THE DOMAIN OF APPLICABILITY OF PERTURBATION THEORY IN THE ELECTRODYNAMICS OF VECTOR MESONS

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From a consideration of two effects—the polarization of the vacuum of vector mesons and the scattering of vector mesons on each other—a criterion is obtained for the energies above which perturbation theory cannot be applied in the electrodynamics of the vector meson.

1. In recent times there has been much discussion of the possibility that the weak interaction is produced by the exchange of a charged vector meson (in what follows we call it the W meson). At present there are no unambiguous experimental indications on this point. The assumed existence of such a meson leads to serious difficulties, in particular in the treatment of electrodynamic corrections to the weak interaction, since the electrodynamics of the W meson is not renormalizable. This has the result that the contribution of virtual processes is completely determined by the region of large energies of the virtual particles (which here are photons and W mesons), or more exactly by the energy region in which the electromagnetic interaction of the W meson cannot in general be regarded as weak. (We remark at once that such assertions have meaning if it is at all possible to formulate the electrodynamics of the W meson within the framework of field theory. All of our further discussion is entirely based on the assumption that this is possible.)

The question, at what energies (by energy we always mean some invariant characteristic quantity, for example the energy in the c.m.s.) the formulas of perturbation theory can be applied in the electrodynamics of vector mesons, has been investigated in a number of papers, the first of which were the papers of Landau [1] and Oppenheimer. [2] From an examination of the Compton effect on the vector meson Landau [1] derived the result that in the electrodynamics of mesons without an anomalous magnetic moment the formulas of perturbation theory are applicable at energies E such that $e^2E^2/m^2 \ll 1$ (m is the mass of the vector meson).

In the present paper we refine this criterion, and give an exact limit on the energies, above which limit the apposite quantities (vertex functions, scattering amplitudes) in the electrodynamics of the W meson cannot be described by the first approximation of perturbation theory without coming into contradiction with the fundamental principles of quantum field theory. This treatment will be carried out both for the case in which the vector meson has an anomalous magnetic moment (of the order of unity), and for the case in which there is no such anomalous moment. We consider two effects: the polarization of vacuum owing to vector mesons, and the scattering of a W^+ meson by a W^- meson. From a consideration of the polarization of vacuum we shall obtain a limit κ_{max} on the mass of a virtual photon, above which the vertex function for the decay of a virtual photon with mass κ into two W mesons with momenta k_1 and k_2 [$k_1^2 = k_2^2 = -m^2$, $(k_1 + k_2)^2 = -\kappa^2$] must differ decidedly from its unperturbed value and decrease with increase of κ^2—that is, the electromagnetic interaction of the meson must begin to be cut off. In the study of the scattering of W^+ by W^- we shall find a limit on the energy, above which the cross section for this scattering, as calculated in the first approximation of perturbation theory, exceeds the unitary limit $\sim \pi\lambda\!\!\!^-{}^2$.

2. We start from the Källén-Lehmann representation for the photon Green's function, whose transverse part can be put in the form

$$D_{\mu\nu}(q) = (\delta_{\mu\nu} - q_\mu q_\nu / q^2) D(q^2), \qquad (1')$$

$$D(q^2) = \frac{1}{q^2} + \int_a^\infty \frac{\rho(\varkappa^2)\, d\varkappa^2}{q^2 + \varkappa^2 - i\varepsilon}. \qquad (1'')$$

Here

$$\rho(\varkappa^2) = \frac{1}{\pi} \operatorname{Im} D(-\varkappa^2)$$

$$= \frac{(2\pi)^3}{3} \sum_{n,\nu} \langle 0 | \acute{A}_\nu(0) | n \rangle \langle n | A_\nu(0) | 0 \rangle$$

is the positive spectral function, A_ν is the Heisenberg operator of the photon field, and $|n\rangle$ is a physical state from the complete system of functions with the four-momentum $k_n^2 = -\kappa^2$. The integration in (1″) begins at the square of the mass of the lowest intermediate state that contributes to $\rho(\kappa^2)$.

