CONTRIBUTION TO THE THEORY OF TURBULENCE IN A TWO-TEMPERATURE PLASMA

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Possible stationary distributions of turbulent fluctuations are studied in a plasma consisting of hot electrons drifting through cold ions at a velocity u exceeding the velocity s of two-temperature sound. It is shown that stationary distributions of fluctuations exist in the long-wave region; no stationary distribution can be established in the short-wave region. The dependence of the amplitude and angular distribution of stationary turbulent fluctuations on the wave vector k is investigated in an unbounded plasma. It is shown that as the wave vector is decreased the squared amplitude of fluctuations in the scalar potential increases as k^{-3}, whereas the angular distribution of the fluctuations changes periodically as k is varied. For certain values of the wave vector all turbulent waves (i.e., waves propagating at an angle θ less than $\cos^{-1}(s/u)$) to the electron flow, are characterized by the same amplitude, which is independent of the angle θ. For some other values of k almost all turbulent waves are propagated along the electron flow or at an angle $\theta = \cos^{-1}(s/u)$ to it.

1. INTRODUCTION

It is well known that the so called ion-acoustic instability can occur in a collisionless plasma consisting of hot electrons drifting through a background of cold ions: if the electron drift-velocity exceeds the two-temperature sound velocity, then acoustic oscillations can be amplified. As Kadomtsev and Petviashvili[1,2] have shown, the growth in the amplitude of random acoustic oscillations can be limited by the nonlinear interaction of these oscillations with particles in the plasma, as a result of which a stationary distribution of fluctuations can be set-up, i.e., the plasma enters into a condition of steady-state turbulence.

Starting with a set of "coupled" equations for the correlation functions, an equation is derived[1,2] for the spectral distribution of stationary fluctuations. This equation describes the balance between a "source" and "sink" of oscillations. The source-strength of the oscillations is proportional to the increment in growth derived from the linear theory, and the sink corresponds to the nonlinear interaction of the waves with particles. In the present paper we derive a more exact equation for the spectral distributions of the established fluctuations by taking into account a second "source" of oscillations—"random forces." This source plays a dominant role at short wavelengths; in this region, as is shown below, a stationary distribution does not exist.

Stationary distributions of fluctuations have been studied in the long-wavelength region by Kadomtsev and Petviashvili[1,2] for the case of a bounded plasma (a discharge of very small cross section). In the present paper we study fluctuations in an unbounded plasma when the wavelengths are not too short. We show that in a turbulent plasma it is possible to have not just one but an infinite number of different fluctuation distributions. This result differs from the result obtained for the case of an equilibrium or quasi-equilibrium plasma. For all the distributions, as the wave vector k is decreased, the squared amplitude of the steady-state turbulent fluctuations increases as k^{-3}, and the angular distribution is also a periodic function of k. For certain values of k, the amplitude of the turbulent fluctuations does not depend on the angle between the direction of **k** and the electron flow. For some other values of **k** the amplitude has a sharp maximum for waves propagating along the flow or at the Cerenkov angle.

A permissible stationary distribution of turbulent acoustic oscillations can be characterized by two parameters: the period Π of the angular-dependence oscillations and the phase of these oscillations. At the same time, the value of Π characterizes the rate at which the amplitude of the steady state oscillations increases as k is decreased.

In limiting the present paper to a study of stationary distributions of fluctuations, we do not consider which one of the permissible distributions is set up by a given initial distribution of random acoustic waves (or whether in general a stationary distribution is set-up at all).

2. EQUATIONS DESCRIBING STATIONARY DISTRIBUTIONS OF FLUCTUATIONS

First of all we derive the equation for the spectral distribution of steady state fluctuations in a plasma consisting of hot electrons drifting through a background of cold ions at a velocity $\mathbf{u}$ that exceeds the two-temperature sound speed s. For this purpose we generalize the method of Kadomtsev and Petviashvili[1,2] to some extent, introducing into the kinetic equations "random forces" that are then normalized in accordance with the general theory of fluctuations (see [3] in this connection).

