RELATIVISTIC COULOMB FUNCTIONS

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Expansions are derived in powers of $(\alpha Z)^2$ for the relativistic Coulomb Green's function and for the wave function of a spinor particle with definite asymptotic momentum p, valid for arbitrary p. For large energies E and small angles $\vartheta \lesssim 1/E$ the expansion contains a supplementary small parameter of the order of 1/E.

Up to the present there has been no derivation of a closed expression for the relativistic Coulomb Green's function G_C, nor for the related Møller function φ, the scattering amplitude T, nor the wave function $|\varphi_p\rangle u_p$ corresponding to a definite asymptotic momentum p. Evidently if there does exist a closed form for these functions it has a very cumbersome analytic form, not suitable for practical purposes.

In the present paper we propose a new expansion of the relativistic functions in terms of the parameter $(\alpha Z)^2 q \ln E/E$, where q is the momentum transferred to the nucleus and E is the energy of the particle. Each term of this expansion has meaning for arbitrary values of the asymptotic momentum p ($p^2 = E^2 - m^2$), including the case $p \to 0$.

In a previous paper by the writer [see [1], Eqs. (4) and (7)] the following equation was derived for the Møller function in the Coulomb field $-\alpha ZV$:

$$\varphi = \varphi^0 + \alpha Z\varphi^1 - (\alpha Z)^2 \varphi G^+ V \varphi^1, \quad (1)$$

$$\varphi^0 = 1 - \alpha Z G^0 V \varphi^0, \quad T^0 = V\varphi^0, \quad (2)$$

$$\varphi^1 = (\alpha Z 2EG^+)^{-1}(1-\varphi^0) = -G^- T^0. \quad (3)$$

Here G^0, φ^0, and T^0 are the "nonrelativistic" free Green's function, Møller function, and scattering amplitude[1]; $G^\pm$ are the Green's functions of the free Dirac equation with positive and negative energy E, which are diagonal in momentum space:

$$\langle \mathbf{k}_2 | G^s | \mathbf{k}_1 \rangle = G^s(\mathbf{k})\,\delta(\mathbf{k}_2 - \mathbf{k}_1), \quad (4)$$

$$G^0(\mathbf{k}) = \frac{-2E}{k^2 - p^2 - i\varepsilon}, \quad G^\pm(\mathbf{k}) = \frac{\tilde{k} \pm E + \beta m}{k^2 - p^2 - i\varepsilon}, \quad \tilde{k} \equiv \mathbf{k}\boldsymbol{\alpha}, \quad (5)$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices. A closed form for the function T^0 in momentum space has been obtained in [1].

Iterating Eq. (1), we obtain φ in the form

$$\varphi = \sum_{n=0}^{\infty} (\alpha Z)^{2n} (\varphi^0 - \alpha Z G^- T^0)(G^+ V G^- T^0)^n. \quad (6)$$

The wave function $|\varphi_p\rangle u_p$ is connected with the φ of Eq. (1) by the relation

$$\varphi |\mathbf{p}\rangle u_p = |\varphi_p\rangle u_p, \quad (7)$$

$$(\tilde{p} - E + \beta m) u_p = 0, \quad \tilde{p} = \boldsymbol{\alpha}\mathbf{p}, \quad \boldsymbol{\alpha}\beta = -\beta\boldsymbol{\alpha}. \quad (8)$$

Using successively the free Dirac equation and the commutation relations of the matrices $\boldsymbol{\alpha}$ and β, Eq. (8), we can put the series (6) for the wave function in the form

$$\langle \mathbf{k} | \varphi_p \rangle u_p = \sum_{n=0}^{\infty} (\alpha Z)^{2n} \langle \mathbf{k} | (\varphi^0 + \alpha Z \tilde{G}^- T^0) \times (\tilde{G}^+ V \tilde{G}^- T^0)^n | \mathbf{p} \rangle u_p, \quad (9)$$

$$\tilde{G}^\pm(\mathbf{k}) = \tilde{Q}_{\mathbf{kp}}{}^\pm / (k^2 - p^2 - i\varepsilon), \quad \tilde{Q}_{\mathbf{kp}}{}^\pm F \equiv \tilde{k} F \pm F \tilde{p} \quad (10)$$

(F is an arbitrary function).

