C_1', and subsequent coincidence of $t_1^{(\mu)}$ with $t_1 = 4\mu^2$ gives $j + 1 = 2\alpha(4\mu^2)$.

Finally, a singularity of $\varphi_j(t)$ which does not depend on j arises for $(t^{1/2} - 2\mu)^2 = 4\mu^2$; this is the ordinary threshold singularity $t = 16\mu^2$.

As was already pointed out in the discussion of the singularities of the unitarity relation (5), we can draw the cut from the singularity I) in such a way that the singularities II) and III) remain on the second sheet with respect to this cut. In the t_1 plane this situation corresponds to the way of drawing the cut from the point t_1^- that is shown by the dashed line in Fig. 9.

Now let us consider $t < 16\mu^2$. Then in the t_1 plane the singularity $(t^{1/2} - 2\mu)^2$ never reaches $t_1 = 4\mu^2$. Coincidence of t_1^+ and t_1^- occurs, as before, at the point $t_1^+ = t_1^- = t/4$. If $j < 2\alpha(t/4) - 1$, then t_1^+ and t_1^- move off into the complex plane after their collision, as can be seen intuitively in the explicit expression for $t_1^\mp$ obtained by expanding $\alpha(t)$ in a Taylor's series:

$$\gamma t_1^\pm = {}^1\!/_2(j-1) \pm {}^1\!/_2\{\gamma t(2(j-1) - \gamma t)\}^{1/2}, \quad \gamma = \alpha'(0).$$

Since for $t < 16\mu^2$ we have $(t^{1/2} - 2\mu)^2 < t/4$, the point t_1^+ cannot coincide with $(t^{1/2} - 2\mu)^2$ on the physical sheet of j with a cut running to the left from $j = 2\alpha(t/4) - 1$. Similarly, t_1^- goes off into the complex plane without reaching $t_1^{(\mu)}$, and consequently for $t < 16\mu^2$ all singularities of types II) and III) arise on the second sheet with respect to the singularity I).

Thus in the j plane there remains a single branching $j = 2\alpha(t/4) - 1$, which arises from the two-reggeon intermediate state and does not contain the masses of the particles.

The behavior of $\varphi_j(t)$ in the neighborhood of this singularity has been determined in the paper by V. N. Gribov and others. In general, however, besides the singularity $j = 2\alpha(t/4) - 1$ the function $\varphi_j(t)$ contains the three-particle singularities considered in the preceding section. We shall show that they go off onto the second sheet with respect to the branching I).

For this purpose we note that the product $N^j N^{j(4)}$ which appears in the integrand in (6) contains as a normalizing factor the quantity [3]

$$\left[\tan\frac{\pi\alpha(t_1)}{2}\ \tan\frac{\pi\alpha(t_2)}{2}\right]^{-1}.$$

If a scalar particle with mass m involved in the three-particle unitarity relation lies on the Regge trajectory $\alpha(t_2)$, then $\alpha(m^2) = 0$ and $N^j N^{j(4)}$ contains a pole in the t_2 plane at $t_2 = m^2$.

In the four-particle unitarity relation (6) we can take instead of the contour C_2 which goes around $t_2 = 4\mu^2$ (see Fig. 10) a new contour C_2^* which goes around the point $t_2 = m^2$ (dashed line in Fig. 10). The change this makes is that the residue at $t_2 = m^2$ is added to the four-particle integral (6). It is easy to see that this residue is the same as the three-particle unitarity relation. This procedure is essentially that which was used by Mandelstam [1] to include the two-particle unitarity relation in the three-particle relation.

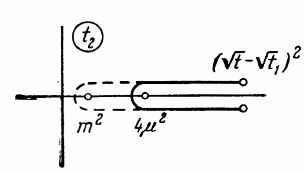

FIG. 10

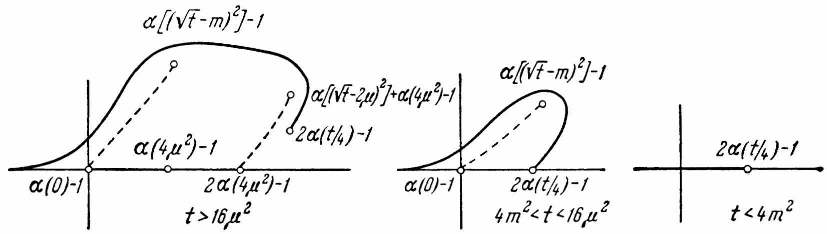

FIG. 11

The singularities of the unitarity relation altered in this way can now be found easily by the same method as before. Besides the singularities 1)—3) the new discontinuity $\Delta_4' \varphi_j(t)$ has the singularities

$$j = \alpha((\sqrt{t} - m)^2) - 1, \quad j = \alpha^*((\sqrt{t} - m)^2) - 1, \quad j = \alpha(4\mu^2) - 1.$$

