SOME FEATURES OF THE ELECTRICAL CONDUCTIVITY OF METALS AT LOW TEMPERATURES

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Submitted to JETP editor April 4, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) 47, 1415-1426 (October, 1964)

The electrical conductivity of metals at low temperatures is treated for the case where, aside from scattering of electrons by static lattice defects and sample boundaries, the only processes with any sizable probability are collisions between electrons and phonons without umklapp. The effect of such collisions is very dependent on the nature of the residual resistance. If scattering of electrons at sample boundaries (or any other macroscopic objects) predominates, then under certain conditions the conduction process resembles the flow of a viscous charged fluid. The dependence of the resistance on the temperature and the transverse dimensions of the sample is then peculiar and, in particular, there should be a temperature minimum in the resistance. A different situation arises when the residual resistance results from microscopic lattice defects like impurity atoms. In this case a plateau appears in the plot of resistance vs. temperature.

1. EFFECT OF DIFFERENT ELECTRON SCATTERING MECHANISMS ON LOW-TEMPERATURE CONDUCTIVITY

IT is known that at sufficiently low temperatures the electrical resistance is usually constant and is determined by the scattering of electrons by impurity atoms and other structural defects of the lattice as well as at sample boundaries. The temperature dependence results from the different interactions in the electron-phonon system. Usually one uses the formula

$$\rho(T) = \rho_0 + \alpha T^2 + \beta T^5, \qquad (1)$$

where ρ_0 is the residual resistance, the second term is related to electron-electron and the third to electron-phonon collisions; for simplicity we have omitted tensor indices. In the following we shall be considering single crystal samples which are sufficiently pure and large so that the residual resistance is reached only at helium temperatures.

A. Electron-electron collisions. A Born approximation calculation shows that at helium temperatures the dependence of the resistance should be given by the second term in (1), i.e., $\rho(T) - \rho_0 \sim T^2$ (Landau and Pomeranchuk,[1] cf. also Ginzburg and Silin [2]). In principle there is no question that such a dependence exists. But if we exclude the transition metals this dependence is not observed experimentally. The problem reduces to estimating the numerical coefficient of T^2.

First we note that the Born approximation gives somewhat too high a value for the electron-electron scattering cross section (according to Abrahams,[3] about a factor of 4 too large). In addition one should take account of the fact that the only collisions that contribute directly to the resistance are those accompanied by umklapp (the so-called U-processes). Obviously such collisions are possible only between electrons located in certain definite regions which are cut out of the Fermi surface by planes perpendicular to the umklapp vector and bisecting the line from the center of the Brillouin zone to its boundary. The most important point is probably the fact that the matrix element for the U-process contains an interference factor which is small compared to unity (cf. Ziman,[4] pp. 181, 369). We note that N-processes (i.e. collisions without umklapp) can also give a term proportional to T^2 if the Fermi surface is nonspherical (analogous to the case of Sec. 3 of the present paper).

From our remarks it is clear that at present theoretical arguments do not permit a sufficiently reliable estimate of the order of magnitude of the coefficient α in (1), nor do they say what experimental accuracy is needed to observe the quadratic dependence of the resistance in any particular metal. In this connection it is of interest to make a comparison with data on infrared absorption of metals. According to [5], the effective frequency

of electron-electron collisions can be written in the following form, valid for both the static and high-frequency cases:

$$\tau_{ee}^{-1} = \tau_0^{-1}[(T/\varepsilon_0)^2 + (\hbar\omega/2\pi\varepsilon_0)^2] \quad (2)$$

(where τ_{ee} is the free time between collisions, ϵ_0 is the limiting energy, and ω is the frequency of the electromagnetic wave). Thus electron-electron collisions in the near infrared region ($\lambda = 2\pi c/\omega \approx 1$—$20\ \mu$, $\hbar\omega \approx 10^3 - 10^4$ °K) are therefore $10^3 - 10^5$ times more probable than in the static case at helium temperatures.

For a not too dirty metal the infrared absorption coefficient can be written as

$$A = A_0 + 2/\omega_0\tau_{ee}, \quad (3)$$

where the quantity A_0, which is independent of ω and T, is related to collisions of the electrons with phonons and sample boundaries, $\omega_0 = (4\pi ne^2/m)^{1/2}$ is the plasma frequency, n is the electron density, e the charge and m the electron mass. On the other hand, at helium temperatures, if we consider the electron-electron collisions as a small contribution along with the impurity scattering, the static resistance can be written as (cf. Sec. 3)

$$\rho = 4\pi\omega_0^{-2}[\tau_{ei}^{-1} + \tau_{ee}^{-1}], \quad (4)$$

where τ_{ei}^{-1} is the effective frequency of collisions with impurities. The important point is that (3) and (4) are obtained under analogous assumptions, the first for $\omega\tau_{ee} \gg 1$, the second for $\tau_{ei}^{-1}\tau_{ee} \gg 1$. This is the reason why (2) is applicable to both cases. True, the interaction with impurities is described by an operator whereas ω is a number, and moreover in the infrared region, unlike the static case, there are Fermi liquid effects. But apparently neither of these has a strong influence on the value of τ_0.

