LOWER CRITICAL FIELD AND NONLINEAR EFFECTS IN THE ELECTRODYNAMICS OF SUPERCONDUCTING ALLOYS

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The lower critical field H_{c1} for highly impure alloys ($l \ll \xi_0$) is found. It is given by formulas (34) and (30), where H_c is the thermodynamic critical field. Corrections to the depth of penetration of a weak magnetic field into a superconducting alloy are also computed.

INTRODUCTION

ONE of the features of the behavior of superconducting alloys in a magnetic field is the appreciable range of their transition from the superconducting into the normal state. Accordingly, the alloys are characterized by the presence of two critical fields—upper H_{c2} at which the first occurrence of the superconducting phase is possible, and so-called lower critical field H_{c1}, corresponding to the start of penetration of the magnetic field into the superconductor.

Within the framework of the phenomenological Ginzburg-Landau theory, i.e., near the critical temperature of the superconducting transition, the resultant state was investigated in detail by Abrikosov[1]. In particular, he obtained expressions for the critical fields H_{c1} and H_{c2}:

$$H_{c1} = H_c(\ln\varkappa + 0.08)/\sqrt{2}\varkappa, \qquad H_{c2} = H_c\sqrt{2}\varkappa,$$

where H_c—thermodynamic critical field and κ—parameter of the Ginzburg-Landau theory

$$\varkappa = \frac{\sqrt{2}\,2e}{\hbar c}[H_c\delta_0^2]_{T=T_c}.$$

Abrikosov obtained his results for the lower critical field under the assumption that $\kappa \gg 1$. Following the formulation of the BCS microscopic theory of superconductivity[2], Gor'kov showed[3] that the noted similarities in the behavior of superconducting alloys in a magnetic field resulted from the change in the superconducting correlation of the conduction electrons scattered by the impurities. It turned out then that the parameter κ of the Ginzburg-Landau theory depends on the mean free path l. With decreasing mean free path, the superconducting alloy loses the properties of a pure superconductor at a certain impurity concentration and acquires the characteristic properties observed experimentally in magnetic fields. For sufficiently large concentrations, κ increases like $1/l$ and can become very large, as assumed also by Abrikosov[1]. There is no doubt at present that the above picture describes well all the features of the behavior of alloys in a magnetic field.

Later on Shapoval[4], likewise within the framework of the microscopic theory, investigated the upper critical field H_{c2} for superconducting alloys over the entire temperature interval.

In the present paper we obtain the lower critical field over the entire temperature interval, for a strongly contaminated alloy ($l \ll \xi_0$, $\xi_0 = \hbar v/2\pi kT_c$—correlation parameter). The condition $l \ll \xi_0$ is equivalent to the condition $\kappa \gg 1$ used in [1] for an analysis of the region near the lower critical field. The possibility of solving this problem is connected with the fact that in the essential region the equations of the superconducting alloys are local, and the quantities characterizing the superconductor differ little from their values in the homogeneous problem in the absence of a field.

We also calculate in this paper corrections quadratic in the field for the depth of penetration of a weak field.

1. LOWER CRITICAL FIELD

Like Abrikosov[1], we assume here that the transition from the superconducting state into the mixed state is a second-order phase transition. We assume also an analogous distribution of the currents in the field, i.e., we assume that the penetration of the field in the superconductor is in the form of thin filaments, separated from one another by large distances, so that they can be regarded as independent. We shall consider henceforth only one filament.

The field distribution in a superconducting alloy is determined with the aid of the equations for the Green's function and Maxwell's equation

$$\begin{pmatrix} i\omega - [i\nabla_x + e\mathbf{A}(\mathbf{x})]^2/2m + \mu - \sum_a u(\mathbf{x}-\mathbf{x}_a) & \Delta(\mathbf{x}) \\ -\Delta^*(\mathbf{x}) & -i\omega - [i\nabla_x - e\mathbf{A}(\mathbf{x})]^2/2m + \mu - \sum_a u(\mathbf{x}-\mathbf{x}_a) \end{pmatrix} \times \begin{pmatrix} \mathfrak{G}_\omega(\mathbf{x},\mathbf{x}') & -\mathfrak{F}_\omega(\mathbf{x},\mathbf{x}') \\ \mathfrak{F}_\omega^+(\mathbf{x},\mathbf{x}') & \mathfrak{G}_{-\omega}(\mathbf{x}',\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \delta(\mathbf{x}-\mathbf{x}') & 0 \\ 0 & \delta(\mathbf{x}-\mathbf{x}') \end{pmatrix}, \quad (1)^*$$

