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GEOMETRIC OPTICS OF OPEN RESONATORS

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Caustics are formed by the multiple reflection of rays from an elliptic mirror. The caustics can form ellipses or hyperbolas that are contained within the elliptical mirror and have the same foci. "Quantization" conditions that must be satisfied by the caustics and the rays are derived and the geometric meaning of these conditions is elucidated. This approach can be used to analyze the characteristic oscillations of certain open resonators; the oscillation regions are bounded by caustics and thus exhibit low diffraction losses. These results are generalized to arbitrary two-dimensional wave fields bounded by caustics. A method of designing mirrors with specified caustics is given.

MANY properties of open resonators follow as a consequence of geometric optics; for example, we can consider the formation of caustics that bound regions in which certain characteristic oscillations can occur. It is of interest to examine the relation between geometric optics and the theory of open resonators. We consider two-dimensional problems, in which case this relation is found to be a very natural one. The simplest example is the problem of elliptic mirrors, which we consider first; we then generalize the results to the case of mirrors and caustics of arbitrary form.

1. GEOMETRIC OPTICS OF ELLIPTIC MIRRORS

Consider the ellipse shown in Fig. 1; light rays lying in its plane can be reflected from this ellipse, which will be called the mirror ellipse. After a number of reflections a beam of light originally coinciding with the segment AB will cover some definite part of the area within the elliptic mirror; part of this area will also remain uncovered by the beam. In Fig. 1 we draw an ellipse which is confocal with the mirror ellipse and tangent to the ray AB (this will be called the inner ellipse). The ray BC, which is the reflection of ray AB in the mirror ellipse, is also tangent to the inner ellipse. This is a consequence of a well known theorem in geometry (cf. for example [1]). It is obvious that in all subsequent reflections the ray will always be tangent to the inner ellipse. Thus, the inner ellipse represents an envelope for this family of rays, that is to say, the inner ellipse is a caustic.

The considerations given above refer to the case in which the initial ray intersects the major axis of the ellipse to the right or left of the two foci. However, if the original ray intersects the major axis of the ellipse between the foci (Fig. 2) the caustics will be hyperbolas that are confocal with the mirror ellipse while the rays will cover the area between the branches of the hyperbola. The case of hyperbolic caustics can lead to certain complications. After being tangent to one branch of the hyperbola the light ray may experience several reflections (rather than a single reflection) from the mirror ellipse before it becomes tangent to the other branch of the hyperbola. Thus, after

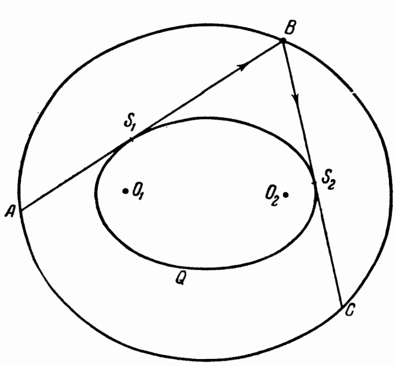

FIG. 1

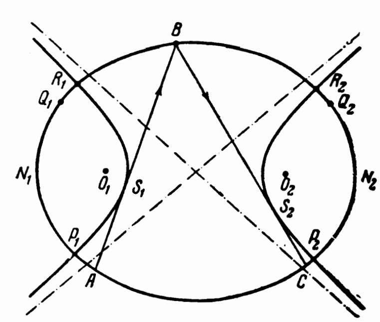

FIG. 2

several reflections the ray itself is not tangent to the hyperbola; rather, its geometric extension outside the area bounded by the ellipse plays the role of the tangent. However, this complication does not introduce any fundamental difference in the problem.

It will be evident that these constructions have direct significance for open resonators. In the case of hyperbolic caustics it is clear that a light ray propagating inside the ellipse will not go beyond the region bounded by the two branches of the hyperbola; hence, certain portions of the ellipse, for example the arcs $R_1N_1P_1$ and $R_2N_2P_2$, can be removed. Thus, one obtains an open resonator in which two mirrors, segments of an ellipse, contain a system of rays bounded by caustics. In going from the geometric-optics description to the more accurate wave description it is found that the wave field as a whole does actually penetrate beyond the caustics; hence, the mirror cannot be formed by the caustic itself but must be extended (for example the upper mirror must reach the points Q_1 and Q_2) in order to keep the radiation losses at a minimum.