From the representation (1″) there follows the inequality

$$\int_{a}^{\infty} \frac{\rho(\varkappa^2)\, d\varkappa^2}{|D(-\varkappa^2)|^2 (\varkappa^2)^2} \leqslant 1. \tag{2}$$

This inequality is an immediate consequence of the general integral representation for the function $-D^{-1}(-\kappa^2)$, which, as can be seen from (1″), is an R-function in the complex plane of κ^2. We shall not give details here of the proof of the inequality (2), which is obtained just as in the cited papers[3,4]; the only thing important for us is that the proof of (2) does not require any additional assumptions beyond those that are the basis of the Källén-Lehmann representation for the Green's function. We shall consider only the contribution made by the two-meson intermediate state to $\rho(\kappa^2)$ [this only strengthens the inequality (2)], and shall calculate this contribution in the lowest approximation in the charge e (here e is the renormalized charge, and $e^2 = \alpha = 1/137$). For this purpose we use the expression for the vertex function $\Gamma^{\nu}_{\beta\alpha}(k_2, k_1)$ in first approximation (cf., e.g., [5]):

$$\Gamma_{\beta\alpha}{}^{\nu}(k_2, k_1) = \delta_{\alpha\beta}(k_1 + k_2)_\nu - \delta_{\beta\nu}k_{2\alpha} - \delta_{\alpha\nu}k_{1\beta} + \gamma[\delta_{\alpha\nu}(k_2 - k_1)_\beta - \delta_{\beta\nu}(k_2 - k_1)_\alpha]. \tag{3}$$

Here γ is the anomalous magnetic moment of the W meson. The corresponding matrix element occurring in the definition of the two-meson contribution to $\rho(\kappa^2)$ is given by

$$\langle k_2\lambda_2; k_1\lambda_1 | A_\nu(0) | 0\rangle = -e(4\pi / 4k_{10}k_{20})^{1/2} D((k_1 + k_2)^2) \times e_\beta^{\lambda_2}(k_2)\Gamma_{\beta\alpha}{}^{\nu}(k_2, -k_1) e_\alpha^{\lambda_1}(k_1), \tag{4}$$

where e^{λ}_{α} is the real polarization vector of the meson. We note its properties:

$$ke^\lambda(k) = 0, \quad \sum_{\lambda=1,2,3} e_\alpha{}^\lambda(k)\, e_\beta{}^\lambda(k) = \delta_{\alpha\beta} + k_\alpha k_\beta / m^2.$$

The calculations give the following expression for $\rho(\kappa^2)$:

$$\rho(\varkappa^2) = \frac{\alpha}{3\pi} |D(-\varkappa^2)|^2 \left(\frac{\varkappa^2 - 4m^2}{\varkappa^2}\right)^{1/2} \left(\frac{\varkappa^2 - 4m^2}{4m^2}\right) \times \left(\varkappa^2 + 3m^2 + 3\varkappa^2\gamma + \varkappa^2 \frac{\varkappa^2 + 4m^2}{4m^2}\gamma^2\right). \tag{5}$$

This expression must be integrated in Eq. (2), beginning at $\kappa^2 \geq 4m^2$. The integral in (2) diverges at the upper limit, however, so that we know that $\rho(\kappa^2)$ cannot be defined by (5) for large κ^2. Cutting off the integral in (2) at a value $\kappa^2_{\max}$, we get an upper limit on the mass of the virtual photon for which the representation (5) for $\rho(\kappa^2)$ is still valid, or in other words for which the formula (3) for the vertex function is valid without the radiative corrections being taken into account.

Thus after substitution of (5) in the "cut off" integral in (2) we get the following:

$$\varkappa^2 = -(k_1 + k_2)^2 \lessapprox \varkappa^2_{max} = 12\pi\alpha^{-1}m^2, \quad \gamma = 0, \tag{6′}$$

$$\varkappa^2 = -(k_1 + k_2)^2 \lessapprox \varkappa^2_{max} = \gamma^{-1}(96\pi / \alpha)^{1/2} m^2, \quad \gamma \neq 0. \tag{6″}$$

Numerically $\kappa_{\max} = 70\text{m}$ $(\gamma = 0)$ and $\kappa_{\max} = 14m\gamma^{-1/2}$ $(\gamma \neq 0)$. If the anomalous magnetic moment of the W meson arises owing to radiative corrections and furthermore is small (smaller in order of magnitude than $\alpha^{1/2}$), $-\kappa^2_{\max}$ will be determined by (6′). A striking feature of (6′) and (6″) are the large numerical coefficients.