A quadratic dependence $A(\omega)$ for the noble metals was seen by Biondi [6] (cf. also Dingle [7]). Comparison of these results with formulas (2), (3) and (4) shows that at helium temperatures the αT^2 term in (1) should be only a few percent of the residual resistance ρ_0 even in the purest samples, where the mean free path for impurity collisions is $l_{ei} \sim 0.1$ cm.

B. Interaction with phonons. The T^5 law for the resistance was found by Bloch on the assumption that the phonons are in equilibrium, i.e., there is some mechanism for dissipation of the quasimomentum in the phonon system. As Peierls showed, one such mechanism is phonon-phonon collisions with umklapp. But as the temperature is lowered the probability for these U-processes falls exponentially like $e^{-\gamma\Theta/T}$, where Θ is the Debye temperature. The numerical coefficient γ is difficult to estimate since it is determined by the behavior of the acoustic branches for sizable values of the quasimomentum. Crudely speaking, $2\gamma\Theta$ is equal to the smallest value of the sum of the energies of the phonons participating in the collision, when we take account of the appropriate momentum conservation law for the U-process. It is clear that for each metal there should be some definite temperature below which the U-processes can no longer guarantee equilibrium of the phonons, and the corresponding contribution to the resistance must drop exponentially. From a comparison with data on thermal conductivity of dielectrics (Peierls [8]) it follows that such a situation should occur over a quite wide range of low temperatures, say for $T/\Theta \lesssim 1/10$.

According to Klemens,[9] U-processes are important down to much lower temperatures. But the treatment given in his paper is hardly justified. We also note that, contrary to Klemens' assertion, the collisions of phonons with metal boundaries cannot guarantee the validity of the Bloch solution at the very lowest temperatures. In fact, when $T \ll \Theta$ the electron-electron free path is much greater than the electron-phonon free path, so that with decreasing temperature the electrons "feel" the presence of the boundaries earlier than do the phonons.

In principle, there is still another possibility, which was also pointed out by Peierls.[8] This is electron-phonon collisions with umklapp. The question whether such U-processes together with N-processes can assure a stationary state of the electron-phonon system in an electric field is related essentially to the topology of the Fermi surface and the direction of the field. According to Peierls, it is necessary that certain sections of the Fermi surface by planes parallel to the field be open. This question requires special consideration, but it is clear beforehand that in many metals such processes are surely unimportant at low temperatures. This is the case, for example, for the alkali metals and aluminum. In the following we shall not consider electron-phonon U-processes.

C. Formulation of the problem. From our remarks it is clear that in many metals, at sufficiently low temperatures, in addition to scattering of electrons by various static objects, the only process with significant probability is the collision of electrons and phonons. The study of the electrical conductivity of metals in this case is the object of our remaining discussion.

We denote by l_{ep} and l_{pe} respectively the effective free paths for scattering of electrons by phonons and of phonons by electrons. The essential point is that although both free paths are related to the same interaction processes, at low temperatures we always have $l_{ep} \gg l_{pe}$ (cf. Sec. 2). Thus the phonon radiated by an electron, after a very small path and consequently very short time will be absorbed by another electron. Thus we are in fact dealing with electron-electron collisions, which are characterized by the free path l_{ep}.

Since quasimomentum is conserved in these collisions, they do not give rise directly to an electrical resistance. It turns out, however, that these collisions can have a very important influence on the process of transfer of momentum from the electron gas to static objects, and thus change the resistance. The mechanism for this is determined by the nature of the residual resistance. More precisely, everything depends on the behavior of the free path l_{ep} and the characteristic separation between the static objects, d, at those temperatures where the probabilities for electron-electron collisions and collisions with static objects become comparable.

If the residual resistance is connected with scattering of electrons by macroscopic objects (sample boundaries, dislocations, etc), the inequality $l_{ep} \ll d$ may be satisfied. A special mechanism then arises for the conductivity, which reminds one of the Poiseuille flow of a viscous fluid. The resistance depends in an unusual way on the parameters d and T, and in particular there should be a minimum in the temperature dependence $\rho(T)$ (cf. Sec. 2).

But if the residual resistance is caused by microscopic lattice defects (like impurity atoms), then in the temperature region of interest we always have $l_{ep} \gg d$. Here N-processes occur only if the electron dispersion law is anisotropic. The resistance first increases with temperature, then saturates, and then again begins to grow (Sec. 3).

2. ELECTRICAL CONDUCTIVITY OF THIN SAMPLES

In this section for concreteness we shall consider the case where the residual resistance arises because of scattering of electrons by the sample boundaries, although it will be clear that the results are qualitatively correct for any macroscopic defects.

In order of magnitude,

$$l_{ep} \approx a\frac{\varepsilon_0}{\Theta}\left(\frac{\Theta}{T}\right)^5, \qquad l_{pe} \approx a\frac{\varepsilon_0}{\Theta}\frac{\Theta}{T}, \tag{5}$$

where a is the lattice constant (cf. Peierls [10]). As already remarked, from the inequality $l_{ep} \gg l_{pe}$ it follows that the electron-phonon interactions practically reduce to electron-electron collisions, characterized by the free path $l_{ep}(T)$.