Neglecting small terms which do not alter the structure of the equation and lead only to renormalization of the main terms, we have

$$-k^2 \operatorname{Im} \varepsilon(\mathbf{k}, \omega) I(\mathbf{k}) + a^2 \omega e^2 \int d\mathbf{v} f_e(\mathbf{v}) \delta(\omega - \mathbf{kv}) = I(\mathbf{k}) \int K(\mathbf{k}, \mathbf{k}') I(\mathbf{k}') d\mathbf{k}', \tag{1}$$

where $I(\mathbf{k})$ is the correlator of the scalar potential φ,

$$\langle \varphi(\mathbf{k}, \omega) \varphi(\mathbf{k}', \omega') \rangle = \delta(\mathbf{k} + \mathbf{k}') \delta(\omega + \omega') \{ I(\mathbf{k}) \delta(\omega - ks) + I(-\mathbf{k}) \delta(\omega + ks) \}, \tag{2}$$

$\epsilon(\mathbf{k}, \omega)$ is the dielectric permittivity of the plasma (calculated from the linear theory)

$$K(\mathbf{k}, \mathbf{k}') = \frac{16\pi e^4}{(M\omega^2)^3} (\mathbf{kk}')^2 \frac{[\mathbf{kk}']^2}{(\mathbf{k} - \mathbf{k}')^2} \left(1 + 3 \frac{\omega - \omega'}{\omega}\right) \times \operatorname{Im} \int \frac{(\omega - \omega') f_i d\mathbf{v}}{(\mathbf{k} - \mathbf{k}')\mathbf{v} - (\omega - \omega')}, \tag{3}$$

$a = (4\pi e^2 n / T_e)^{-1/2}$ is the Debye radius, $f_{e,i}$ are the distribution functions, m and M the masses, and $T_{e,i}$ the temperatures of the electrons and ions. Equation (1) differs from Eq. (2) of [2] by the presence in the left hand side of the second term, which describes a wave source due to "random forces."[1)]

The integration of the right hand side of (1) can be limited to the region $\mathbf{k}' \cdot \mathbf{u} > k's$ by taking into account the fact that the level of turbulent fluctuations (for which $\mathbf{k} \cdot \mathbf{u} > ks$) is much higher than the level of stable fluctuations ($\mathbf{k} \cdot \mathbf{u} < ks$). By observing that the function K differs significantly from zero only when

$$(\omega - \omega')^2 \sim (\mathbf{k} - \mathbf{k}')^2 T_i / M \ll (\mathbf{k} - \mathbf{k}')^2 s^2$$

and assuming for simplicity that $1 - s/u \ll 1$, we transform (1) to

$$\frac{a^2 T_e}{(2\pi)^2} - \xi(\theta) I(k, \theta) = -B I(k, \theta) k \times \frac{\partial}{\partial k} \int_{s/u}^{1} (1 - \cos^2\theta \cos^2\theta') k^3 I(k, \theta') d\cos\theta', \tag{4}$$

where $\theta(\theta')$ is the angle between $\mathbf{k}(\mathbf{k}')$ and $\mathbf{u}$,

$$\xi(\theta) = \frac{T_e}{nm} \sqrt{\frac{2\pi M}{m}} \int \delta(\omega - \mathbf{kv}) \mathbf{k} \frac{\partial f_e}{\partial \mathbf{v}} d\mathbf{v}, \qquad B = 8\pi e^2 T_i T_e^{-3} (2\pi M / m)^{1/2}. \tag{5}$$

We now evaluate the function $\xi(\theta)$ in (4). By taking into account the existence of a plateau in the electron distribution function (Vedenov, Velikhov, and Sagdeev[5]) in the region for which $\partial f_e / \partial v_{\parallel} = \epsilon m v_{\parallel} / T_e$ where $v_{\parallel} = \mathbf{v} \cdot \mathbf{u}/u$ and ϵ is a small parameter which characterizes the slope of the plateau, we obtain

$$\xi(\theta) = \begin{cases} 1 - u\cos\theta/s & (\cos\theta < s/u), \\ \varepsilon(1 - u\cos\theta/s) & (\cos\theta > s/u). \end{cases} \tag{6}$$

In order of magnitude $\epsilon \sim (e^2 T_i \Lambda / a T_e^2)^{1/2}$, where Λ is the Coulomb-logarithm (see [6] in this connection).