Elliptic caustics can be realized in an open resonator formed by the arc of a mirror ellipse if it is bounded at the ends by mirrors that lie along hyperbolas that are confocal with the mirror ellipse and extend from the mirror ellipse to the inner ellipse and somewhat beyond. In a resonator of this kind the caustic is a segment of an ellipse confocal with the mirror ellipse (Fig. 3). Elliptic caustics are analogous to circular caustics in a cylindrical resonator and the open resonator shown in Fig. 3 is an open sector of such a resonator (cf. [2]).

2. QUANTIZATION CONDITIONS

Thus, according to geometric optics any ellipse or hyperbola that is confocal with a mirror ellipse can play the role of a caustic. Actually, however, at a given frequency only those ellipses or hyperbolas that satisfy a "quantization" condition can actually be caustics. These conditions determine the resonance frequencies of the open resonator.

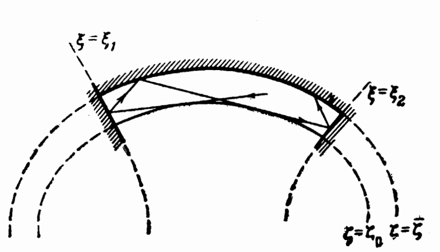

FIG. 3

In view of its two-dimensional nature one expects that the system will have two quantum numbers. It is evident that those quantities will be quantized which are invariant, i.e., quantities that are independent of the initial direction of the ray. The first condition, which can almost be written by inspection, is that the length of the closed caustic must equal a whole number of wavelengths; the length of the caustic contiguous to the reflecting mirror must be a whole number of half wavelengths. The second condition is less obvious. It is known from geometry [1] that the difference between the total length of two tangents to an ellipse $S_1B + S_2B$ and the length of elliptic arc S_1S_2 is a constant if the tangents intersect on a confocal ellipse (see Fig. 1). It is then reasonable to set this difference also equal to an integral number of wavelengths.

The application of the first condition to the elliptic caustic (Fig. 1) leads to the formula

$$\gamma \int_0^{2\pi} \sqrt{\mathrm{ch}^2 \zeta_0 - \sin^2 \xi}\, d\xi = 2m\pi, \qquad (1)^*$$

where ξ and ζ are elliptical coordinates; these are related to the Cartesian coordinates x and z by the expressions

$$x = d\, \mathrm{ch}\, \zeta \sin \xi, \qquad z = d\, \mathrm{sh}\, \zeta \cos \xi, \qquad (2)^\dagger$$

where d is half the distance between the foci while the parameter ν is

$$\gamma = kd,$$

where $k = \omega/c = 2\pi/\lambda$ is the wave number corresponding to the angular frequency ω or the wavelength λ.

In order to express the second condition in analytic form we calculate the difference between the total length of the two tangents S_1B and BS_2 and the elliptic arc S_1S_2. To calculate this difference we determine the difference of the phase advances $kds_1 - kds_2$. It follows from Eq. (2) that the phase change along the tangents is

$$kds_1 = \gamma \sqrt{(\mathrm{ch}^2 \zeta - \sin^2 \xi)\,[1 + (d\xi / d\zeta)^2]}\, d\zeta,$$

whereas the phase change along the inner ellipse $\zeta = \zeta_0$ is

$$kds_2 = \gamma (d\xi / d\zeta) \sqrt{\mathrm{ch}^2 \zeta_0 - \sin^2 \xi}\, d\zeta,$$

where the derivative $(d\xi/d\zeta)$ is computed along the tangent S_1B. Thus

*ch = cosh.

†sh = sinh.

$$kds_1 - kds_2 = \gamma\{\sqrt{(\mathrm{ch}^2\zeta - \sin^2\xi)[1 + (d\xi/d\zeta)^2]} - (d\xi/d\zeta)\sqrt{\mathrm{ch}^2\zeta_0 - \sin^2\xi}\}d\zeta. \tag{3}$$

The equation for the tangent to an ellipse passing through ξ_0 and ζ_0 has the following form in elliptical coordinates:

$$A\,\mathrm{ch}\,\zeta\sin\xi + B\,\mathrm{sh}\,\zeta\cos\xi = C, \qquad A = \mathrm{sh}\,\zeta_0\sin\xi_0,$$
$$B = \mathrm{ch}\,\zeta_0\cos\xi_0, \qquad C = \mathrm{sh}\,\zeta_0\,\mathrm{ch}\,\zeta_0. \tag{4}$$