When so written the wave function does not depend on the matrix β, and the dependence on the energy E comes in only through the parameter $\xi = \alpha Z E/p$ contained in the function T^0. The first term of the sum in (9) depends on an even number of matrices α, and the second on an odd number. Using the fact that the problem involves only the two vectors $\mathbf{p}$ and $\mathbf{k}$, by invariance arguments we can write

$$\sum_{n=0}^{\infty} (\alpha Z)^{2n} \langle \mathbf{k} | \varphi^0 (\tilde{G}^+ V \tilde{G}^- T^0)^n | \mathbf{p} \rangle = A_1 + (\alpha Z)^2 A_2 \tilde{k}\tilde{p}, \quad (11)$$

$$\alpha Z \sum_{n=0}^{\infty} (\alpha Z)^{2n} \langle \mathbf{k} | \tilde{G}^- T^0 (\tilde{G}^+ V \tilde{G}^- T^0)^n | \mathbf{p} \rangle = \alpha Z (A_3 \tilde{k} + A_4 \tilde{p}). \quad (12)$$

The invariant functions A_i are related to each other by the condition of symmetry with respect to time reversal, so that there are only three independent functions.

Each term of the series (11) and (12) has meaning for arbitrary p, as can be verified easily by writing out the integrals in the momentum space over the d^3k_i and making the change of variables $k_i = pn_i$. Then all of the integrals are still convergent and depend only on the direction of the momentum, $\mathbf{n}_0 = \mathbf{p}/p$ and the parameter $\xi = \alpha ZE/p$ in the function T^0, which is defined [see Eq. (25)] for all ξ.

At large energies $E \sim p \gg m$ and small $q = |\mathbf{k}-\mathbf{p}| \lesssim m$ ($h = |\mathbf{k}+\mathbf{p}| \lesssim m$) the expansion (12) contains, besides $(\alpha Z)^2$, an additional small parameter q/E (or h/E). The appearance of these small parameters can be understood from the following considerations. The expansion (9) differs from the Born series for φ^0 (see [1]) by the replacement of the quantity 2E in the numerator of the Green's function G^0 of Eq. (5) by $\widetilde{Q}_{kp\pm}$, Eq. (10), and replacement of some of the potentials V by the function T^0. At high energies all of the terms of the expansion for φ^0 are of the same order in the energy, because the parameter of the expansion for φ^0 is the quantity $\xi = \alpha ZE/p \approx \alpha Z$. For the same reason T^0 and V are of the same order in E. Taking into account the fact that the main contributions to all the integrals come from the region $\mathbf{k}_i \sim \mathbf{k}$, we conclude that the presence of $\widetilde{G}^-$ (or $\widetilde{G}^+$) in (9) corresponds to the appearance of an additional small parameter q/E (or h/E).

The quantity q/E is small at small angles, and the quantity h/E is small at angles close to 180°. At high energies most of the processes occur mainly at small angles, and the cross section at large angles is small and of no practical interest. In the intermediate range of angles the two quantities q/E and h/E are of the order of unity and give no small factors. A more detailed analysis shows that the effective small factor that appears at small angles is of the order of q ln E/E.

Thus the effective small parameter in the expansion (9) at small angles is the quantity $(\alpha Z)^2 q \times \ln E/E$. We hope to give a rigorous proof of this statement in another place.

The meaning of the expansion we obtain becomes clearer on comparison with the well known properties of the expansion of the Coulomb function in partial waves.[2,3] The dependence on Z and E comes into the partial waves through the two parameters $\xi = \alpha ZE/p$ and $\gamma_l = (l^2 - \alpha^2 Z^2)^{1/2}$. Terms of the Born series that diverge for $p \to 0$ arise only from the expansion in the parameter ξ. The fact that (9) does not depend explicitly on E and that all of the terms of the expansion are finite for $p \to 0$ indicates that the series (9) does not contain any expansion with respect to the parameter ξ. Consequently the expansion (9) is taken only with respect to the parameter[1)] $(\alpha Z)^2/l$. The effective value of l is equal to pr, where $r \sim 1/q$, so that $l \sim E/q$. This leads to the expansion parameter which we have obtained.

Everything that has been said applies also to the expansion (6) of the Møller function. To verify this, we write out the expression for the product $G^+(\mathbf{k})G^-(\mathbf{k}')$ (we are interested only in the matrix structure, and therefore we omit V and T^0):

$$G^+(\mathbf{k})\,G^-(\mathbf{k}') = \frac{\hat{k}\hat{k}' - p^2 - (\hat{k} - \hat{k}')(E - \beta m)}{(k^2 - p^2 - i\varepsilon)(k'^2 - p^2 - i\varepsilon)} \tag{13}$$

Owing to the commutation rules (8) of the matrices α and β and the equation

$$(E + \beta m)(E - \beta m) = p^2 \tag{14}$$

the product of an arbitrary number of the combinations (13) will contain $(E - \beta m)$ only to the first degree. Therefore the general expression for the Møller function can be written in the form

$$\langle \mathbf{k}_2|\varphi|\mathbf{k}_1\rangle = B_1 + (\alpha Z)^2 B_2\hat{k}_2\hat{k}_1 + \alpha Z(B_3\hat{k}_2 + B_4\hat{k}_1) + \{B_1' + (\alpha Z)^2 B_2'\hat{k}_2\hat{k}_1 + \alpha Z(B_3'\hat{k}_2 + B_4'\hat{k}_1)\}(E - \beta m). \tag{15}$$

Here the functions B_i and B_i' depend on E only through the parameter ξ.

