PROOF OF THE MANDELSTAM REPRESENTATION FOR A SIXTH-ORDER LADDER DIAGRAM

D. Ya. PETRINA

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The Mandelstam representation is proved for a sixth-order ladder diagram. General sufficient conditions for the validity of the representation are established.

1. After the work of Landau [1] and of Polkinghorne and Screaton [2] it was made clear that the singularities of the contributions of Feynman diagrams are located on analytical surfaces—the Landau surfaces. It is still unclear, however, how points of the Landau surface can be singular points of the contribution on the "physical" sheet. This question has come to be of primary importance in connection with the proof of the Mandelstam representation in perturbation theory, and the following problem has arisen: what are the criteria which will indicate which points of the Landau surface can be singular points of a contribution on the "physical" sheet, and whether the Mandelstam representation is valid for an arbitrary diagram?

In the proofs of the Mandelstam representation for particular diagrams and classes of diagrams given by a number of authors, original methods have been proposed for investigating the analytical properties of the contributions of Feynman diagrams. We may mention the papers of Mandelstam, [3] Tarski, [4] Vladimirov, [5] Gribov and Dyatlov, [6] and Rudik and Simonov. [7] Without belittling the significance of these papers, we note that they fail to some extent to utilize the fact that the singularities of the contributions are located on the Landau analytical surfaces (the paper of Tarski [4] is an exception), and do not establish any general criterion for the validity of the Mandelstam representation.

In a paper by Eden and others [8] the proposition is developed that the Mandelstam representation is valid if the Landau curves have no isolated singularities in a definite region. This criterion is presented in [8] without proof, and for the proof reference is made to another paper by Eden, [9] in which there are several incorrect statements and inaccuracies. It seems to us that this criterion is not true in the general case, and a contribution can have complex singularities even when there are no isolated singular points.

In recent papers by the present writer [10] a continuity theorem has been used to obtain a general criterion for the validity of the Mandelstam representation. It was found that from the behavior of the Landau curves (i.e., from the intersections of the Landau surfaces with the real plane) one can determine whether the Mandelstam representation is valid. Roughly speaking, this criterion states that the Mandelstam representation holds for the contribution of an arbitrary diagram if the real points of intersection of the Landau surfaces with definite planes do not go over into complex points (whether this situation is possible is completely determined by the behavior of the Landau curves). In the present paper this criterion is made more precise, and the Mandelstam representation is proved for a sixth-order ladder diagram (particles of equal masses $m = 1$ are considered).

2. The contribution $F(s,t)$ of the sixth-order ladder diagram (see figure, diagram a) is a holomorphic function in the region $B(s,t \mid s < 4, t < 9)$, and also in a region D which contains: 1) all points (s,t) with $\operatorname{Im} s$ and $\operatorname{Im} t$ of the same sign, and 2) all those points with $\operatorname{Im} s$ and $\operatorname{Im} t$ of opposite signs that lie on the analytic surfaces

$$as + bt = c, \quad a > 0, \qquad b > 0, \quad a^2 + b^2 = 1 \qquad (1)$$

with [10] $c < 4a + 9b$. In addition to this, the func-

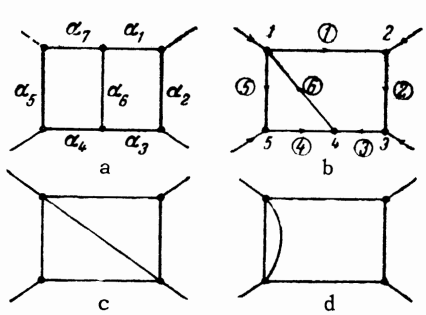

FIG. 1

tion F(s, t) can be analytically continued to any point of the space C^2 of the two complex variables s, t, with the possible exceptions of points located on the Landau surfaces:

$$t = \frac{s-4}{2}\left[4\left(1+\frac{2}{s-4}\right)^3 - 3\left(1+\frac{2}{s-4}\right) - 1\right], \tag{2}$$

$$s = 4\frac{t^2-5t+4}{t^2-8t+4}, \tag{3}$$

$$t = 1 + 2\left[1+\cos\left(\alpha+\frac{\pi}{3}\right)\right]^2 - 2(1+\cos\alpha)\left[1+\cos\left(\alpha+\frac{\pi}{3}\right)\right]\frac{\sin(\alpha+\pi/3)}{\sin\alpha},$$