Differentiation with respect to ζ yields

$$\frac{d\xi}{d\zeta} = \frac{B\,\mathrm{ch}\,\zeta\cos\xi + A\,\mathrm{sh}\,\zeta\sin\xi}{B\,\mathrm{sh}\,\zeta\sin\xi - A\,\mathrm{ch}\,\zeta\cos\xi}. \tag{5}$$

This relation is easily transformed by means of the identities

$$B\,\mathrm{ch}\,\zeta\cos\xi + A\,\mathrm{sh}\,\zeta\sin\xi = \sqrt{(\mathrm{ch}^2\zeta_0 - \sin^2\xi_0)(\mathrm{ch}^2\zeta_0 - \sin^2\xi)},$$
$$B\,\mathrm{sh}\,\zeta\sin\xi - A\,\mathrm{ch}\,\zeta\cos\xi = \sqrt{(\mathrm{ch}^2\zeta_0 - \sin^2\xi_0)(\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0)}. \tag{6}$$

In order to prove the first identity we square both sides of Eq. (4):

$$A^2\,\mathrm{ch}^2\zeta\sin^2\xi + 2AB\,\mathrm{ch}\,\zeta\,\mathrm{sh}\,\zeta\sin\xi\cos\xi + B^2\,\mathrm{sh}^2\zeta\cos^2\xi = C^2.$$

Carrying out an obvious transformation we find

$$(B\,\mathrm{ch}\,\zeta\cos\xi + A\,\mathrm{sh}\,\zeta\sin\xi)^2 = C^2 - A^2\sin^2\xi + B^2\cos^2\xi.$$

Substituting the values of A, B, C, on the right side and taking the square root of both sides we obtain the first identity (6). The second identity is proved in the same way.

Substituting these identities in Eq. (5) we have

$$d\xi/d\zeta = \sqrt{(\mathrm{ch}^2\zeta_0 - \sin^2\xi)/(\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0)}.$$

Correspondingly, Eq. (3) becomes

$$kds_1 - kds_2 = \gamma\sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\,d\zeta.$$

The same expression is obtained for the tangent BS_2. Hence the difference between the total length of the tangents and the length of the elliptical arc S_1S_2 is

$$k(S_1B + BS_2 - S_1S_2) = 2\gamma\int_{\zeta_0}^{\bar{\zeta}}\sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\,d\zeta,$$

where $\zeta = \bar{\zeta}$ is the equation of the mirror ellipse. The second quantization condition becomes

$$\gamma\int_{\zeta_0}^{\bar{\zeta}}\sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\,d\zeta = n\pi \qquad (n = 1, 2, \ldots). \tag{7}$$

If the elliptic caustic is only partially realized $\xi_1 < \xi < \xi_2$ (cf. Fig. 3) the first quantization condition (1) is modified as follows:

$$\gamma\int_{\xi_1}^{\xi_2}\sqrt{\mathrm{ch}^2\zeta_0 - \sin^2\xi}\,d\xi = m\pi, \tag{8}$$

while the second quantization condition (7) remains unchanged.

In the case of hyperbolic caustics the first quantization condition

$$\gamma\int_{-\bar{\zeta}}^{\bar{\zeta}}\sqrt{\mathrm{ch}^2\zeta - \sin^2\xi_0}\,d\zeta = n\pi \qquad (n = 1, 2, \ldots) \tag{9}$$

means that the length of the caustic contiguous to the mirror is equal to a whole number of half wavelengths. The second quantization condition

$$\gamma\int_{-\xi_0}^{\xi_0}\sqrt{\sin^2\xi_0 - \sin^2\xi}\,d\xi = m\pi \qquad (m = 0, 1, 2, \ldots) \tag{10}$$

is equivalent to the statement that the length of the curve $P_1S_1BS_2P_2$ (cf. Fig. 2) is equal to a whole number of half wavelengths. The equation $\zeta = \pm\bar{\zeta}$ defines the position of the elliptic mirrors while the equation $\xi = \pm\xi_0$ defines the position of the hyperbolic caustics.

The quantization conditions introduced above can be used to compute both the position of the caustics (i.e., the parameters ζ_0 or ξ_0) as well as the oscillation frequency (i.e., γ, k, or ω). These conditions are analogous to the Bohr-Sommerfeld conditions in quantum mechanics. In the next section we show that these conditions are essentially the same as those obtained by a more rigorous analysis.