The terms of the expansion of B_i in powers of $(\alpha Z)^2$ have meaning for arbitrary p and contain the additional small parameter $|\mathbf{k}_2 - \mathbf{k}_1|/E$; when (13) and (14) are taken into account this can be verified in precise analogy with the case of the wave function. By means of Eq. (1) we can express the functions B_i' in terms of the B_i. The condition of symmetry under time reversal imposes one further relation between the functions B_i, so that only three independent functions B_i remain.

The corresponding representation for the relativistic Coulomb Green's function G_C and the scattering amplitude T can be obtained from (6) by means of the relations

$$G_C = \varphi G^+, \quad T = V\varphi, \quad \varphi = 1 - \alpha Z G^+ T. \tag{16}$$

In the Furry-Sommerfeld-Maue approximation the wave function $|\varphi_p\rangle$, the Møller function φ, and the Green's function G_C can be represented in the form[1]

1)See the note added in proof on page 773 in[2], where it is indicated that the expansion parameter is the quantity $(\alpha Z)^2/l$, and not $(\alpha Z)^2/l^2$.

$$\langle \mathbf{k} | \varphi_p{}^{\mathrm{FSM}} \rangle = \langle \mathbf{k} | \varphi_p{}^0 \rangle + \alpha Z \langle \mathbf{k} | \varphi_p{}^1 \rangle = \frac{N}{2\pi^2} \left(-\frac{\partial}{\partial \eta} - \alpha Z \frac{\widetilde{\nabla}_p}{2i\xi} \right) \langle \mathbf{k} | \Phi_p(i\eta) \rangle |_{\eta \to 0}, \tag{17}$$

$$\langle \mathbf{k} | \Phi_p(i\eta) \rangle \equiv \frac{1}{q^2+\eta^2} \left\{ \frac{q^2+\eta^2}{(\mathbf{q}+\mathbf{p})^2 - (p+i\eta)^2} \right\}^{i\xi}, \tag{18}$$

$$\mathbf{q} = \mathbf{k} - \mathbf{p}, \quad N = e^{\pi\xi/2}\Gamma(1-i\xi);$$

$$\varphi^{\mathrm{FSM}} = \varphi^0 + \alpha Z \varphi^1 = 1 - \alpha Z G^+ T^0, \tag{19}$$

$$G^{\mathrm{FSM}} = \varphi^{\mathrm{FSM}} G^+ = G^+ - \alpha Z G^+ T^0 G^+. \tag{20}$$

The operator ∇_p in (17) does not act on q and ξ.

By means of the representation for φ^0 [see Eq. (13b) in [1]] the third and fourth terms of the series for the wave function (9) can be written in the form

$$\langle \mathbf{k} | \varphi^0 \tilde{G}^+ V | \varphi_p{}^1 \rangle = -\frac{\partial}{\partial \eta} S(\eta) |_{\eta \to 0}, \tag{21}$$

$$\langle \mathbf{k} | \varphi^1 \tilde{G}^+ V | \varphi_p{}^1 \rangle = \tilde{G}^-(\mathbf{k}) S(0), \tag{22}$$

$$S(\eta) = \langle \mathbf{k} | T_{i\eta}{}^0 \tilde{G}^+ V_0 | \varphi_p{}^1 \rangle, \quad T_{i\eta}{}^0 = V_{i\eta} \varphi^0, \tag{23}$$

$$\langle \mathbf{k}_2 | V_{i\eta} | \mathbf{k}_1 \rangle \equiv \frac{1}{2\pi^2} \frac{1}{(\mathbf{k}_2 - \mathbf{k}_1)^2 + \eta^2}, \quad V_0 \equiv V, \quad T_0{}^0 \equiv T^0. \tag{24}$$

The momentum representation $\langle \mathbf{k}_2 | T^0_{i\eta} | \mathbf{k}_1 \rangle$ with "free" left-hand momentum k_2 has been obtained in [1] by applying the procedure of displacement of the Born series, beginning at the right side, i.e., from the position of the momentum $\mathbf{k}_1$. This representation is

$$\langle \mathbf{k}_2 | T_{i\eta}{}^0 | \mathbf{k}_1 \rangle = \int_0 dx_1 \exp\left\{ i\xi \int_{x_1} \frac{dx_1'}{x_1' \Lambda_1'} \right\} \times \frac{\partial}{\partial x_1} (\langle \mathbf{k}_2 | V_{p\Lambda_1 + i\eta} | \mathbf{k}_1 x_1 \rangle), \tag{25}$$

$$\Lambda_1^2 = (1 - n_1^2 x_1)(1 - x_1), \quad \mathbf{n}_i = \mathbf{k}_i / p, \tag{26}$$

where the matrix element in the integrand is defined by (24).

