EFFECT OF MULTIPLE SCATTERING ON TRANSITION RADIATION

V. E. PAFOMOV

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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Transition radiation is studied by taking into account the effect of multiple scattering. Results are obtained which give a quantitative description of the spectral density and the total radiation energy associated with the presence of a material interface; this radiation supplements the usual bremsstrahlung radiation.

1. Passage of a charged particle through a medium interface produces radiation[1] known as transition radiation. The spectrum of the transition radiation of a relativistic particle is enriched, upon increase in the particle energy, by new frequencies ($\omega \lesssim \omega_{cr} = \omega_0 E/mc^2$, $\omega_0 = (4\pi n Ze^2/m)^{1/2}$), which leads to a proportional increase in the total radiation energy.[2,3] When $E > L(mc^2)^3/cE_s^2$ (L is the radiation unit length, $E_s = 21 \times 10^6$ eV) the effect of multiple scattering becomes significant;[4] this scattering enriches the spectrum even more rapidly with new "harder" quanta ($\omega \lesssim \omega_{cr}^* = E_s^2 E^2 c/(mc^2)^4 L$) and leads to a quadratic dependence of the transition radiation energy on the energy of the particle. This transition radiation supplements the ordinary bremsstrahlung radiation in an unbounded medium.[5] When $E > L(mc^2)^4/E_s^2\hbar c$, the energy of the radiated quanta becomes comparable with the energy of the particle, and the quadratic dependence becomes linear.

Previously,[5] we gave an estimate of the effect of multiple scattering on the transition radiation. A number of researches have been devoted to the solution of this problem.[6-8] However, a detailed analysis shows that the results obtained in these researches for the spectral density contain negative values, as a result of which, upon neglect of quantum effects and of the polarization of the medium, the integrated result over all frequencies turns out to be equal to zero. This discrepancy is associated with the fact that the subdivision of the total radiation into transition radiation and bremsstrahlung contains a certain element of arbitrariness in this case, since both these parts are interrelated. In order that the calculation of one of these have a physical meaning, it is necessary to precisely define the other. In the present work we give a method of calculating the transition radiation based on a division of this kind.

2. In an inhomogeneous medium, if the energy loss is small, the value of the spectral density of energy radiation per unit length is constant along the path, and on a path $\mathscr{L}'$, it is therefore equal to a certain value $W''_\omega \mathscr{L}'$. We assume that the entire space is filled with the medium, with the exception of a layer $\mathscr{L}$, where $\mathscr{L}$ is large in comparison with the length of coherent interaction of the particle with the electric field (for the region of frequencies under consideration). In this case, the particle radiation generated along the path $\mathscr{L}' + \mathscr{L}$ (with intersection of a vacuum gap of width $\mathscr{L}$ in the middle of the trajectory), consists of two noninterfering parts, and the dependence on $\mathscr{L}$ drops out. We divide the intensity of each of these parts into $W''_\omega \mathscr{L}'/2$ and a remainder which we call the transition radiation. Its calculation, inasmuch as W''_ω is known, enables us to determine the total radiation.

We shall consider the radiation of waves whose frequencies are much higher than the optical frequencies. Reflection and refraction of these waves at the separation boundaries can be neglected; radiation at large distances is determined only by the trajectory of the particle. If the energy loss is small the trajectory is determined by the identical scattering both before and after passage through the vacuum gap. In particular, it then follows that the intensity of the transition radiation does not depend on whether the particle is moving from the medium into the vacuum or in the opposite direction. We note that in the absence of interference the radiation from the separate trajectories can be summed independently. We shall use this circumstance below, and for each of the possible trajectories we shall measure the quantum-emission angles from the direction of particle motion produced in the vacuum gap.

The averaging of the radiation intensity over all trajectories, which corresponds to such a superposition, is equivalent to averaging over the set of trajectories shifted and turned in space in such a way that the coordinate and the velocity of the particle are uniquely determined in the vacuum gap. We shall call this the averaging relative to the produced direction of particle motion in the vacuum. The "distribution function" for the set of trajectories shifted in this manner satisfies the ordinary kinetic equation on the path beyond the gap, and an equation obtained from the kinetic equation by a time reversal ahead of the gap. This function is symmetric relative to the vacuum gap, from which follows directly the equality of the intensities of the noninterfering parts of the radiation (for large $\mathscr{L}$). We describe the spectral density of the radiation energy per unit solid angle of one of these, as $\mathscr{L}' \to \infty$, in the following way:

$$W_{n\omega}' = c\overline{\left|\int_{-\infty}^{0} \mathbf{A}\,dt + \int_{0}^{\infty} \tilde{\mathbf{A}}\,dt\right|^2}, \qquad (1)$$

where

$$\mathbf{A} = \frac{e\omega}{2\pi c^2}[\mathbf{nv}]\,e^{i(\mathbf{kr}-\omega t)}, \qquad (2)^*$$