It follows from (3) and (5) that for $\kappa^2 \gtrsim \kappa^2_{\max}$ the vertex function must fall off in such a way that the integral in (2) will converge. Since roughly speaking $\rho(\kappa^2)$ is proportional to $|\Gamma(\kappa)|^2$ (roughly, because $\Gamma^{\nu}_{\beta\alpha}$ has a spin structure), for large κ^2 the quantity $\Gamma(\kappa)$ must fall off if $\gamma = 0$ and must fall off more rapidly than $(\kappa^2)^{-1/2}$ if $\gamma \neq 0$.

Assuming that the vertex function does not differ from its unperturbed value (13) for $\kappa^2 < \kappa^2_{\max}$, we can easily find the minimum admissible deviation of the photon Green's function $D(q^2)$ from the free function $D_0(q^2) = q^{-2}$. For this purpose we write down the general expression for $D^{-1}(q^2)$ as an R-function of q^2 (cf. [4]):

$$D^{-1}(q^2) = q^2 \left\{1 - q^2 \int_{a}^{\infty} \frac{\rho(\varkappa^2)\, d\varkappa^2}{|D(-\varkappa^2)|^2 (\varkappa^2 + q^2)(\varkappa^2)^2} - q^2 \sum_n R_n \frac{1}{(\varkappa_n{}^2)^2 (\varkappa_n{}^2 + q^2)}\right\}, \tag{7}$$

where the constants R_n and κ^2_n are real and positive. From (7) we have

$$q^2 D(q^2) > \left\{1 - q^2 \int_{4m^2}^{\varkappa^2 max} \frac{\rho(\varkappa^2)\, d\varkappa^2}{|D(-\varkappa^2)|^2 (\varkappa^2 + q^2)(\varkappa^2)^2}\right\}^{-1}. \tag{8}$$

Substituting (5) and (8), we find that for $q^2 \gg m^2$

$$q^2 D(q^2) > \left[1 - \frac{\alpha}{12\pi}\frac{q^2}{m^2} \ln \frac{\varkappa^2_{max} + q^2}{q^2}\right]^{-1} \quad (\gamma = 0), \tag{9′}$$

$$q^2 D(q^2) > \left[1 - \frac{\alpha\gamma^2}{48\pi}\frac{q^2}{m^4}\left(\varkappa^2_{max} - q^2 \ln \frac{\varkappa^2_{max} + q^2}{q^2}\right)\right]^{-1} \quad (\gamma \neq 0). \tag{9″}$$

The relations (9′) and (9″) provide a possibility of estimating when the effect of polarization of vacuum owing to vector mesons can appear in actual experiments. For example, it follows from (9′) that in experiments on electron-electron scattering (with clashing beams) we can expect that when there is a momentum transfer $q \approx 20m$ there will be an increase of the differential cross section by more than 20 percent in comparison with the value when the polarization of the mesonic vacuum is not taken into account. For $\gamma \neq 0$ the vacuum-polarization effect becomes manifest at much smaller momentum transfers. Indeed, if $\gamma = 1$, a 20 percent effect appears already at $q \approx 5m$.

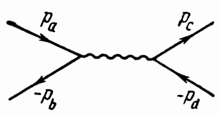

FIG. 1

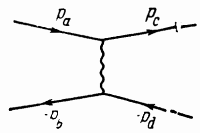

FIG. 2

3. Let us now examine the amplitude for scattering of a W^- on a W^+ meson; in lowest order this is given by the diagrams of Figs. 1 and 2. Here p_a, p_c are the momenta of the W^- before and after scattering, and p_b, p_d are the corresponding momenta of the W^+. Our notations are for the c.m.s.: $\mathbf{p} = \mathbf{p}_a = -\mathbf{p}_b$, $\mathbf{k} = \mathbf{p}_c = -\mathbf{p}_d$; ω is the energy of the mesons in this system, $\omega = p_{0j}$ $(j = a, b, c, d)$. The differential cross section for scattering in states with definite helicity is

$$\frac{d\sigma}{d\Omega} = |\varphi_{\lambda_c\lambda_d;\lambda_a\lambda_b}(\omega, \theta, \Phi)|^2,$$

where φ is the helicity amplitude defined by Jacob and Wick[6]:

$$\varphi_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{1}{2p}\sum (2J+1)\langle\lambda_c\lambda_d|T^J(\omega)|\lambda_a\lambda_b\rangle$$
$$\times\, d^J_{\lambda_a-\lambda_b;\lambda_c-\lambda_d}(\theta)\exp[i(\lambda_a-\lambda_b-\lambda_c+\lambda_d)\Phi]. \quad (10)$$