This representation (25), however, is unsuitable for the calculation of the matrix element (23), where we must have available a "free" right-hand momentum $\mathbf{k}_1$. The required representation can be obtained by applying the procedure of displacement of the Born series for $\langle \mathbf{k}_2 | T^0_{i\eta} | \mathbf{k}_1 \rangle$, [1] beginning on the left, i.e., from the side of the momentum $\mathbf{k}_2$. We thus get

$$\langle \mathbf{k}_2 | T_{i\eta}{}^0 | \mathbf{k}_1 \rangle = \int_0 dx_2 \exp\left\{ i\xi \int_{x_2} \frac{dx_2'}{x_2' \Lambda_2'} \right\} \times \frac{\partial}{\partial x_2} (\langle \mathbf{k}_2 x_2 | V_{p\Lambda_2} | \mathbf{k}_1 \rangle), \tag{27}$$

$$\Lambda_2^2 = (1 - n_2^2 x)(1 - x) - \mu^2 x_2, \quad \mu = \eta / p. \tag{28}$$

The two representations can be reduced to each other by means of the rather nontrivial change of variables

$$x = \frac{1}{1+y_1}, \quad y_1 = (1 - n_1^2) \frac{z-1}{4z},$$

$$z = \frac{2 - (1 + n_2^2 + \mu^2) x_2 + 2\Lambda_2}{x_2 (1 - n_2^2 - \mu^2 + 2i\mu)}$$

$$(dx_1 / x_1 \Lambda_1 = dz / z = dx_2 / x_2 \Lambda_2). \tag{29}$$

By means of (27) we can put (23) in the form

$$S(\eta) = \int_0 dx_2 \exp\left\{ i\xi \int_{x_2} \frac{dx_2'}{x_2' \Lambda_2'} \right\} \times \frac{\partial}{\partial x_2} (\langle \mathbf{k}_2 x_2 | V_{p\Lambda_2} G^+ V_0 | \varphi_p{}^1 \rangle). \tag{30}$$

The matrix element in the integrand in (30) has been calculated previously, [4] and is of the form

$$\langle \mathbf{k}_2 x_2 | V_{p\Lambda_2} G^+ V_0 | \varphi_p{}^1 \rangle = \frac{N}{4i\pi^2} \int_0^\infty d\lambda \times \int_0^1 \frac{dx_3}{p\Lambda_3} \frac{y\tilde{k}_2 \tilde{c}_y + \tilde{c}_y \tilde{p}(1+\Lambda_3)}{k_2^2 y^2 - (p + p\Lambda_3 + i\lambda)^2} \langle \mathbf{k}_2 y | \Phi_p(p\Lambda_3 + i\lambda) \rangle,$$

$$\Lambda_3 = \Lambda_2 |_{x_2 = y}, \quad y = x_2 x_3, \tag{31}$$

$$\mathbf{c}_y = \mathbf{k} y - \mathbf{p} - (p\Lambda_3 + i\lambda) \mathbf{p} / p. \tag{32}$$

The symbol written in the integrand is defined by (18).

Substituting (31) in (30), replacing the variable x_3 by y, and denoting x_2 by x, we get

$$S(\eta) = \frac{N}{2\pi^2} \int_0^\infty d\lambda \int_0^1 \frac{dx}{\Lambda} \exp\left\{ i\xi \int_x^1 \frac{dx'}{x'\Lambda'} \right\} \times \frac{1}{2pi} \frac{x\tilde{k}_2 \tilde{c}_x + \tilde{c}_x \tilde{p}(1+\Lambda)}{k^2 x^2 - (p + p\Lambda + i\lambda)^2} \langle \mathbf{k}_2 x | \Phi_p(p\Lambda + i\lambda) \rangle, \tag{33}$$

$$\Lambda^2 = (1 - n_2^2 x)(1 - x) - \mu^2 x,$$

$$\mathbf{c}_x = \mathbf{k} x - \mathbf{p} - (p\Lambda + i\lambda) \mathbf{p} / p. \tag{34}$$

When in (33) we go over to the variable z of (29) it is not hard to verify that (33) has meaning for arbitrary p. Furthermore the main contribution to the integral (33) comes from $x \sim 1$ and $\lambda \lesssim q$. Taking into account the fact that for large $E \sim p \gg m$ the part of the integrand in (33) that is after the factor 1/2pi is of the order of magnitude of unity, we come to the conclusion that the entire integral is of the order q ln E/E (the ln E comes from the integration of dx/Λ for $x \sim 1$). We note that the smallness so obtained is lost when we replace the c_x of (34) by 2E.

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