Equations (4), (5), and (6) permit the function $I(k, \theta)$ to be determined. We observe that (4) contains both the function I and its derivative $\partial I / \partial k$ and can therefore have an infinite number of solutions. It can be shown that even the original integral equation (1) also has an infinite number of solutions; this result is due to the degeneracy of the kernel K. The integral $\int K(\mathbf{k}, \mathbf{k}') I(\mathbf{k}') d\mathbf{k}'$ does not define a unique function $I(\mathbf{k})$ (in particular any function of the form $I(\mathbf{k}) = k^{-3} F(\theta)$ reduces the right hand side of the equation to zero).

In solving (4) it is convenient to consider the regions of turbulence ($\cos\theta > s/u$) and stability ($\cos\theta < s/u$) separately. In the stable region, by neglecting the term which is nonlinear in the function I, we obtain the result previously derived from linear theory[7,8]

$$I(k, \theta) = a^2 T_e (2\pi)^{-2} (1 - u\cos\theta / s)^{-1}. \tag{7}$$

For values of k which are not too large, it is not difficult to show that the nonlinear effects described by the right hand side of (4) contribute to the correlator of stable fluctuations a term proportional to ϵ.

Passing on to the investigation of fluctuations in the region of turbulence we note that, according to (4) and (6), for $\cos\theta > s/u$ the function $I(k, \theta)$ can be written in the form

1) The author has since learnt that an equation of the same form as (1) has been derived independently by Silin[4].

$$I(k,\theta)=\frac{a^2T_e s}{(2\pi)^2\varepsilon u}\frac{\rho(z)}{z}\left\{(1-\cos\theta)+\left(1-\frac{s}{u}\right)\lambda(z)\right\}^{-1}, \tag{8}$$

where ρ and λ are functions of the variable $z = \zeta(ak)^3$ and,

$$\zeta = 12e^2T_i s^2 a^{-1}(T_e u\varepsilon)^{-2}(2M/\pi m)^{1/2} \tag{9}$$

(the order of magnitude of ζ is $\sim \Lambda^{-1}\sqrt{M/m}$).

The function ρ characterizes the amplitude of the steady state acoustic oscillations, and the function λ characterizes their angular distribution. It follows from (4) that these functions satisfy the equations

$$\rho^2\frac{d\lambda}{dz}=\Phi_0+\frac{\rho}{z}\Psi_0,\qquad \rho\frac{d\rho}{dz}=\Phi_1+\frac{\rho}{z}\Psi_1 \tag{10}$$

where the coefficients Φ and Ψ are functions of λ alone and have the form

$$\Phi_0=D^{-1}\left\{2\ln\left(1+\frac{1}{\lambda}\right)-1\right\},$$

$$\Psi_0=D^{-1}\left\{(1-\lambda)\ln\left(1+\frac{1}{\lambda}\right)+1\right\}$$

$$\Phi_1=D^{-1}\left\{\frac{2}{1+\lambda}-\ln\left(1+\frac{1}{\lambda}\right)\right\},$$

$$\Psi_1=D^{-1}\left\{\frac{1-\lambda}{\lambda(1+\lambda)}+\ln\left(1+\frac{1}{\lambda}\right)\right\},$$

$$D=\ln^2\left(1+\frac{1}{\lambda}\right)-\frac{1}{\lambda(1+\lambda)}. \tag{11}$$

3. SPECTRAL AND ANGULAR DISTRIBUTIONS OF STEADY STATE FLUCTUATIONS

We start our study of the spectral distribution of turbulent fluctuations with the case of very short wavelength fluctuations, $k \to \infty$. From (10) it follows that as $z \to \infty$ the function ρ increases like $z^{1/2}$, but the value of λ tends to a constant limit $\lambda \to \lambda_\infty$, which is defined by the equation $\Phi_0(\lambda_\infty) = 0$. It is readily shown that $\Phi_1(\lambda_\infty) < 0$ and therefore as $z \to \infty$ the value of $\rho^2 = 2z\Phi_1(\lambda_\infty)$ is negative. (Numerical calculations give $\lambda_\infty \approx 0.4$ and $\rho \approx 1.3\,iz^{1/2}$). Thus for very large values of k real solutions of (10) do not exist, i.e., according to (8), stationary fluctuation distributions do not exist.