3. RIGOROUS DERIVATION OF THE QUANTIZATION CONDITIONS

In order to carry out a rigorous analysis of the wave field we start with the two-dimensional wave equation

$$\partial^2\Phi/\partial x^2 + \partial^2\Phi/\partial z^2 + k^2\Phi = 0 \tag{11}$$

using the boundary conditions at the mirrors. In what follows we shall make use of the simplest boundary condition

$$\Phi = 0. \tag{12}$$

In elliptic coordinates (2) Eq. (11) becomes

$$\partial^2\Phi/\partial\xi^2 + \partial^2\Phi/\partial\zeta^2 + \gamma^2(\mathrm{ch}^2\zeta - \sin^2\xi)\Phi = 0$$

and allows separation of variables, i.e., we can obtain particular solutions of the form

$$\Phi = X(\xi)Z(\zeta).$$

The functions X and Z are the solutions of the ordinary differential equations

$$d^2X / d\xi^2 + \gamma^2(\varkappa - \sin^2\xi)X = 0, \qquad (13)$$

$$d^2Z / d\zeta^2 + \gamma^2(\mathrm{ch}^2\zeta - \varkappa)Z = 0. \qquad (14)$$

When $\kappa < 1$ we put $\kappa = \sin^2 \xi_0$ and the coefficient of X in Eq. (13) vanishes at $\xi = \xi_0$. When $\kappa > 1$ we put $\kappa = \cosh^2 \zeta_0$ and correspondingly the coefficients of Z in Eq. (14) vanish when $\zeta = \zeta_0$. It will become clear below that the equations $\xi = \pm \xi_0$ and $\zeta = \zeta_0$ determine the shape and position of the caustics.

Let us now consider the solution of Eq. (13) when $\kappa = \cosh^2 \zeta_0$. Since we assume that all dimensions of the mirror ellipse are much greater than the wavelength, Eq. (13) will contain the large parameter $\nu \gg 1$; this situation can be used to simplify the solution of the equation. We expand the solution in powers of the reciprocal of this large parameter, obtaining as a first approximation,

$$X = \mathrm{const}\,[\gamma^2(\mathrm{ch}^2\zeta_0 - \sin^2\xi)]^{-1/4} \times \exp\left[\pm\, i \int_0^{\xi} \sqrt{\mathrm{ch}^2\zeta_0 - \sin^2\xi}\, d\xi\right]. \qquad (15)$$

This is the semiclassical or WKB approximation. Evidently the solution (15) must be periodic in ξ:

$$X(\xi) = X(\xi + 2\pi),$$

which immediately leads to the quantization condition in (1).

For the resonators shown in Fig. 3, the periodicity condition is replaced by the boundary conditions (12) at the mirrors $\xi = \xi_1$ and $\xi = \xi_2$ in which case the function X is taken in the form

$$X = \mathrm{const}[\gamma^2(\mathrm{ch}^2\zeta_0 - \sin^2\xi)]^{-1/4} \sin\left[\gamma \int_{\xi_1}^{\xi} \sqrt{\mathrm{ch}^2\zeta_0 - \sin^2\xi}\, d\xi\right].$$

In this case the quantization condition in (8) is obtained.

When $\kappa = \cosh^2 \zeta_0$ and $\zeta > \zeta_0$ we can solve Eq. (14) in similar fashion, obtaining

$$Z = C_0[\gamma^2(\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0)]^{-1/4} \times \sin\left[\gamma \int_{\zeta_0}^{\zeta} \sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\, d\zeta + \varphi_0\right]. \qquad (16)$$

When $\zeta < \zeta_0$ the solution is of the form

$$Z = [\gamma^2(\mathrm{ch}^2\zeta_0 - \mathrm{ch}^2\zeta)]^{-1/4} \times \left\{C_1 \exp\left[-\gamma \int_{\zeta}^{\zeta_0} \sqrt{\mathrm{ch}^2\zeta_0 - \mathrm{ch}^2\zeta}\, d\zeta\right] + C_2 \exp\left[\gamma \int_{\zeta}^{\zeta_0} \sqrt{\mathrm{ch}^2\zeta_0 - \mathrm{ch}^2\zeta}\, d\zeta\right]\right\}, \qquad (17)$$