$$\Delta(\mathbf{x}) = gT\sum_\omega \mathfrak{F}_\omega(\mathbf{x},\mathbf{x}), \qquad \mathrm{rot}\,\mathrm{rot}\,\mathbf{A}(\mathbf{x}) = -4\pi\mathbf{j}(\mathbf{x}), \quad (2)$$

where $\mathbf{A}(\mathbf{x})$—vector potential, $\omega = 2\pi(n + \frac{1}{2})T$, $u(\mathbf{x} - \mathbf{x}_\alpha)$—potential of interaction with the impurity atoms, g—constant of interaction of the electrons with one another, μ—chemical potential, and $\mathfrak{G}_\omega(\mathbf{x},\mathbf{x}')$, $\mathfrak{F}_\omega(\mathbf{x},\mathbf{x}')$ and $\mathfrak{F}^+_\omega(\mathbf{x},\mathbf{x}')$—temperature Green's functions, defined, for example, in [5]. The current density $\mathbf{j}(\mathbf{x})$ is connected with $\mathfrak{G}_\omega(\mathbf{x},\mathbf{x}')$ by the relation

$$\mathbf{j}(\mathbf{x}) = 2T\sum_\omega \left\{\frac{ie}{2m}(\nabla_{x'} - \nabla_x)\mathfrak{G}_\omega(\mathbf{x},\mathbf{x}') - \frac{e^2\mathbf{A}(\mathbf{x})}{m}\mathfrak{G}_\omega(\mathbf{x},\mathbf{x}')\right\}_{\mathbf{x}'\to\mathbf{x}} \quad (3)$$

Equation (1) was written out in matrix form, with the aggregate of the Green's function forming a single matrix Green's function for the operator in the left side of (1)

$$\hat{G}_\omega(\mathbf{x},\mathbf{x}') = \begin{pmatrix} \mathfrak{G}_\omega(\mathbf{x},\mathbf{x}') & -\mathfrak{F}_\omega(\mathbf{x},\mathbf{x}') \\ \mathfrak{F}_\omega^+(\mathbf{x},\mathbf{x}') & \mathfrak{G}_{-\omega}(\mathbf{x}',\mathbf{x}) \end{pmatrix}.$$

Inasmuch as the problem has cylindrical symmetry, we choose the direction of the filament along the z axis, and direct the vector potential $\mathbf{A}$ perpendicular to the radius vector. We exclude the phase dependence from among the quantities in (1), (2), and (3), assuming, as in [1], that it is of the form

$$\Delta(\mathbf{x}) = \Delta'(\mathbf{x})e^{i\varphi}, \qquad \mathfrak{G}_\omega(\mathbf{x},\mathbf{x}') = \mathfrak{G}_\omega'(\mathbf{x},\mathbf{x}')e^{i(\varphi-\varphi')/2},$$
$$\mathfrak{F}_\omega(\mathbf{x},\mathbf{x}') = \mathfrak{F}_\omega'(\mathbf{x},\mathbf{x}')e^{i(\varphi+\varphi')/2},$$
$$\mathfrak{F}_\omega^+(\mathbf{x},\mathbf{x}') = \mathfrak{F}_\omega^{+\prime}(\mathbf{x},\mathbf{x}')e^{-i(\varphi+\varphi')/2}, \quad (4)$$

where φ—polar angle of the point with coordinate x.

Substituting expressions (4) in (1), (2), and (3) we obtain a system of equations in which primed quantities will now be involved. Leaving off the primes, we obtain a system of equations that differs from (1)—(3) only in the fact that $\mathbf{A}(\mathbf{x})$ is replaced by

$$\mathbf{Q}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) - \nabla\psi/2e.$$

We shall have in mind this function in what follows.