It is also not difficult to come to the latter conclusion directly by using (4). Indeed, it is evident from this equation that $I \sim k^{-3/2}$ as $k \to \infty$, so that the right hand side of the equation is negative, i.e., the nonlinear interaction of the waves with particles acts as a source of waves rather than as a "sink."

The value of the wave vector k_c at which the solutions of (4) and (10) cease to be real is determined by the initial distribution of the fluctuations. It is evident that if $ak_c > 1$ (i.e., if $z_c = \zeta(ak_c)^3 > \Lambda^{-1}\sqrt{M/m}$) then a stationary distribution of fluctuations exists for all physically allowable values of the wave vector.

We now pass on to a study of the spectral distribution of fluctuations in the region where the wavelengths are not extremely short. By taking into account the fact that in this region $\Phi_{0,1} \ll \Psi_{0,1}\rho/z$, one can obtain a solution of (10) in closed form:

$$\rho = C_\rho e^{f(\lambda)},\qquad z = C_z e^{C_\rho g(\lambda)}, \tag{12}$$

where the functions f and g are determined by the formulae

$$f(\lambda)=\int\limits_{C_f}\Psi_0^{-1}\Psi_1 d\lambda,\qquad g(\lambda)=\int\limits_{C_g}\Psi_0^{-1}e^{f(\lambda)}d\lambda \tag{13}$$

and C_ρ, C_z, C_f, and C_g are constants. From the explicit form of the functions Ψ [formulae (11)] it follows that the functions $e^{f(\lambda)}$ and $g(\lambda)$ have branch points at $\lambda = 0$ and -1, and that these functions are real for $\lambda > 0$ and for $\lambda < -1$. The function $g(\lambda)$ is bounded for all values of λ; the function $e^{f(\lambda)}$ tends to infinity as $\lambda \to \infty$.

As a consequence of these results, it is convenient to introduce two forms of the functions f and g: the functions f^+ and g^+ for $\lambda > 0$ are defined by Eqs. (13) with $C_f = +1$ and $C_g = 0$, and the functions f^- and g^- for $\lambda < -1$ are defined by the same equations with $C_f = -2$ and $C_g = -1$. We note that $f^\pm > 0$ for $\lambda > 1$ and $\lambda < -2$, $f^\pm < 0$ for $0 < \lambda < 1$ and $-2 < \lambda < -1$, and $g^\pm < 0$ for $\lambda > 0$ and $\lambda < -1$. By knowing the position and nature of the singularities of these functions we can see how the values of ρ and λ change as z is varied.

For C_z we choose the value of the variable $z = z_0$ for which $\rho = 0$ and set $C_\rho = \rho_0 > 0$. It is easily seen that $\lambda \to 0$ as $z \to z_0 - 0$. If $\ln(z_0/z) \ll 1$, then

$$\rho = \ln(z_0/z),\qquad \lambda = \exp\{-\rho_0/\ln(z_0/z)\}. \tag{14}$$

As z is reduced the functions ρ and λ increase. As $z \to z_1$ where $z_1 = z_0\exp\{\rho_0 g^+(\infty)\}$ we have $\lambda \to \infty$ and $\rho \to \infty$, where in the limit $\rho = \rho_0\lambda$.

We now examine the variation of ρ and λ when $z > z_0$. In order to do this we set $C_z = z_0$, $C_\rho = \rho_0' < 0$ in (12). It is easily seen that $\rho \to 0$ and $\lambda \to -1$ as $z \to z_0 + 0$. If $\ln(z/z_0) \ll 1$, then

$$\rho = -2\ln(z/z_0),$$

$$\lambda + 1 = -\exp\{\rho_0'/2\ln(z/z_0)\}. \tag{15}$$

As z is increased the absolute values of the functions ρ and λ increase and they remain negative. As $z \to z_{-1}$ where $z_{-1} = z_0 \exp\{\rho_0 g^+(\infty)\}$, we have $\lambda \to -\infty$ and $\rho \to -\infty$, where in the limit $\rho = |\rho_0'|\lambda$.