where C_0, C_1, and C_2 are constants. None of these solutions apply near $\zeta = \zeta_0$ because at this point the large parameter γ^2 is multiplied by the small quantity $\cosh^2 \zeta - \cosh^2 \zeta_0$. Since the solution cannot grow exponentially as ζ diminishes, the constant C_2 must be set equal to zero. Thus, Z is oscillatory when $\zeta > \zeta_0$ and dies out exponentially when $\zeta < \zeta_0$. The equation $\zeta = \zeta_0$ defines the caustic. In order to connect the solution on both sides of the point ζ_0, i.e., to choose the phase constant φ_0 in Eq. (16), it is necessary to carry out a more complete analysis.

For large values of γ, an approximate representation of the function Z over the entire range $0 \le \zeta \le \bar{\zeta}$ (including the point ζ_0) is

$$Z = C_0[-t / \gamma^2(\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0)]^{1/4} v(t), \qquad (18)$$

where the variables t and ζ satisfy the relations

$$\tfrac{2}{3}(-t)^{3/2} = \gamma \int_{\zeta_0}^{\zeta} \sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\, d\zeta \quad \text{for } \zeta > \zeta_0,$$

$$\tfrac{2}{3}t^{3/2} = \gamma \int_{\zeta}^{\zeta_0} \sqrt{\mathrm{ch}^2\zeta_0 - \mathrm{ch}^2\zeta}\, d\zeta \quad \text{for } \zeta < \zeta_0, \qquad (19)$$

where v(t) is the Airy function, which falls off exponentially as $t \to \infty$. When $t \to -\infty$ this function can be given by the asymptotic expression

$$v(t) = (-t)^{-1/4} \sin[\tfrac{2}{3}(-t)^{3/2} + \pi/4]. \qquad (20)$$

There is also a second Airy function u(t) which increases exponentially when $t \to \infty$. This solution is discarded for the same reason that we have written $C_2 = 0$ in Eq. (17). The properties of the Airy function are given by Fock[3,4]. The derivation of Eqs. (18) and (19) can also be found in [3].

If $\zeta > \zeta_0$, and we consider points far away from the caustic so that $-t \gg 1$, using Eq. (20), we can write Eq. (18) in the form

$$Z = C_0[\gamma^2(\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0)]^{-1/4} \times \sin\left[\gamma \int_{\zeta_0}^{\zeta} \sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\, d\zeta + \pi/4\right]. \qquad (21)$$

Thus, the constant $\varphi_0 = \pi/4$ in Eq. (16). If the solution in (21) is to satisfy the boundary condition at $\zeta = \bar{\zeta}$ the following condition must be satisfied:

$$\gamma \int_{\zeta_0}^{\bar{\zeta}} \sqrt{\mathrm{ch}^2\zeta - \mathrm{ch}^2\zeta_0}\, d\zeta = (n - \tfrac{1}{4})\pi. \qquad (22)$$

The condition (22) differs from (7) in that it contains $n - \frac{1}{4}$ rather than n. The significance of both expressions is the same, but (22) takes account of the fact that the geometric-optics analysis [Eq. (16)] does not apply near the caustic $\zeta = \zeta_0$.

If one considers only those rays for which $\zeta > \zeta_0$ (far from the caustic) in accordance with Eq. (20) it is necessary to take account of the fact that the phase of a ray going away from the caustic differs by $\pi/2$ from that computed on the basis of geometric optics using the phase of the incoming ray. This is the so-called "phase jump" at a caustic (cf. for example [5]) and it means that n is replaced by $n - 1/4$ in the quantization condition (22).

Now let $\kappa = \sin^2 \xi_0$. We first consider Eq. (14). Since the coefficient of Z does not vanish at any point the semiclassical approximation applies over the whole range of variation of ζ so that

$$Z = \text{const}\,[\gamma^2(\text{ch}^2\zeta - \sin^2\xi_0)]^{-1/4} \times \frac{\cos}{\sin}\left[\gamma\int_0^{\zeta}\sqrt{\text{ch}^2\zeta - \sin^2\xi_0}\,d\zeta\right]. \tag{23}$$

The quantization condition (9) is obtained from the boundary condition (12) at the mirrors $\zeta = \pm\bar{\zeta}$; for odd values of n the cosine is used in Eq. (23) while even values of n require the sine.