Starting from the diagrams of Figs. 1 and 2, we get the following expression for this amplitude:

$$\varphi_{\lambda_c\lambda_d;\lambda_a\lambda_b} = \frac{\alpha}{4\omega}\xi_\nu^{\lambda_c*}\eta_\mu^{\lambda_d*}\xi_\alpha^{\lambda_a}\eta_\beta^{\lambda_b}[M^{(1)}_{\nu\mu;\alpha\beta} + M^{(2)}_{\nu\mu;\alpha\beta}], \quad (11)$$

$$M^{(1)}_{\nu\mu;\alpha\beta} = [\Gamma^\sigma_{\beta\alpha}(-p_b, p_a)\Gamma^\sigma_{\nu\mu}(p_c, -p_d)]/(p_a+p_b)^2, \quad (12')$$

$$M^{(2)}_{\nu\mu;\alpha\beta} = [\Gamma^\sigma_{\nu\alpha}(p_c, p_a)\Gamma^\sigma_{\beta\mu}(-p_b, -p_d)]/(p_c-p_a)^2. \quad (12'')$$

Here (12′) and (12″) are the contributions from diagrams 1 and 2, respectively. The vertex $\Gamma^\nu_{\beta\alpha}$ is given by Eq. (3). The time components of the helical-polarization vectors in (11) are given by the relations

$$\xi_0^{\lambda_j} = \mathbf{p}_j\boldsymbol{\xi}^{\lambda_j}/\omega, \qquad \eta_0^{\lambda_j} = \mathbf{p}_j\boldsymbol{\eta}^{\lambda_j}/\omega;$$

and their space components are

$$\boldsymbol{\xi}^{\lambda_a} = \frac{\omega}{m}\mathbf{e}_3(1-|\lambda_a|) - \frac{\lambda_a}{\sqrt{2}}(\mathbf{e}_1 + i\lambda_a\mathbf{e}_2),$$
$$\boldsymbol{\eta}^{\lambda_b} = -\frac{\omega}{m}\mathbf{e}_3(1-|\lambda_b|) + \frac{\lambda_b}{\sqrt{2}}(\mathbf{e}_1 - i\lambda_b\mathbf{e}_2),$$
$$\boldsymbol{\xi}^{\lambda_c*} = \frac{\omega}{m}\mathbf{e}_3'(1-|\lambda_c|) - \frac{\lambda_c}{\sqrt{2}}(\mathbf{e}_1' - i\lambda_c\mathbf{e}_2'),$$
$$\boldsymbol{\eta}^{\lambda_d*} = -\frac{\omega}{m}\mathbf{e}_3'(1-|\lambda_d|) + \frac{\lambda_d}{\sqrt{2}}(\mathbf{e}_1' + i\lambda_d\mathbf{e}_2'). \quad (13)$$

Here $\mathbf{e}_j$ and $\mathbf{e}_j'$ are unit vectors for linear polarizations connected with two systems of axes in which the respective directions of the z axis are along $\mathbf{p}$ and $\mathbf{k}$. We note that $\mathbf{p} = p\mathbf{e}_3$, $\mathbf{k} = k\mathbf{e}_3'$.

The vectors $\mathbf{e}_j'$ are obtained from $\mathbf{e}_j$ by means of a rotation $R(\Phi, \theta, -\Phi)$. Therefore all of the scalar products that arise in the calculation of concrete helicity amplitudes from (11) are given by the formulas

$$\mathbf{e}_j\mathbf{e}_i = \delta_{ji}, \qquad \mathbf{e}_j'\mathbf{e}_i' = \delta_{ji},$$

$$\mathbf{e}_k\mathbf{e}_i' = [R(\Phi,\theta,-\Phi)]_{ki} = \left|\begin{matrix} \sin^2\Phi + \cos\theta\cos^2\Phi & -\sin\Phi\cos\Phi(1-\cos\theta) & \cos\Phi\sin\theta \\ -\sin\Phi\cos\Phi(1-\cos\theta) & \cos^2\Phi + \cos\theta\sin^2\Phi & \sin\Phi\sin\theta \\ -\cos\Phi\sin\theta & -\sin\Phi\sin\theta & \cos\theta \end{matrix}\right|.$$