To understand the physical reason for the influence of normal collisions on the electrical conductivity we consider the limiting cases: $l_{ep} \gg d$ and $l_{ep} \ll d$. In the first case the electron, between two collisions with the walls, covers a distance d with essentially no collisions in the interior. Consequently $\rho \sim d^{-1}$, and is independent of the temperature. If $l_{ep} \ll d$, the electron in the bulk of the sample undergoes a large number of collisions with other electrons before reaching the wall. It is clear that the path covered by the electron between two collisions with the boundaries is markedly increased. During all this time the electron is accelerated by the electric field. The fact that in the collisions there is an exchange of momentum with other electrons does not change the situation.[1)] Using the standard formulas for Brownian motion it is easy to show that the length of the trajectory between two collisions with the boundary is of order d^2/l_{ep}. Since this quantity has the meaning of an effective mean free path, $\rho \sim l_{ep}/d^2$. Thus, in contrast to the Knudsen case, the resistance contains the square of the sample thickness and, most important, has an unusual temperature dependence—together with the free path l_{ep} it drops with increasing temperature.

Starting from these qualitative arguments it is now easy to find the qualitative behavior of the resistance (cf. Fig. 1). At very low temperatures, when $l_{ep} \gg d$, the resistance is roughly speaking independent of temperature[2)] and is proportional to d^{-1}. In addition, starting from the temperature where $l_{ep} \approx d$, the resistance drops with increasing T according to the law $\rho \sim T^{-5}d^{-2}$ (for

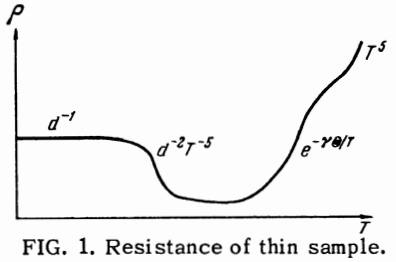

FIG. 1. Resistance of thin sample.

[1)] Similar arguments were presented briefly by the author [11] for the case of the ordinary electron-electron N-collisions.

[2)] The relatively small increase of the resistance in this region may be related to the fact that volume collisions limit the free path for electrons which move almost parallel to the boundaries of the sample (cf., for example, [12]).

$l_{ep} \ll d$). This drop will continue so long as d^2/l_{ep} is not comparable to one of the lengths which characterize the volume collisions with momentum loss. If this is the impurity length the resistance will remain constant and equal to the resistance for an infinitely large sample. A rise in the resistance should occur later because of phonon-phonon U-processes. The corresponding free path l^U varies essentially as $e^{-\gamma\Theta/T}$ and will be estimated below. There should be an exponential increase starting from the temperature where $l_{ei} \approx l^U(T)$, up to the temperature where $l^U(T) \approx l_{ep}(T)$, and then the resistance should vary as T^5.

Let us formulate the conditions for the appearance of the diffusion mechanism for the electrical conductivity:

a) the sample thickness should be small compared to the electron-impurity free path: $d \ll l_{ei}$;

b) the phonon-phonon U-processes should still be unimportant when $l_{ep}(T) \approx d$. More precisely, the inequalities $l_{ep}(T) \ll d \ll l^U(T)$ should be satisfied in a certain temperature range;

c) electron-phonon U-processes are unimportant, which is surely true for the case of closed Fermi surfaces (cf. Sec. 1).

For a rough orientation we note that, when $l_{ei} \sim 1$ cm and $d \approx 10^{-2}$—10^{-3} cm, condition b) should be satisfied over the region between helium and hydrogen temperatures, though the situation depends on the precise properties of the metal.

For a quantitative description of this mechanism it is natural to use a hydrodynamic approach. This method is especially suited to the present case, since it enables us to take account of bulk collisions with momentum loss (when they are less probable than N-processes), as well as the influence of the boundaries. The hydrodynamic equation for the electron-phonon gas can be obtained in the usual fashion, starting from the corresponding kinetic equations (cf, for example, Landau and Lifshitz [13]). We shall write the equations in the form

$$\mathbf{v}\partial f/\partial \mathbf{r} + e\mathbf{E}\mathbf{v}\partial f_0/\partial\varepsilon = \hat{J}_{ep}\{f, N\} + \hat{J}_{ei}\{f, N\},$$

$$\mathbf{s}\partial N/\partial \mathbf{r} = \hat{J}_{pe}\{f, N\} + \hat{J}_{pp}^{N}\{N\} + \hat{J}_{pp}^{U}\{N\}. \qquad (6)$$

Here $f(\mathbf{p}, \mathbf{r})$ and $N(\mathbf{q}, \mathbf{r})$ are the distribution functions for the electrons and phonons;

$$f_0(\varepsilon) = [1 + \exp\{(\varepsilon - \varepsilon_0)/T\}]^{-1},$$

$$\mathbf{v} = \partial\varepsilon_{\mathbf{p}}/\partial\mathbf{p},\ \mathbf{s} = \partial(h\nu_{\mathbf{q}})/\partial\mathbf{q};$$

$h\nu_{\mathbf{q}}$ is the phonon energy; $\hat{J}_{ep}$ is the electron-phonon collision operator, $\hat{J}_{pe}$ the collision operator for phonons with electrons, $\hat{J}_{ei}$ for electrons with impurities; $\hat{J}_{pp}^{N}$ and $\hat{J}_{pp}^{U}$ respectively describe phonon-phonon N- and U-processes. The scattering of phonons by impurities is not taken into account. We know that at low temperatures the probability for this process is roughly $(T/\Theta)^4$ times smaller than the probability for scattering of electrons by impurities.[3)] In addition, as will become clear from the sequel, the contribution of phonon-impurity scattering to the hydrodynamic equation is further reduced by an additional factor $(v/s)(T/\Theta)^4$.