$\mathbf{r}$ and $\mathbf{v}$ are the coordinate and velocity of the particle, respectively, at the time t, $\mathbf{k}$ is the wave vector of the photon and $\mathbf{n} = \mathbf{k}/k$. Here and below we shall designate with a tilde the vector $\mathbf{A}$ which determines the radiation field generated by the particle on a path in the medium in the case of a specific trajectory determined by the scattering. It is assumed in (1) that the particle moves from the vacuum into the medium, crossing the medium interface at the time $t = 0$.

The effect of the interface on the radiation is found by comparison with the radiation in a homogeneous medium, allowing the width of the vacuum gap $\mathscr{L}$ to approach zero. Here, inasmuch as (1) corresponds to one half of the intensity of radiation for large values of $\mathscr{L}$, (1) must be compared with half the intensity of radiation in the limiting case $\mathscr{L} \to 0$:

$$W_{n\omega}'' = \frac{c}{2}\overline{\left|\int_{-\infty}^{0} \tilde{\mathbf{A}}\,dt + \int_{0}^{\infty} \tilde{\mathbf{A}}\,dt\right|^2}. \qquad (3)$$

The transition radiation is described by the equality $W_{n\omega} = W_{n\omega}' - W_{n\omega}''$. We shall write it in the following form:

$$W_{n\omega} = c\left|\int_{-\infty}^{0} \mathbf{A}dt\right|^2 + c\,\overline{\int_{-\infty}^{0} \mathbf{A}\,dt\int_{0}^{\infty} \tilde{\mathbf{A}}^* dt} + c\,\overline{\int_{-\infty}^{0} \mathbf{A}^* dt\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt}$$
$$+ c\overline{\left|\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt\right|^2} - \frac{c}{2}\overline{\left|\int_{-\infty}^{0} \tilde{\mathbf{A}}\,dt\right|^2} - \frac{c}{2}\overline{\int_{-\infty}^{0} \tilde{\mathbf{A}}\,dt\int_{0}^{\infty} \tilde{\mathbf{A}}^* dt}$$
$$- \frac{c}{2}\overline{\int_{-\infty}^{0} \tilde{\mathbf{A}}^* dt\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt} - \frac{c}{2}\overline{\left|\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt\right|^2}. \qquad (4)$$

We note that for motion from the vacuum into the medium (we assume this direction in (1) only for definiteness, since the effect does not depend upon it), the particle does not experience the retarding force associated with the radiation of the waves under consideration on a path in the vacuum far from another interface, inasmuch as these waves are not reflected from the interface. It then follows that the generation of transition radiation is brought about in the case in question by a different prior history of the motion (relative to the time $t = 0$) in comparison with the case of motion in an inhomogeneous medium. In this connection, the result of integration of (4) over the photon angles and over the frequency must be interpreted as the difference in the work of the retarding force in the cases under comparison over a time t from zero to infinity.

In Eq. (4), it is assumed that the path of travel of the particle in the medium $\mathscr{L}'$ is infinite: therefore, the separate terms (namely, the fourth, fifth and the last) contain an infinite intensity due to the ordinary bremsstrahlung. They mutually cancel one another. Furthermore, they cancel even for finite $\mathscr{L}'$, and the general result (4) does not depend on $\mathscr{L}'$, if $\mathscr{L}'$ is much larger than the length of coherent interaction. As a consequence of the symmetry of the "distribution function" relative to $t = 0$, the following equality is satisfied:

$$\overline{\int_{-\infty}^{0} \tilde{\mathbf{A}}\,dt} = -\overline{\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt}. \qquad (4a)$$

In the sixth and seventh terms of (4), the integral factors determine the radiation field generated by particles on different parts of the trajectory relative to the coordinate of the particle at the time $t = 0$, and the integrands there are connected only by the unique definition of $\mathbf{r}$ and $\mathbf{v}$ at $t = 0$. The independence of the averaging of the integrals then follows. An account of this fact, and also of Eq. (4a), leads to the following result:

$$W_{n\omega} = c\left|\int_{-\infty}^{0} \mathbf{A}\,dt + \overline{\int_{0}^{\infty} \tilde{\mathbf{A}}\,dt}\right|^2. \qquad (5)$$

The integrals within the absolute-value signs denote the amplitudes of the radiation field when the particle stops on the medium interface and is

*$[\mathbf{nv}] = \mathbf{n} \times \mathbf{v}$.

ejected into the other medium. Thus, the additional intensity of radiation, associated with the presence of the interface, is proportional to the integral of the square of the modulus of the sum of the corresponding amplitudes over the solid angle.