The function X (with $\kappa = \sin^2 \xi_0$) can no longer be given by the semiclassical approximation

$$X = \frac{C}{[\gamma^2(\sin^2\xi_0 - \sin^2\xi)]^{1/4}}\,\frac{\cos}{\sin}\left[\gamma\int_0^{\xi}\sqrt{\sin^2\xi - \sin^2\xi_0}\,d\xi\right] \tag{24}$$

since the difference $\sin^2\xi - \sin^2\xi_0$ vanishes when $\xi = \pm\xi_0$, i.e., on the hyperbolic caustics. The use of the sine or cosine is determined by the symmetry of the problem since the characteristic functions must be either even or odd functions of ξ. Using the Airy function we obtain the following expression for the function X for the range $0 \le \xi < \xi_0$

$$X = \frac{C'}{[\gamma^2(\sin^2\xi_0 - \sin^2\xi)]^{1/4}} \times \sin\left[\gamma\int_{\xi}^{\xi_0}\sqrt{\sin^2\xi_0 - \sin^2\xi}\,d\xi + \frac{\pi}{4}\right], \tag{25}$$

this coincides with Eq. (24) when

$$\gamma\int_0^{\xi_0}\sqrt{\sin^2\xi_0 - \sin^2\xi}\,d\xi = \left(m + \frac{1}{2}\right)\frac{\pi}{2} \quad (m = 0, 1, 2, \ldots), \tag{26}$$

where odd values of m correspond to the cosine in Eq. (24) and even values of m to the sine. This condition differs from (10) in that m is replaced by $m + 1/2$; as before, this as associated with the phase jump at the caustic, i.e., the difference arises because geometric optics does not apply near the caustic.

If the parameter ξ_0 is small Eqs. (9) and (26) reduce to the approximate expression derived by one of the authors [6] for cylindrical mirrors. In this case $\sin\xi$ can be replaced by ξ and Eq. (26) assumes the form

$$\gamma\int_0^{\xi_0}\sqrt{\xi_0^2 - \xi^2}\,d\xi = \left(m + \frac{1}{2}\right)\frac{\pi}{2} \qquad (m = 0, 1, 2, \ldots)$$

or

$$\tfrac{1}{2}\gamma\xi_0^2 = m + \tfrac{1}{2}. \tag{27}$$

Making the analogous approximations in Eq. (9) we find

$$\gamma\int_0^{\bar{\zeta}}\text{ch}\,\zeta\left(1 - \frac{\xi_0^2}{2\,\text{ch}^2\zeta}\right)d\zeta = n\frac{\pi}{2}$$

or

$$\gamma\,\text{sh}\,\bar{\zeta} - \tfrac{1}{2}\gamma\xi_0^2\,\text{arc sin th}\,\bar{\zeta} = n\pi/2. \tag{28*}$$

Introducing the notation

$$l = d\,\text{sh}\,\bar{\zeta}$$

and

$$\alpha = \text{arc sin th}\,\bar{\zeta} = \text{arc sin}\sqrt{l/r_0},$$

where $r_0 = d\sinh^2\bar{\zeta}/\cosh\bar{\zeta}$ is the radius of curvature of the mirrors at the point $\xi = 0$, and $2l$ is the distance between mirrors, using the relation in (27) we can write Eq. (28) in the form

$$2kl = n\pi + (2m + 1)\alpha, \tag{29}$$

in accordance with [6].

It should be emphasized that the approximate formulas written above apply only if the wavelengths are small. This means that the quantization numbers m and n must be large (either one or both). Equation (29) applies for large n and reasonably small m.

As shown in [6] there is a rough analogy between the quantum mechanics of the harmonic oscillator and the wave optics of cylindrical mirrors when $\xi_0 \ll 1$. Equation (27) corresponds to the "half-integral" quantization of the oscillator energy in contrast with the "integral" quantization that follows from Eq. (10).

4. MIRROR AND CAUSTICS

A number of general conclusions can be drawn from the preceding sections if one takes account of the following considerations. Assume that we are given a closed convex curve in a plane (Fig. 4). From a point A two tangents are extended to this curve. Next, consider the geometric location of

*th = tanh.