For an approximate solution of (6) it is natural to use the fact that N-processes are the most probable. The solution will be in the form of a series in the small parameters l_{ep}/d, l_{ep}/l_{ei} and l_{ep}/l^U. The method of successive approximations leads to the following system of equations:

$$\hat{J}_{ep}\{f^{(0)}, N^{(0)}\} = 0, \qquad \hat{J}_{pe}\{f^{(0)}, N^{(0)}\} + \hat{J}_{pp}^{N}\{N^{(0)}\} = 0; \quad (7)$$

$$\mathbf{v}\partial f^{(0)}/\partial\mathbf{r} = \hat{J}_{ep}\{f^{(1)}, N^{(1)}\},$$

$$\mathbf{s}\partial N^{(0)}/\partial\mathbf{r} = \hat{J}_{pe}\{f^{(1)}, N^{(1)}\} + \hat{J}_{pp}^{N}\{N^{(1)}\}; \quad (8)$$

$$\mathbf{v}\partial f^{(1)}/\partial\mathbf{r} + e\mathbf{E}\mathbf{v}\partial f_0/\partial\varepsilon = \hat{J}_{ep}\{f^{(2)}, N^{(2)}\} + \hat{J}_{ei}\{f^{(0)}\},$$

$$\mathbf{s}\partial N^{(1)}/\partial\mathbf{r} = \hat{J}_{pe}\{f^{(2)}, N^{(2)}\} + \hat{J}_{pp}^{N}\{N^{(2)}\} + \hat{J}_{pp}^{U}\{N^{(0)}\}. \quad (9)$$

From the pair of equations (7) it follows that $f^{(0)} = f_0(\epsilon - \mathbf{p}\cdot\mathbf{u})$ and $N^{(0)} = N_0(h\nu - \mathbf{u}\cdot\mathbf{q})$, where N_0 is the Bose distribution function and $\mathbf{u}(\mathbf{r})$ represents the drift velocity of the equilibrium electron-phonon system as a whole.

To solve Eqs. (8), which now take the form

$$-v_i p_k \frac{\partial u_k}{\partial x_i}\frac{\partial f_0}{\partial\varepsilon} = \hat{J}_{ep}\{f^{(1)}, N^{(1)}\},$$

$$-s_i q_k \frac{\partial u_k}{\partial x_i}\frac{\partial N_0}{\partial h\nu} = \hat{J}_{pe}\{f^{(1)}, N^{(1)}\} + \hat{J}_{pp}^{N}\{N^{(1)}\}, \quad (8')$$

we must use the explicit form of the collision integrals. We write the Hamiltonian for the electron-phonon interaction in the form

$$\hat{H} = \sum_{\mathbf{p},\mathbf{q}} A(\mathbf{p}, \mathbf{q})\, a_{\mathbf{p}}^{+} a_{\mathbf{p}+\mathbf{q}} b_{\mathbf{q}}^{+} + \text{c.c.},$$

where $a_{\mathbf{p}}$ and $b_{\mathbf{q}}$ are Fermi and Bose operators. For the following it is important that for small $|\mathbf{q}|$ we have $|A(\mathbf{p}, \mathbf{q})|^2 \sim |\mathbf{q}|$. It is easily

[3)] The situation is different if the metal contains a considerable quantity of isotopes. But the probability of scattering of phonons in this case contains the additional small factor $(\Delta M/M)^2$ (where ΔM is the mass difference between the isotope and the main constituent atoms). The direct scattering of electrons from isotopes, caused by the inhomogeneity of the zero-point oscillations (Pomeranchuk [14]), may be even more important at helium temperatures.

shown that in the approximation linear in $f^{(1)}$ and $N^{(1)}$,

$$\hat{J}_{ep}\{f^{(1)}, N^{(1)}\} = \int d\mathbf{q}\, L_{\mathbf{p},\mathbf{q}}^{\mathbf{p}+\mathbf{q}}(\varphi_{\mathbf{p}+\mathbf{q}} - \varphi_{\mathbf{p}} - \psi_{\mathbf{q}}) + \int d\mathbf{q}\, L_{\mathbf{p}}^{\mathbf{p}-\mathbf{q},\mathbf{q}}(\varphi_{\mathbf{p}-\mathbf{q}} - \varphi_{\mathbf{p}} + \psi_{\mathbf{q}}),$$