3. For what follows, it is necessary to average Eq. (2) over all possible particle trajectories. We shall use the result of Gol'dman,[6] who shows that in the small angle approximation, which is sufficient for consideration of the radiation of a relativistic particle,1) we have

$$\int w(\mathbf{r}, \vartheta, t) \exp [i(\mathbf{kr} - \omega t)] d\mathbf{r} = \exp (\alpha_1 \theta^2 + \alpha_2 \theta\theta_0 + \alpha_3);$$
$$\alpha_1 = -(i\omega / 8q)^{1/2} \operatorname{cth} (2i\omega q)^{1/2} t,$$
$$\alpha_2 = (i\omega / 2q)^{1/2} \operatorname{sh}^{-1} (2i\omega q)^{1/2} t,$$
$$\alpha_3 = -i\omega\gamma t / 2g^2 - \ln [\operatorname{sh}(2i\omega q)^{1/2} t] - \theta_0^2 (i\omega / 8q)^{1/2} \operatorname{cth} (2i\omega q)^{1/2} t + \ln (i\omega / 8\pi^2 q)^{1/2},$$
$$\vartheta = \theta - \theta_0, \qquad \omega \gg \omega_0, \tag{6*}$$

where $w(\mathbf{r}, \vartheta, t)$ is the distribution function which satisfies the ordinary kinetic equation in the Fokker-Planck approximation, with the initial condition

$$w(\mathbf{r}, \vartheta, 0) = \delta(\mathbf{r})\delta(\theta - \theta_0), \qquad g = E / mc^2,$$
$$q = \omega_{cr}^* / 4g^4, \qquad \gamma = 1 + \omega_{cr}^2 / \omega^2;$$

θ and θ_0 are the angles between the wave vector $\mathbf{k}$ of the radiated photons and $\mathbf{v}$ and $\mathbf{v}_0$, respectively.

Making use of (6), we have

$$\widetilde{\mathbf{A}} = \frac{e\omega}{2\pi c} \int \theta w (\mathbf{r}, \theta - \theta_0, t) e^{i(\mathbf{kr} - \omega t)} d\mathbf{r}\, d\theta$$
$$= \frac{\theta_0}{\operatorname{ch}^2 (2i\omega q)^{1/2} t} \exp \left[-\theta_0^2 \left(\frac{i\omega}{8q}\right)^{1/2} \operatorname{th} (2i\omega q)^{1/2} t - \frac{i\omega\gamma t}{2g^2} \right]. \tag{7†}$$

Transforming in (5) to integration over a nondimensional quantity x, which is connected with t by the relation $x = (2i\omega q)^{1/2} t$, making use of (7), and carrying out simple transformations, we get the following formula:

$$W_{n\omega} = \frac{e^2 g^2}{\pi^2 cz} \left| \frac{1}{1+z} - \sigma\gamma \int_0^\infty e^{-\sigma z \operatorname{th} x - \sigma\gamma x} dx \right|^2; \tag{8}$$

$$z = \theta_0^2 g^2, \quad g = E / mc^2, \quad \sigma = (i\omega / 2\omega_{cr}^*)^{1/2},$$
$$\gamma = 1 + \omega_{cr}^2 / \omega^2, \quad \omega_{cr} = \omega_0 g, \quad \omega_{cr}^* = E_s^2 g^2 / m^2 c^3 L.$$