$$s = 1 + 2[1+\cos\alpha]^2 - 2(1+\cos\alpha) \times \left[1+\cos\left(\alpha+\frac{\pi}{3}\right)\right]\frac{\sin\alpha}{\sin(\alpha+\pi/3)}, \tag{4}$$

$$t = 4\frac{t^2-6t+9}{t^2-10t+9}, \tag{5}$$

$$s = 4, \qquad s = 9, \qquad t = 9. \tag{6}$$

The surfaces (2)—(5) give the singularities of the diagrams a, b, c, d, respectively. Equation (2) was obtained in a paper by Okun' and Rudik,[11] and Eq. (4) in a paper by Kolkunov, Okun', and Rudik.[12] Equations (3) and (5) can be obtained by an elementary calculation.

It must be pointed out that the surface (3) does not contain any point that would correspond to positive Feynman parameters. It turns out that the Landau equations for diagram b are soluble only for $\alpha_4 = -\alpha_6$, where α_i is the parameter for the i-th line of diagram b. The fact that this diagram and even entire classes of diagrams that are in a certain sense related to it have no essential singularities which would correspond to positive Feynman parameters has been pointed out in papers by Eden,[13] Islam,[14] and Rudik and Simonov.[7]

We wish to prove that the function F(s, t), or more exactly that branch of it which is defined in a region $B \cup D$ by means of Feynman integrals and is holomorphic in this region, and whose possible singularities are located on the surfaces (2)—(6), admits of analytic continuation to the region consisting of the direct product of the s and t planes with the cuts Im s = 0, Re s ≥ 4 and Im t = 0, Re t ≥ 9 excluded. When this is true we shall say that the function F(s, t) is holomorphic on the "physical" sheet.

For this purpose we use the theorem of continuity. We shall formulate this theorem[15-17] in a way convenient for this use. We consider a sequence of analytic planes E_j: $a_j s + b_j t = c_j$, $a_j > 0$, $b_j > 0$, which converge to an analytic plane E_I: $a_I s + b_I t = c_I$, where

$$\lim_{j\to\infty} a_j = a_I, \qquad \lim_{j\to\infty} b_j = b_I, \qquad \lim_{j\to\infty} c_j = c_I.$$

We mark out on the planes E_j, E_I finite regions G_j, G_I such that

$$\lim_{j\to\infty} G_j = G_I, \qquad \lim_{j\to\infty} \partial G_j = \partial G_I.$$

The continuity theorem states that: If the regions G_j and their boundaries ∂G_j belong to the region of holomorphy of the function F(s, t) and the boundary ∂G_I also belongs to the region of holomorphy, then the region G_I also belongs to the region of holomorphy.

Previously[10] we considered a continuous layer of planes as + bt = c, with a and b fixed and the parameter c varying continuously, $c < c_I$. In this case the statement of the continuity theorem was: If a function F(s, t) is holomorphic in the regions G(c) for $c < c_I$ and at a single point of the region $G(c_I)$, then it is holomorphic in the entire region $G(c_I)$. Here $\lim G(c) = G(c_I)$. By means of this theorem we established for $c = c_I$ criteria for the holomorphy of the contributions on the "physical" sheet which allow us to prove the Mandelstam representation for some comparatively complicated diagrams. As will be shown below, this latter type of continuity theorem is not adequate to prove for the ladder diagram a that the function F(s, t) is holomorphic on the "physical" sheet. One can prove this, however, by applying the more general continuity theorem.

Let us consider the intersections of the planes (1) with c > 4a + 9b with the surfaces (2)—(5). It can be shown by elementary manipulations that there exists a number a_0, $0 < a_0 < 1$, such that as the parameter c > 4a + 9b increases the real points of intersection of the surfaces (2)—(5) with the planes (1) do not become complex for $0 < a < a_0$ [we call a point (s, t) with |Im s| ≠ 0, |Im t| ≠ 0 complex, and a point with Im s = 0, Im t = 0 real; for 0 < a < 1 only such points can lie on the planes (1)]. If, on the other hand, $a_0 \le a < 1$, then there exists a number $c = c_0(a, b)$ such that two points of intersection of the surface (4) with the planes (1) that are real for $c = c_0(a, b)$ become complex for $c > c_0(a, b)$. This is a consequence of the fact that the curve (4) consists of two branches; one of them is located in the region s > 9, t > 9 and has negative slope and is convex downwards, and the other, located in the region 4 < s < 9, t < 9, has negative slope and is convex upward. A straight line $as + bt = c_0(a, b)$, $a_0 \le a < 1$, touches this part of the branch. It can also be shown that for c > 4a

+ 9b all complex points of intersection are finite. As the parameter $c > 4a + 9b$ increases, complex points of intersection can become real. We denote by $c_1, c_2, \ldots$ those values of the parameter c at which complex points of intersection become real.