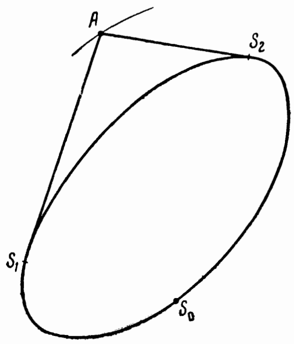

FIG. 4

the point A for which the perimeter of the figure $AS_1S_0S_2A$ is a constant; this curve may be called a bi-involute. It can be shown that the tangents S_1A and S_2A make equal angles at A with the normal to the bi-involute.

The inverse statement also holds. If the normal to a curve forms at each point equal angles with the tangents to some other curve $S_0S_1S_2$ the perimeter of the figure $AS_1S_0S_2A$ is a constant regardless of the position of A.

If the bi-involute is a mirror then the original curve is a caustic, that is to say, the original curve is the envelope of the family of rays reflected from the bi-involute. As we have already seen in the case of the elliptic mirror, only those caustics can be realized which satisfy the quantization conditions.

There are also quantization conditions that apply in the general case. The first quantization condition requires that the perimeter of the closed caustic must be equal to a whole number of wavelengths; the perimeter of the caustic contiguous to the mirror must also be equal to a whole number of half wavelengths. The second quantization condition, when modified by the general corrections obtained in Sec. 3, states that in the case of a single closed caustic (Fig. 4) the perimeter of the figure $AS_1S_0S_2A$ must consist of a whole number of wavelengths minus a quarter wavelength. However, if the caustic consists of two branches (as in Figs. 2 and 5) the length of the curve $P_1S_1BS_2P$ must be equal to an odd number of quarter wavelengths. The first condition follows from the fact that at sufficiently short wavelengths the phase velocity along the caustic is c. The second condition follows from the requirement that the field must be single-valued; in this formulation it is assumed that the caustic has finite curvature (and that the radius of curvature is appreciably greater than the wavelength).

The considerations given above offer a method for graphical construction of mirrors with specified caustics. This problem is encountered, for example, in the design of quantum oscillators.

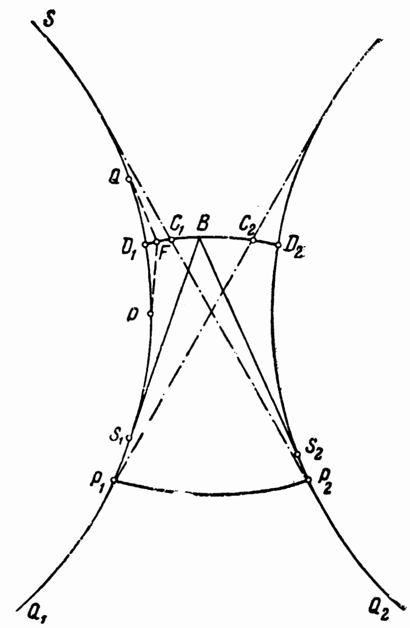

FIG. 5

Suppose that the caustics are to be two circles (Fig. 5). Using this example we shall show how the method of construction must be modified when the ray experiences more than one reflection from a given mirror in going from one caustic to another. We first stretch a flexible wire along $O_1S_1BS_2O_2$. Then, inserting a pencil at point B and holding the wire taut we trace out the curve between points C_1 and C_2. These points lie on the common tangents to the two specified circles. The small segments of the mirror C_1D_1 and C_2D_2 are constructed in a different way; specifically, the wire is stretched from one of the circles. For example, to plot the segment C_1D_1 the wire must be stretched along $SQFPO_1$. Then we trace out the segment C_1D_1 from the condition that the difference of the lines SQF and FPO_1 is a constant. In the present case, in which the caustics are circles, the segment C_1D_1 is a straight line. In the last construction we use the fact that the plotted curve bisects the angle between the wires (or the tangents).

CONCLUSION

We have shown in this work that oscillations in open resonators bounded by caustics can be analyzed in terms of geometric optics. This approach is highly instructive and allows us to compute the characteristic frequency and field distribution for a given mode, but obviously does not give diffraction losses. However, these losses are unimportant for these modes because in all practical cases the diffraction losses are smaller than the losses due to nonideal reflection from the mirrors. The method given in Sec. 4 for designing mirrors for specified caustics can be used to design kinematic mechanisms for the fabrication and polishing of mirrors.

The authors wish to thank E. L. Kosarev for valuable comments, especially for the discussion of the geometric constructions.

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[1] G. Salmon, Conic Sections, 1860, Secs. 194 and 356 (Reprinted by Chelsea, N.Y.).

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