It is now easy to calculate the amplitude φ in (11) and to determine when, as the energy increases, this quantity comes to exceed the limiting value imposed on it in the general formula (10) because of the unitarity of the scattering matrix $S^J(\omega) = 1 + iT^J(\omega)$. For $\omega^2/m^2 \gg 1$ the amplitude (11) always contains a small number of partial waves, since the dependence on the scattering angle in the denominator of the function $M^{(2)}_{\nu\mu;\alpha\beta}$ [see (12″)] cancels out at high energies. Moreover, it is natural to expect that the helicity amplitudes (11) will increase most rapidly with the energy for $\lambda_j = 0$, since, as can be seen from (13), there is an additional power of the energy ω for these (longitudinal) polarizations. For $\gamma \neq 0$ and $\omega^2/m^2 \gg 1$ we have

$$\varphi_{00;00} = \gamma^2\frac{\alpha\omega^3}{4m^4}\left(\frac{8}{3} - 6P_1(\cos\theta) - \frac{2}{3}P_2(\cos\theta)\right). \quad (14)$$

Since

$$|\langle 0.0|T^J|0.0\rangle| = |\langle 0.0|i^{-1}(S^J-1)|0.0\rangle|^2 < 2,$$

when we compare (14) with (10) we find $\omega^4 < 3m^4/2\alpha\gamma^2$.

As can be seen from (14), for $\gamma = 0$ the quantity $\varphi_{00;00}$ is identically zero. This can also be seen easily from the form of the vertex $\Gamma^\nu_{\beta\alpha}(k_2, k_1)$ in Eq. (3), which for $\gamma = 0$ has the property $k_{2\beta}\Gamma^\nu_{\beta\alpha}(k_2, k_1)k_{1\alpha} = 0$. Therefore for $\gamma = 0$ the amplitudes that increase most rapidly with the energy are those for which at each of the vertices of the diagram of Fig. 1 or 2 one of the mesons is transversely polarized and the other is longitudinally polarized. An example of such an amplitude is $\varphi_{00;11}$, which at high energies comes only from the one diagram of Fig. 2. For $\varphi_{00;11}$ we get at high energies

$$\varphi_{00;\,11} = \alpha\omega / 4m^2. \tag{15}$$

Again using (10) and noting that $|\langle 0, 0 | T^J | 1, 1\rangle| < 1$, since the process is inelastic, we get $\omega^2 < 2m^2/\alpha$.

We get analogous results for the other helicity amplitudes. For example, from an examination of $\varphi_{0-1;10}$ we have (for $\gamma = 0$) $\omega^2 < 3m^2/\alpha$.

It seems to us convenient to formulate the final result in terms of the variable $s^2 = -(p_a + p_b)^2$, which coincides with the variable $\kappa^2 = -(k_1 + k_2)^2$ which was considered in connection with the photon Green's function in Sec. 2. We finally get

$$s^2 = -(p_a + p_b)^2 \lesssim s^2_{max} = 8m^2/\alpha, \qquad \gamma = 0, \tag{16$'$}$$

$$s^2 = -(p_a + p_b)^2 \lesssim s^2_{max} = \gamma^{-1} m^2\sqrt{24/\alpha}\,, \quad \gamma \neq 0. \tag{16$''$}$$

Thus the actual scattering of W mesons must in any case not be described in terms of only the contribution of the two diagrams of Figs. 1 and 2 for $s^2 \lesssim s^2_{max}$. It can be seen from a comparison of Eqs. (6′), (6″) and (16′), (16″) that the limitations obtained from a consideration of actual scattering are somewhat stronger (by about a factor four in the size of s^2) than those that arise from a consideration of the vacuum polarization. It must be pointed out, however, that whereas from the consideration of the vacuum polarization we can draw the conclusion that the vertex function $\Gamma^{\nu}_{\beta\alpha}(k_2, k_1)$ falls off for $\kappa^2 > \kappa^2_{max}$, no such conclusion can be drawn from the consideration of the scattering amplitude, since for $s^2 > s^2_{max}$ the unitarity can be restored not only through a change of Γ, but also through diagrams with the exchange of two or more photons.

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