Making use of the results of Ternovskiĭ[8] and carrying similar calculations and transformations, we obtain a quantum-mechanical generalization of Eq. (8):

$$W_{n\omega} = \frac{e^2 g^2}{2\pi^2 p_0^2 \{k^2 + [p_0^2 + (p_0 - k)^2] z\}}$$
$$\times \left| \frac{p_0^2 + (p_0 - k)^2 - k^2}{1+z} + k^2 \sigma \int_0^\infty \frac{dx}{\operatorname{ch} x} e^{-\sigma z \operatorname{th} x - \sigma\gamma x} - \sigma\gamma [p_0^2 + (p_0 - k)^2] \int_0^\infty e^{-\sigma z \operatorname{th} x - \sigma\gamma x} dx \right|^2; \tag{9}$$

$$g = p_0 - k / 2, \qquad \sigma = \sqrt{ik / 8qg^2 p_0 (p_0 - k)},$$
$$\gamma = 1 + p_0 (p_0 - k) k_0^2 / k^2, \qquad k_0^2 = 4\pi n Z e^2,$$
$$qg^2 = 2\pi n Z^2 e^4 \ln (\theta_2 / \theta_1), \qquad m = \hbar = c = 1,$$

where k is the momentum of the radiated photon and p_0 the momentum of the particle at the time t = 0.

Generalization of the result (9) to the case of two media takes the form

$$W_{n\omega} = \frac{e^2 g^2}{2\pi^2 p_0^2 \{k^2 + [p_0^2 + (p_0 - k)^2] z\}}$$
$$\times \left| k^2 \sigma_1 \int_0^\infty \frac{dx}{\operatorname{ch} x} e^{-\sigma_1 z \operatorname{th} x - \sigma_1 \gamma_1 x} - \sigma_1 \gamma_1 [p_0^2 + (p_0 - k)^2] \right.$$
$$\times \int_0^\infty e^{-\sigma_1 z \operatorname{th} x - \sigma_1 \gamma_1 x} dx - k^2 \sigma_2 \int_0^\infty \frac{dx}{\operatorname{ch} x} e^{-\sigma_2 z \operatorname{th} x - \sigma_2 \gamma_2 x}$$
$$\left. + \sigma_2 \gamma_2 [p_0^2 + (p_0 - k)^2] \int_0^\infty e^{-\sigma_2 z \operatorname{th} x - \sigma_2 \gamma_2 x} dx \right|^2, \tag{10}$$

where the indices 1 and 2 denote that the corresponding quantities refer to the first or second medium, respectively.

The spectral density and the total energy of transition radiation are equal to

$$W_\omega = \frac{\pi}{g^2} \int_0^\infty W_{n\omega} dz, \qquad W = \pi \int_0^{p_0} \frac{dk}{g^2} \int_0^\infty W_{n\omega} dz. \tag{11}$$

4. We now consider the transition radiation at frequencies $\omega \ll E/\hbar$. In the case of motion of the charged particle through the boundary of separation of vacuum and the medium, the study of the spectral density of the transition radiation leads to the following results:

For $\omega_{cr}^* \ll \omega_{cr}$ we have

$$W_\omega = \frac{e^2}{\pi c} \left[\left(1 + 2\frac{\omega^2}{\omega_{cr}^2}\right) \ln \left(1 + \frac{\omega_{cr}^2}{\omega^2}\right) - 2 \right], \quad \omega \lesssim \omega_{cr}, \tag{12}$$

$$W_\omega = \frac{e^2}{\pi c} \left[\frac{1}{6} \left(\frac{\omega_{cr}}{\omega}\right)^4 + \frac{8}{21} \left(\frac{\omega_{cr}^*}{\omega}\right)^2 \right], \quad \omega \gg \omega_{cr}; \tag{13}$$

for $\omega_{cr}^* > \omega_{cr}$ we have

1)Misprints in the work cited[6] are corrected here, and polarization of the medium is taken into account.

*sh = sinh, cth = coth.

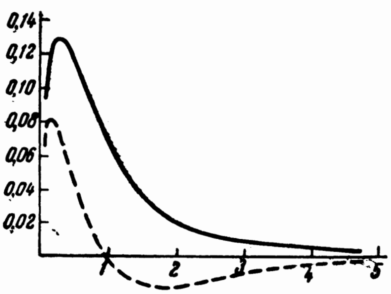
†ch = cosh, th = tanh.

$$W_\omega = \frac{e^2}{\pi c} \ln \frac{\omega_{cr}^2}{\omega^2}, \quad \omega < \omega_{cr}^{4/3} \omega_{cr}^{*\,-1/3}, \tag{14}$$

$$W_\omega = \frac{e^2}{\pi c} \ln \frac{2}{3}\left(\frac{\omega_{cr}^*}{\omega}\right)^{1/2}, \quad \omega_{cr}^{4/3} \omega_{cr}^{*\,-1/3} < \omega \leqslant \omega_{cr}^*, \tag{15}$$

$$W_\omega = \frac{e^2}{\pi c} \frac{8}{21}\left(\frac{\omega_{cr}^*}{\omega}\right)^2, \quad \omega \geqslant \omega_{cr}^*. \tag{16}$$