3. After these preparatory remarks we proceed to the proof that the function $F(s,t)$ has no complex singular points on the "physical" sheet. Possible complex singular points of the function $F(s,t)$ on the "physical" sheet can lie only on the intersections of the surfaces (2)—(5) with the plane (1) for $c > 4a + 9b$. We shall show that the existence of complex singular points on the "physical" sheet contradicts the continuity theorem.

Let us consider the planes (1) for a fixed value of a in the interval $0 < a < a_0$. On these planes we mark out a region G_1 containing all complex points with $\operatorname{Im} s > 0$, $\operatorname{Im} t < 0$. On the planes (1) with $4a + 9b \leq c < c_1$ we take finite simply connected regions $\bar{G}(c) \subset G_1$ which depend continuously on the parameter c and contain all of those complex points of intersection of the plane (1) with the surfaces (2)—(5) for which $\operatorname{Im} s > 0$, $\operatorname{Im} t < 0$. We construct a region $D'(a, c_1)$ consisting of the union of the regions G_1 for $c < 4a + 9b$ and the regions $G_1 - G(c)$ together with some neighborhood of the boundaries $\partial G(c)$ for $c_1 > c \geq 4a + 9b$. The region $D'(a, c_1)$ is simply connected, and for a suitable choice of the neighborhoods of the boundaries $\partial G(c)$ it does not contain any singular points of the function $F(s,t)$. Therefore the function $F(s,t)$ is holomorphic in the region $D'(a, c_1)$. The boundaries $\partial G(c)$ of the regions $G(c)$ are contained in the region $D'(a, c_1)$, and consequently in the region of holomorphy of the function $F(s,t)$. Also for $c < 4a + 9b$ we can mark out regions $\bar{G}(c) \subset G_1$ which depend continuously on c and are such that $\lim \bar{G}(c) = \bar{G}(4a + 9b)$ for $c \to 4a + 9b$. These regions are contained in the region of holomorphy of the function $F(s,t)$. Applying the continuity theorem, we find that also for $c_1 > c \geq 4a + 9b$ the regions $G(c)$ are contained in the region of holomorphy of the function $F(s,t)$, and the function $F(s,t)$ will be holomorphic in the regions G_1 for $c < c_1$. We can now repeat the arguments given above for $c_1 \geq c > c_2$, and so on.

Thus for any fixed a in the interval $0 < a < a_0$ the function $F(s,t)$ is holomorphic in a region $D(a, \infty)$ which consists of the union of the regions G_1 for $-\infty < c < \infty$. The union of the regions $D(a, \infty)$ for $0 < a < a_0$ is a simply connected region which we denote by D_a, and the function $F(s,t)$ is holomorphic in the region D_{a^0}. If, on the other hand, the parameter a is in the interval $a_0 \leq a < 1$, then for $c > c_0(a,b)$ there arise new complex points of intersection of the surface (4) with the plane (1), which were real for $c = c_0(a, b)$. Therefore for $c > c_0(a,b)$ it is in general impossible to construct simply connected regions $G(c)$ which depend continuously on the parameter c and contain all of the complex points of intersection, and which together with their boundaries are contained in the regions G_1, with the boundaries $\partial G(c)$ not intersecting the surfaces (2)—(5).

By the method just described we can only show that there are no complex singular points in the regions G_1 for $c \leq c_0(a,b)$. [The fact that the function $F(s,t)$ is holomorphic in the region D^a can be proved by using only the second variant of the continuity theorem.[10]] It turns out, however, that the complex points of intersection of the planes (1) with the surfaces (2)—(5) for $a_0 \leq a < 1$ and $c > c_0(a,b)$ are also not singular points of $F(s,t)$.