It is readily shown that the behavior of the functions ρ and λ for $z < z_1$ is similar to their behavior for $z > z_0$. As $z \to z_1 - 0$ both these functions tend to $-\infty$, where in the limit $\rho = |\rho_0''|\lambda$ (ρ_0'' is a negative constant). As $z \to z_2$, where $z_2 = z_1 \exp\{-\rho_0'' g^-(\infty)\}$, the functions ρ and $\lambda+1$ tend to zero in the manner determined by (15) (in which ρ_0' must be replaced by ρ_0'').

The connection between the constants ρ_0 and ρ_0'' can readily be established by noting that the function $I(\mathbf{k})$ must be continuous at $z = z_1$: the values of ρ_0 and ρ_0' can be related to each other by the condition that the function $\int I(\mathbf{k})do$ must be continuous at $z = z_0$. As a result we obtain $\rho_0' = \rho_0'' = -\rho_0$.

Thus the functions ρ and λ are periodic in $\ln(z_0/z)$ with a period $\{\rho_0|g^+(\infty) + g^-(\infty)|\}$. As $\ln(z_0/z)$ is increased the function ρ increases continuously within its period from $-\infty$ to $+\infty$; in a similar way, λ varies from $-\infty$ to -1, passes through a discontinuity at the point where $\rho = 0$, and varies from 0 to $+\infty$ as $\ln(z_0/z)$ is increased further.

Using (8) we can now examine how the amplitude and angular distribution of turbulent acoustic waves change as k is varied. The function $k^3 I$ is a periodic function of the variable $\ln(k_0/k)$ with a period,

$$\Pi = {}^1\!/_3\rho_0|g^+(\infty) + g^-(\infty)|.$$

If $\ln(k_0/k) = n\Pi$ ($n = 0, \pm 1 \ldots$) then the angular distribution has the extremely singular form:

$$I(k,\theta) = \frac{a^2 T_e s}{(2\pi)^2 \varepsilon \zeta u} \times \rho_0 (ak)^{-3} \begin{cases} \delta(1-\cos\theta), & \ln(k_0/k) \to n\Pi + 0 \\ \delta(\cos\theta - s/u), & \ln(k_0/k) \to n\Pi - 0. \end{cases} \quad (16)$$

In such circumstances almost all turbulent waves are propagated along the current or at the Cerenkov angle to it. If $\ln(k_0/k) = \Pi(n+\nu)$, where

$$\nu = g^+(\infty)\{g^+(\infty) + g^-(\infty)\}^{-1},$$

then all turbulent waves are characterized by only a single amplitude which is independent of the angle θ:

$$I(k,\theta) = \frac{a^2 T_e s}{(2\pi)^2 \varepsilon \zeta (u-s)} \rho_0 (ak)^{-3} \quad \left(\cos\theta > \frac{s}{u}\right). \quad (17)$$

The intensity of turbulent sound waves with frequencies in the interval ω to $\omega + d\omega$ is conveniently characterized by the function

$$J(\omega) = k^2 \frac{dk}{d\omega} \int I(\mathbf{k})\, do. \quad (18)$$

According to (8) we have

$$J(\omega) = \frac{T_e s \rho_0}{2\pi\zeta\varepsilon u a\omega}\alpha, \quad \alpha = \ln\left(1 + \frac{1}{\lambda}\right) e^{f(\lambda)} \operatorname{sign}\lambda, \quad (19)$$

where the function f is defined by (13). We see that as ω is reduced, the function $J(\omega)$ increases like ω^{-1} (if we disregard the oscillatory factor α, whose order of magnitude is unity).[2)]

We note that the permissible stationary distribution is characterized by two parameters: the value of ρ_0, which determines the amplitude of the fluctuations and the oscillation period of their angular distribution, and the value of k_0 which determines the phase of the oscillations. In order to evaluate these parameters it is necessary to know the initial state of the system.

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Translated by F. L. Curzon
326

2)The dependence $J(\omega) \approx \omega^{-1}$ was first established by Kadomtsev and Petviashvili[1,2] for the case of a plasma column of small cross section.