To find the spectral density of the radiation energy at frequencies $\omega \sim \omega_{cr}^*$ for $\omega_{cr}^* \gg \omega_{cr}$, a numerical integration was carried out. The results are shown in the drawing, where the function $\pi c e^{-2} (\omega/\omega_{cr}^*)^{1/2} W_\omega$ is plotted along the ordinate, and the quantity $(\omega/\omega_{cr}^*)^{1/2}$ along the abscissa.



The solid line denotes our results, the dashed line the results of a number of previous researches.[6-8] 2) It turns out that the spectral density of the transition radiation energy can be approximated by the following formula

$$W_\omega = \frac{e^2}{\pi c}\left[\ln \frac{a_1\sqrt{\omega} + (\omega + a_2\omega_{cr}^*)^{1/2}}{a_3\sqrt{\omega}} + \frac{a_3\sqrt{\omega}}{a_1\sqrt{\omega} + (\omega + a_2\omega_{cr}^*)^{1/2}} - 1\right]; \tag{17}$$

$$a_1 = 6\sqrt{3/7} - 1, \quad a_2 = 48/7, \quad a_3 = 6\sqrt{3/7},$$
$$\omega_{cr}^{4/3}\omega_{cr}^{*\,-1/3} < \omega \leqslant E/\hbar,$$

the maximum deviations from which do not exceed several per cent.

Equation (17) is valid under the condition that no electrons are absorbed along the path $s \sim (E/mc^2)^2 c/\omega$ of formation of the transition radiation $(s \ll L)$; they undergo a large number of collisions with the atoms of the medium $(s \gg n^{-1/3})$. The region of applicability of the result (17) is then

$$(mc^2)^2 / E_s^2 \ll \omega / \omega_{cr}^* \ll n^{1/3} L (mc^2)^2 / E_s^2. \tag{18}$$

For the electron, the quantity on the left side of (18) is of the order of 10^{-3}, and on the right side of the order of 10^5; therefore, the relation (18) can be considered both for high $(\omega \gg \omega_{cr}^*)$ and comparatively low $(\omega \ll \omega_{cr}^*)$ frequencies, for which (17) goes over into (16) and (15), respectively.

Integrating (17) over the frequency, we get the radiation energy:

$$W = \frac{e^2\omega_{cr}^*}{\pi c} \frac{a_2}{a_3(a_3 - 2)}\left(\frac{a_3}{a_3 - 2}\ln\frac{a_3}{2} - 1\right) \approx \frac{0.34 e^2 E_s^2 E^2}{\pi L (mc^2)^4}. \tag{19}$$

It is proportional to the square of the particle energy. Equation (19) is valid in the range

$$\omega_0 L (mc^2)^3 / c E_s^2 < E \leqslant L (mc^2)^4 / E_s^2 \hbar c, \tag{20}$$

beyond the limits of which the fundamental role is played either by polarization of the medium or by quantum effects, and the energy of transition radiation is proportional to the energy of the particle.

A comparison of these results with the bremsstrahlung radiation in a condensed medium[9-14] is of interest. It shows that in those regions of frequency in which the intensity of the transition radiation does not vanish and is determined by polarization of the medium or by multiple scattering, and is proportional to $\ln(\omega_{cr}^2/\omega^2)$ or $\ln \frac{2}{3}(\omega_{cr}^*/\omega)^{1/2}$, the bremsstrahlung radiation is weaker than in the case of a rarefied medium by the respective factors ω^2/ω_{cr}^2 or $\frac{3}{2}(\omega/\omega_{cr}^*)^{1/2}$. Equality of the corresponding frequency intervals indicates that the polarization of the medium and multiple scattering produce the same result, which leads to an increase in the intensity of transition radiation and a decrease in the bremsstrahlung. It consists in the decrease of the path of coherent interaction of the particle with the electromagnetic waves of given frequency. In the case of transition radiation a differential effect appears, and in the case of bremsstrahlung the effect is proportional to the relative change in the value of this path.

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2) Comparison of the graphs shown in the drawing shows that the results obtained by a number of authors[6-8] give a description of the total boundary effect of radiation at frequencies $\omega \ll \omega_{cr}^*$.

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