To prove this, we proceed in the following way. We consider regions G_1 located on the planes (1) for $a_0 \leq a < a_0 + \epsilon < 1$, $c_0(a,b) < c < c_0(a,b) + \delta$, where the numbers ϵ, δ are chosen so that in the interval $c_0(a,b) < c < c_0(a,b) + \delta$ and for $a_0 < a < a_0 + \epsilon$ the complex points of intersection do not become real. We mark out finite simply connected regions $\bar{G}(c,a) \subset G_1$ which depend continuously on the parameters a and c and contain all of the complex points of intersection with $\operatorname{Im} s > 0$, $\operatorname{Im} t < 0$. [For $c > c_0(a,b)$ this construction can be carried out.] We now adjoin to the region D_a the regions $G_1 - G(c,a)$ together with neighborhoods of the boundaries $\partial G(c,a)$. The resulting region will be simply connected, and with a suitable choice of the neighborhoods of the boundaries $\partial G(c,a)$ will not contain any singular points of the function $F(s,t)$. Consequently, the function $F(s,t)$ is holomorphic in this region. It is clear from the construction that the boundaries $\partial G(c,a)$ are contained in this region—that is, they are contained in the region of holomorphy. For $a < a_0$ we mark out regions $\bar{G}(c,a) \subset G_1$ which depend continuously on the parameters a and c and are such that $\lim \bar{G}(c,a) = \bar{G}(c, a_0)$ for $a \to a_0$. The regions $\bar{G}(c,a)$ for $a < a_0$ are contained in the region of holomorphy of the function $F(s,t)$. Applying the continuity theorem, we find that the regions $\bar{G}(c,a)$ for $a_0 \leq a < a_0 + \epsilon$ and $c_0(a,b) < c < c_0(a,b) + \delta$ are contained in the region of holomorphy of the function $F(s,t)$.

Thus the function $F(s,t)$ will be holomorphic in the regions G_1 also for $a_0 \leq a < a_0 + \epsilon$ and $c < c_0(a,b) + \delta$. For $c > c_0(a,b) + \delta$ the real points of intersection do not become complex, and we can carry out the same arguments as for $a < a_0$. The final result is that the function $F(s,t)$

is holomorphic in the regions $D(a, \infty)$ for $a_0 \leq a < a_0 + \epsilon$. Adjoining these regions to the region D_a we get a simply connected region $D_{a+\epsilon}$ in which the function F(s, t) is holomorphic. Continuing this process, we prove that the function F(s, t) is holomorphic in a region D_1 which contains all complex points with Im s > 0, Im t < 0.

By precisely similar agruments we show that the function F(s, t) is holomorphic in a region D_2 which contains all complex points with Im s < 0, Im t > 0. The function F(s, t) was holomorphic in a region $B \cup D$. It admits of analytic continuation to the regions D_1 and D_2. Therefore the function F(s, t) is holomorphic everywhere with the exception of points that are located on the hypersurfaces $s = c > 4$, $t = c > 9$; i.e., it is holomorphic on the "physical" sheet and has a Mandelstam representation (under the condition that it behave suitably at infinity).

4. An attentive reader has probably not failed to note the fact that our arguments are of a very general nature and can also be transferred to other cases without any essential changes. In fact, by using the continuity theorem, one can prove the following general result. Suppose we are given the contribution F(s, t) of an arbitrary diagram, and let the function F(s, t) be holomorphic in the real region $B(s, t \mid s < n_1^2,\ t < n_2^2)$; then complex singular points of the function F(s, t) can exist on the "physical" sheet only if there exists a function $c_0(a, b) > n_1^2 a + n_2^2 b$ which is defined and continuous for all a in the interval $0 < a < 1$, and if also there are real points of intersection of all planes $as + bt = c_0(a, b)$ with the Landau surfaces which for $c > c_0(a, b)$ become complex points of intersection. Geometrically this means that complex singular points of the function F(s, t) can appear only if outside the region B there is a part of a Landau branch which is convex upward and has negative slope, and is such that: 1) its tangents are straight lines $as + bt = c_0(a, b)$; 2) the parameter a runs through all values in the interval $0 < a < 1$; and 3) $c_0(a, b) > n_1^2 a + n_2^2 b$, or else in the region $s > n_1^2$, $t > n_2^2$ there is an isolated singular point or another singular point which "generates" complex points of intersection. (This geometrical interpretation is of course incomplete.)

In previous papers [10] a theorem has been proved, according to which the function F(s, t) is holomorphic in the "physical" sheet provided that real points of intersection do not become complex for $c > n_1^2 a + n_2^2 b$ and all $0 < a < 1$. From the results of the present paper it follows that the function F(s, t) will be holomorphic on the "physical" sheet also in cases in which for $c > n_1^2 a + n_2^2 b$ and some values of a in the interval $0 < a < 1$ real points of intersection become complex; all that is important is that real points of intersection should not become complex for all a in the interval $0 < a < 1$.

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77