$$\hat{J}_{pe}\{f^{(1)}, N^{(1)}\} = 2\int d\mathbf{p}\, L_{\mathbf{p}}^{\mathbf{p}-\mathbf{q},\mathbf{q}}(\varphi_{\mathbf{p}-\mathbf{q}} - \varphi_{\mathbf{p}} - \psi_{\mathbf{q}}),$$

$$L_{\mathbf{p},\mathbf{q}}^{\mathbf{p}+\mathbf{q}} = \frac{2\pi}{\hbar}\frac{1}{h^3}|A(\mathbf{p},\mathbf{q})|^2\delta(\varepsilon_{\mathbf{p}+\mathbf{q}} - \varepsilon_{\mathbf{p}} - h\nu_{\mathbf{q}}) f_0(\varepsilon_{\mathbf{p}}) N_0(h\nu_{\mathbf{q}}) \times [1 - f_0(\varepsilon_{\mathbf{p}+\mathbf{q}})],$$

$$L_{\mathbf{p}}^{\mathbf{p}-\mathbf{q},\mathbf{q}} = \frac{2\pi}{\hbar}\frac{1}{h^3}|A(\mathbf{p}-\mathbf{q},\mathbf{q})|^2\delta(\varepsilon_{\mathbf{p}-\mathbf{q}} - \varepsilon_{\mathbf{p}} + h\nu_{\mathbf{q}}) \times f_0(\varepsilon_{\mathbf{p}})[1 - f_0(\varepsilon_{\mathbf{p}-\mathbf{q}})][1 + N_0(h\nu_{\mathbf{q}})],$$

$$f^{(1)} = -T\frac{\partial f_0}{\partial\varepsilon}\varphi_{\mathbf{p}}, \qquad N^{(1)} = -T\frac{\partial N_0}{\partial h\nu_{\mathbf{q}}}\psi_{\mathbf{q}}. \tag{10}$$

The phonon-phonon collision integral has not been written since the term $J_{pp}^{N}\{N^{(1)}\}$ can be neglected compared to $\hat{J}_{pe}\{f^{(1)}, N^{(1)}\}$. As the analysis shows, the ratio of these terms is of order $(T/\Theta)^4$, a result that could have been foreseen from the ratio of the free paths:

$$l_{pe} \sim a\frac{\varepsilon_0}{\Theta}\frac{\Theta}{T}, \qquad l_{pp^N} \sim a\frac{Ms^2}{\Theta}\left(\frac{\Theta}{T}\right)^5$$

(cf, for example, [15]).

Now the second of Eqs. (8′) can be solved for $\psi_{\mathbf{q}}$ (cf. (10)) and the result substituted in the first equation. The equation obtained for the function $\varphi_{\mathbf{p}}$ can be solved in principle for the case of arbitrary dispersion laws for the electrons and phonons. We shall, however, limit ourselves to the isotropic case, because the result of the general treatment still contains undetermined numerical coefficients of order unity. The angular dependence is then obviously eliminated, and the solution is conveniently sought in the form

$$f^{(1)} = -\frac{\partial f_0}{\partial\varepsilon}v_i p_k\frac{\partial u_k}{\partial x_i}\tau_0\left(\frac{\Theta}{T}\right)^3\left(\frac{\varepsilon_{\mathbf{p}} - \varepsilon_0}{T}\right),$$

where τ_0 is of the same order of magnitude as the electron-phonon mean free time when $T = \Theta$. After rather involved transformations, the equation for χ can be written in the form

$$\int_{-\infty}^{+\infty} K_{xx'}\chi(x')\,dx' + \left(\frac{T}{\Theta}\right)^2\int_{-\infty}^{+\infty} Q_{xx'}\chi(x')\,dx = e^x f_0^2(x), \tag{11}$$

where

$$K_{xx'} = \delta(x - x')\int_{-\infty}^{+\infty} R_{xx''}\,dx'' - R_{xx'} + S_{xx'}^{(1)},$$

$$Q_{xx'} = (x - x')^2 R_{xx'} + S_{xx'}^{(3)},$$

$$R_{xx'} = (x - x')^2 f_0(x) f_0(x')\,|e^{-x} - e^{-x'}|^{-1},$$

$$S_{xx'}^{(n)} = f_0(x) f_0(x')\exp(x + x') \times\int_{-\infty}^{+\infty}|z|^n f_0(x + z)[e^z f_0(x' + z) - f_0(x' - z)]\,dz,$$

$$f_0(x) = (1 + e^x)^{-1}, \qquad x = (\varepsilon_p - \varepsilon_0)/T.$$

Unimportant numerical factors of order unity have been dropped. The important thing for us is that all the kernels, K, Q, R and $S^{(n)}$ are symmetric, and

$$\int_{-\infty}^{+\infty} K_{xx'}\,dx' = 0, \qquad \int_{-\infty}^{+\infty} S_{xx'}^{(n)}\,dx' = 0. \tag{12}$$