Again reconstructing $\varphi_j(t)$ from $\Delta_4' \varphi_j(t)$ by means of a dispersion relation, we find singularities of $\varphi_j'(t)$ in three different regions:

$$t \geqslant 16\mu^2, \quad (m + 2\mu)^2 \leqslant t \leqslant 16\mu^2, \quad t \leqslant (m + 2\mu)^2.$$

The function $I(t, t_1, j)$ is now again defined by the formula (9), in which the contour C_2' passes to the left of $t_2 = m^2$ (Fig. 8). It is easy to see that in addition to the singularities (10)—(12) of the function I there are singularities $t_1 = (t^{1/2} - m)^2$ and $j + 1 = \alpha(t_1) + \alpha(m^2)$. An analysis of the expression (13) then leads to the conclusion that besides the singularities I)—III) there are new ones for $t \geq 16\mu^2$:

$$j = \alpha((\sqrt{t} - m)^2) - 1, \quad j = \alpha(4\mu^2) - 1, \quad j = \alpha(0) - 1.$$

For $(m + 2\mu)^2 \leq t \leq 16\mu^2$ the singularities II) and III) pass off from the physical sheet, as before, and the additional singularities (the three-particle ones) listed above remain, since although $(t^{1/2} - 2\mu)^2$ is always smaller than t_1^+, still $(t^{1/2} - m)^2$ can coincide with t_1^+.

Finally, in the region $t \leq 4m^2$, for our special case of identical reggeons it is impossible for $(t^{1/2} - m)^2$ and t_1^+ to coincide, and the three-particle singularities also pass off from the physical sheet. In this region of t values the j plane contains only the branching at $j = 2\alpha(t/4) - 1$. Figure 11 shows the singularities of $\varphi_j(t)$ in the j plane which appear when the three-particle and four-particle intermediate states are taken into account. The cuts are drawn in such a way that the singularities that depend on the masses of the particles, and also the three-particle singularities, have always remained on the other sheet (these singularities and the cuts associated with them are indicated with dashed lines).

We have treated the case of identical reggeons. If $\alpha(t)$ is a vacuum trajectory and $\alpha(0) = 1$, there can be no scalar particle on it. Therefore our three-particle unitarity relation here can be included in the four-particle relation with different reggeons, $\alpha(t)$ and $\beta(t)$.

All of the conclusions reached above remain unchanged, provided only that there exists a singularity of type I) for the trajectories $\alpha(t)$ and $\beta(t)$. The condition for the existence of this singularity is that it be possible for the roots of the equation

$$j + 1 = \alpha(t_1) + \beta[(\sqrt{t} - \sqrt{t_1})^2], \tag{10'}$$

to coincide, i.e., that the derivative of the right member of (10′) with respect to t_1 can be equal to zero:

$$\alpha'(t_1) - \frac{\sqrt{t} - \sqrt{t_1}}{\sqrt{t_1}} \beta'[(\sqrt{t} - \sqrt{t_1})^2] = 0. \tag{14}$$

Using the fact that for (10′) the values of t_1 are restricted, $0 \leq t_1 \leq t$, and that over this range the coefficient of β' varies from zero to $+\infty$, we arrive at the following condition for the existence of a singularity of type I): the derivatives of $\alpha(t)$ and $\beta(t)$ must have the same sign. Then the simultaneous solution of (10) and (14) determines a singularity $j = j(t)$, which for sufficiently small t remains on the first sheet of the j plane. If t_1^* is the root of the equation (14), then $t = (m + t_1^{*1/2})^2$ is the value at which the three-particle singularities pass off onto the second sheet.

The author is grateful to V. N Gribov, I. Ya. Pomeranchuk, and K. A. Ter-Martirosyan for helpful discussions.

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Translated by W. H. Furry
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