Proceeding to an approximate solution of (11), we note that if we drop the term with $(T/\Theta)^2$ the resulting equation is not solvable. In fact, because of the symmetry of the kernel $K_{xx'}$ and the first of Eqs. (12), the corresponding homogeneous transposed equation has the solution $\chi = \text{const}$, which is not orthogonal to the right hand side, $e^x f_0^2(x)$. Therefore the successive approximation method must be developed as follows:

$$\int_{-\infty}^{+\infty} K_{xx'}\chi^{(0)}(x')\,dx' = 0,$$

$$\int_{-\infty}^{+\infty} K_{xx'}\chi^{(1)}(x')\,dx' = e^x f_0^2(x) - \left(\frac{T}{\Theta}\right)^2\int_{-\infty}^{+\infty} Q_{xx'}\chi^{(0)}(x')\,dx', \ldots$$

From the first equation it follows that $\chi^{(0)} = \text{const}$. The condition for solvability of the second equation for $\chi^{(1)}$ gives $\chi^{(0)} = \alpha(\Theta/T)^2$, where

$$\alpha^{-1} = \int_{-\infty}^{+\infty}\int Q_{xx'}\,dx\,dx' = \int_{-\infty}^{+\infty}\int (x - x')^2 K_{xx'}\,dx\,dx'$$

$$\approx 240\,\zeta(5) \approx 250.$$

Finally we multiply the first of Eqs. (9) by $\mathbf{p}$, the second by $\mathbf{q}$, integrate and add. Then the terms containing $f^{(2)}$ and $N^{(2)}$ cancel and we get the required hydrodynamic equation for $\mathbf{u}(\mathbf{r})$. To terms of higher order in T/Θ we find

$$e\mathbf{E}/m = -\nu\Delta\mathbf{u} + \mathbf{u}/\tau_V, \tag{13}$$

where

$$\nu \approx {}^1/_{15}v_0 l_{ep}(T), \qquad 1/\tau_V = 1/\tau^U + 1/\tau_{ei},$$

$$\tau^U \approx (s/v_0)(\Theta/T)^4\tau_{pp}{}^U, \quad \tau_{pp}{}^U \approx a(Ms/\Theta)(T/\Theta)^{5/2}e^{\gamma\Theta/T}$$

(cf., for example, [15]).

The quantity $l_{ep}(T)$ coincides precisely with the mean free path which appears in the Bloch theory of electrical conductivity. As was to be expected from qualitative considerations, the kinematic viscosity ν is determined in first ap-

proximation by the electron-phonon length l_{ep} alone. It is not difficult to show that the correction for the fact that l_{pe} is finite is of order $\nu(T/\Theta)^8$. The effect of phonon-phonon U-processes is reduced by a factor $(v_0/s)(T/\Theta)^4$, which obviously applies to all collisions occurring in the phonon system.

The solution of Eq. (13) must satisfy the boundary condition $\mathbf{u}(\mathbf{r}) = 0$ at the sample boundaries, which corresponds to diffuse scattering of the electrons. As is easily verified, the electric current density $\mathbf{j}(\mathbf{r}) = ne\mathbf{u}(\mathbf{r})$, where $n = \frac{8}{3}\pi(p_0/h)^3$ is the electron density. In computing the resistance this expression should be averaged over the cross section of the sample.

As an example we give the results for samples in the shape of plates and cylinders. If we write the electrical conductivity coefficient in the usual form: $\rho^{-1} = ne^2 p_0^{-1} l_{eff}$, then for plates

$$l_{eff} = l_V(1 - z^{-1}\tanh z)$$

and for a cylinder,

$$l_{eff} = l_V(1 - 2z^{-1}I_1(z)/I_2(z)).$$

Here $z = \frac{1}{2}d(\nu\tau_V)^{1/2}$, $l_V = v_0\tau_V$, d is the plate thickness of the cylinder diameter; $I_n(z)$ is the Bessel function of imaginary argument ($I_n(z) = i^{-n}J_n(iz)$). We note that the formulas given differ only superficially, and actually give very similar results. Thus, for $z \ll 1$, for a plate $l_{eff} \approx \frac{5}{4}d^2/l_{ep}$, and for a cylinder $l_{eff} = \frac{15}{32}d^2/l_{ep}$, while for $z \gg 1$ in both cases $l_{eff} \approx l_V$, which corresponds to a massive sample. Thus the function $\rho(T, d)$ is insensitive to the shape of the sample and has the form predicted earlier from qualitative arguments. Here d is to be regarded as some characteristic transverse dimension of the sample or as the distance between macroscopic defects.

3. ELECTRICAL CONDUCTIVITY OF MASSIVE SAMPLES

The residual resistance of sufficiently massive metallic samples is as a rule associated with the scattering of electrons by microscopic defects of the crystal lattice such as impurity atoms. As already pointed out, the hydrodynamic case cannot occur, since the length l_{ep} is then always substantially larger than the average separation of atoms, Δ. In fact, even in the purest metals where the electron-impurity length $l_{ei} \sim 1$ cm, the quantity $\Delta \lesssim 10^{-5}$ cm ($l_{ei} \approx aC^{-1}$, $\Delta \approx aC^{-1/3}$, where C is the concentration of impurity atoms). Thus the presence of normal collisions in no way affects the probability for the electron to encounter an impurity atom, even if the former collisions occur much more frequently then the latter. But as we have seen, this does not mean that the normal collisions cannot affect the resistance, though the results of this influence cannot be interpreted as easily as in the case of thin samples.

Proceeding to the computations, we note that in this case the electron dispersion law need not be assumed to be isotropic. In fact, for $\epsilon = \epsilon(|\mathbf{p}|)$, $\mathbf{p} = m\mathbf{v}$, and it follows from the momentum conservation law that the normal collisions have no effect whatsoever on the electrical current $\mathbf{j} \sim e\Sigma\mathbf{v}$. We write the system of kinetic equations in the form

$$e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial\varepsilon} = \hat{J}_{ep}\{\varphi,\psi\} + WT\frac{\partial f_0}{\partial\varepsilon}\varphi_1, \qquad \hat{J}_{pe}\{\varphi,\psi\} = 0. \quad (14)$$

Here, in contrast to the preceding Section, the operator for collision with the impurities is denoted by $\hat{W}$. We assume as usual that these collisions are elastic. Then any function of the energy can be commuted through the operator $\hat{W}$; in particular,

$$\hat{W}\frac{\partial f_0}{\partial\varepsilon}\varphi_{\mathbf{p}} = \frac{\partial f_0}{\partial\varepsilon}\hat{W}\varphi_{\mathbf{p}}.$$

Furthermore, from the hermiticity of the corresponding interaction Hamiltonian it follows that the operator $\hat{W}$ is symmetric and real:

$$W_{\mathbf{p},\mathbf{p}'} = W_{\mathbf{p}',\mathbf{p}} = W^*_{\mathbf{p},\mathbf{p}'} \qquad \left(\hat{W}\varphi_{\mathbf{p}} = \int d\mathbf{p}'\, W_{\mathbf{p},\mathbf{p}'}\varphi_{\mathbf{p}'}\right).$$

Using the second of Eqs. (14), we eliminate $\psi_{\mathbf{q}}$ and write the equations for $\varphi_{\mathbf{p}}$ in expanded form (cf. (10)),

$$e\mathbf{E}\mathbf{v}\frac{\partial f_0}{\partial\varepsilon} - T\frac{\partial f_0}{\partial\varepsilon}\hat{W}\varphi_{\mathbf{p}}$$
$$= \int d\mathbf{q}\, L^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}\left[\varphi_{\mathbf{p}+\mathbf{q}} - \varphi_{\mathbf{p}} - \frac{1}{M(\mathbf{q})}\int d\mathbf{p}'\, L^{\mathbf{p}'+\mathbf{q}}_{\mathbf{p}',\mathbf{q}}(\varphi_{\mathbf{p}'+\mathbf{q}} - \varphi_{\mathbf{p}'})\right]$$
$$+ \int d\mathbf{q}\, L^{\mathbf{p}-\mathbf{q},\mathbf{q}}_{\mathbf{p}}\left[\varphi_{\mathbf{p}-\mathbf{q}} - \varphi_{\mathbf{p}} + \frac{1}{M(\mathbf{q})}\int d\mathbf{p}'\, L^{\mathbf{p}'+\mathbf{q}}_{\mathbf{p}',\mathbf{q}}(\varphi_{\mathbf{p}'+\mathbf{q}} - \varphi_{\mathbf{p}'})\right],$$

$$M(\mathbf{q}) = \int d\mathbf{p}\, L^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}. \quad (15)$$

Since this equation cannot be solved in the general case, we shall consider limiting cases.

A. We start with the region of very low temperatures, where the N-processes can be regarded as a small correction ($l_{ei} \ll l_{ep}(T)$). In zeroth approximation

$$\varphi_0 = T^{-1}(\partial f_0/\partial\varepsilon)\hat{W}^{-1}\mathbf{E}\mathbf{v}.$$

The corresponding current density

$$\mathbf{j}_0 = -\frac{2e^2}{h^3}\int d\mathbf{p}\,\mathbf{v}\frac{\partial f_0}{\partial\varepsilon}\hat{W}^{-1}\mathbf{E}\mathbf{v} \quad (16)$$

is independent of the temperature. The first approximation gives a current j_1 which is proportional to T^5.

To get the sign of the corrections to the principal values of the resistance tensor, we study the scalar $j_1 \cdot E$. This quantity can be represented in the form

$$\mathbf{j}_1\mathbf{E} = -\frac{2e^2}{h^3}\int d\mathbf{q}[(F, F) - (F, G)^2].$$

Here

$$F(\mathbf{p}, \mathbf{q}) = \hat{W}^{-1}\mathbf{E}\mathbf{v}_{\mathbf{p}+\mathbf{q}} - \hat{W}^{-1}\mathbf{E}\mathbf{v}_{\mathbf{p}}, \qquad G^{-2} = \int d\mathbf{p}\, L^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}},$$

where the parentheses denote the scalar product with the positive weighting function $L^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}}$,

$$(F, G) = \int d\mathbf{p}\, F(\mathbf{p}, \mathbf{q})\, G(\mathbf{p}, \mathbf{q})\, L^{\mathbf{p}+\mathbf{q}}_{\mathbf{p},\mathbf{q}},$$

so that, for example, $(G, G) = 1$.

Because of the Schwartz-Bunyakovskiĭ inequality the quantity $j_1 \cdot E$ is nonnegative, and one can show that it vanishes only for an isotropic dispersion law for the electrons. Thus the principal values of the resistance tensor vary according to the law $\rho = \rho_0 + \alpha T^5$, where $\alpha > 0$ for any anisotropic dispersion law.

B. From general arguments it is clear that such an increase of resistance with temperature cannot persist too long. In fact, with increasing temperature N-processes become the most probable, and they cannot of themselves give rise to resistance. Let us find the solution of (15) in this limiting case.

In zeroth approximation $\varphi' = -T^{-1}\mathbf{u} \cdot \mathbf{p}$. The vector $\mathbf{u}$, which has the significance of a drift velocity, can be found from the condition for solvability of the equation for the first approximation. If we choose the cartesian axes along the principal axes of the symmetric tensor

$$\gamma_{ik} = \int d\mathbf{p}\frac{\partial f_0}{\partial \varepsilon} p_i \hat{W} p_k,$$

then

$$u_i = e\gamma_{ii}^{-1}\int d\mathbf{p}\frac{\partial f_0}{\partial \varepsilon} p_i \mathbf{E}\mathbf{v}.$$

The corresponding current density

$$\mathbf{j}_0' = \frac{2e^2}{h^3}\int d\mathbf{p}\, \mathbf{v}(\mathbf{up})\frac{\partial f_0}{\partial \varepsilon}$$

is independent of T, but differs from j_0 (cf. (16)). The difference $j_0 \cdot E - j_0' \cdot E$ can be written in the form

$$\mathbf{j}_0\mathbf{E} - \mathbf{j}_0'\mathbf{E} = \frac{2e^2}{h^2}\left[(\Phi, \Phi) - \sum_{i=1}^{3}(\Phi, G_i)^2\right], \qquad (17)$$

where $\Phi(\mathbf{p}) = \mathbf{E} \cdot \mathbf{v}_{\mathbf{p}}$, $G_i(\mathbf{p}) = (\hat{W}p_i, \hat{W}p_i)^{-1/2}\hat{W}p_i$, and the scalar product is defined as follows:

$$(\Phi, G) = \int d\mathbf{p}\left(-\frac{\partial f_0}{\partial \varepsilon}\right)\Phi(\mathbf{p})\hat{W}^{-1}G(\mathbf{p}).$$

It is easy to verify that this definition satisfies all the necessary conditions. In particular, the commutativity follows from the symmetry and the "elasticity" of $\hat{W}$. One can verify that the norm is positive by using the fact that

$$W_{\mathbf{pp}'} = -B_{\mathbf{pp}'} + \delta(\mathbf{p} - \mathbf{p}')\int d\mathbf{p}'' B_{\mathbf{pp}''},$$

where the quantities $B_{\mathbf{pp}'} > 0$ are proportional to the squares of the matrix elements of the electron-impurity interaction. Then

$$(\Phi, \Phi) = \iint d\mathbf{p}\, d\mathbf{p}' B_{\mathbf{pp}'}[\tilde{\Phi}(\mathbf{p}) - \tilde{\Phi}(\mathbf{p}')]^2,$$

$$\tilde{\Phi}(\mathbf{p}) = (-\partial f_0/\partial \varepsilon)^{1/2}\hat{W}^{-1}\Phi(\mathbf{p}).$$

Since the functions $G_i(\mathbf{p})$ are orthonormal, the difference (17) is nonnegative and vanishes only in the case of an isotropic dispersion law. In the latter case, $G_i(\mathbf{p}) \sim p_i$ and the function $\Phi(\mathbf{p}) = m^{-1}E_i p_i$ can be expanded in the G_i. Thus in this temperature range the resistance in first approximation is constant and larger than the residual resistance. It is not difficult to show that in the next approximation one gets a correction proportional to T^{-5}.

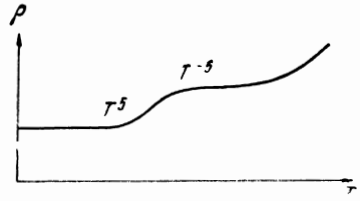

FIG. 2. Resistance of bulk sample.

For convenience we introduce the temperature T_0 at which the probabilities of electron-phonon and electron-impurity collisions are of the same order ($l_{ep}(T_0) \sim l_{ei}$). Then the results of this section can be written in the form

$$\rho(T) = \begin{cases} \rho_0[1 + \gamma(T/T_0)^5], & T \ll T_0 \\ \rho_0'[1 - \delta(T_0/T)^5], & T \gg T_0 \end{cases},$$

where $\gamma, \delta > 0$ and $\rho_0' > \rho_0$ for any isotropic dispersion law for the electrons. It is clear that, with further increase in temperature, phonon-phonon U-processes are included, and the resistance again begins to increase (cf. Sec. 2). One can hardly expect that in experiment the resistance in the intermediate temperature range will be strictly constant. Electron-electron collisions may have an effect there.

The author is deeply indebted to I. M. Lifshitz, M. Ya. Azbel', M. I. Kaganov and V. M. Tsukernik for valuable discussions.

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Translated by M. Hamermesh
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