We shall show below that when $\kappa \gg 1$, as in [1], the main contribution is made by the region of large distances from the center of the filament, where the field is small. Therefore, after excluding from the Green's functions the phase dependence (4), we can solve (1)—(3) by expanding all the quantities in terms of small deviations from their values in the absence of a field. In the expansion of $\hat{G}_\omega(\mathbf{x},\mathbf{x}')$ it is sufficient here to retain the terms that are linear in $\Lambda(\mathbf{x}) = \Delta(\mathbf{x}) - \Delta$ (Δ—value of the gap in the absence of the field) and quadratic in $\mathbf{Q}(\mathbf{x})$. As will be shown below, such expansions are valid at distances $r \gg \delta_0/\kappa \sim (\xi_0 l)^{1/2}$.

For the current density in this region we have [5]

$$\mathbf{j}(\mathbf{x}) = -\frac{\mathbf{Q}(\mathbf{x})}{4\pi\delta_0^2}, \qquad \delta_0 = \frac{1}{2\pi}\left[\frac{1}{\Delta\sigma\,\mathrm{th}\,(\Delta/2T)}\right]^{1/2}, \quad (5)^*$$

where $\sigma = Ne^2\tau_{tr}/m$—residual resistance of normal alloy.

The equation for $Q(\mathbf{x}) = |\mathbf{Q}(\mathbf{x})|$ takes then the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Q}{\partial r}\right) - \frac{Q}{r^2} = \frac{Q}{\delta_0^2}. \quad (6)$$

Its solution under the boundary conditions $Q(\infty) = 0$, $Q(r \ll \delta_0) = (2e)^{-1}\nabla\varphi = (2er)^{-1}$ is

$$Q(r) = \frac{1}{2e\delta_0}K_1\left(\frac{r}{\delta_0}\right), \quad (7)$$

where $K_1(x)$—Bessel function of imaginary argument.

Substituting (7) in (1) we obtain a system of equations for the Green's functions, which we can rewrite in integral form by using the matrix notation:

$$\hat{G}_\omega(\mathbf{x},\mathbf{x}') = \hat{G}_{0\omega}(\mathbf{x},\mathbf{x}') + \int \hat{G}_{0\omega}(\mathbf{x},\mathbf{l})\hat{\Lambda}(\mathbf{l})\hat{G}_\omega(\mathbf{l},\mathbf{x}')\,d\mathbf{l} + \int \hat{G}_{0\omega}(\mathbf{x},\mathbf{l})\hat{Q}(\mathbf{l})\hat{G}_\omega(\mathbf{l},\mathbf{x}')\,d\mathbf{l},$$
$$\frac{\hat{\Lambda}(\mathbf{x})}{g} = T\sum_\omega\{\hat{G}_\omega(\mathbf{x},\mathbf{x}) - \hat{G}_{0\omega}(\mathbf{x},\mathbf{x})\}. \quad (8)$$

We have used here also the following notation:

$$\hat{\Lambda}(\mathbf{x}) = \Lambda(\mathbf{x})\hat{I}, \qquad \hat{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\hat{Q}(\mathbf{x}) = \frac{ie}{m}Q(\mathbf{x})\mathbf{n}_x\hat{\sigma}\nabla_x,$$
$$\hat{\sigma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathbf{n}_x = \frac{\mathbf{Q}(\mathbf{x})}{Q(\mathbf{x})}; \quad (9)$$

*rot = curl.

*th = tanh.

$\hat{G}_{0\omega}(\mathbf{x}, \mathbf{x}')$—Green's function of the superconducting alloy in the absence of a field.

Equation (8) is solved by expanding $\hat{G}_\omega(\mathbf{x}, \mathbf{x}')$ in a series in $\Lambda(\mathbf{x})$ and $Q(\mathbf{x})$. Each term of the expansion can be set here in correspondence with a definite graph, thus arriving at a diagram technique. The difference between this technique and that used for superconductors lies in the fact that each line between x and x′ is set in correspondence with a matrix Green's function $\hat{G}_{0\omega}(\mathbf{x}, \mathbf{x}')$, and two types of vertices are produced, one containing the matrix $\hat{\sigma}$ and the other $\hat{I}$.

Expanding $\hat{G}_\omega(\mathbf{x}, \mathbf{x}')$ with the required accuracy, and using the definition of $\hat{\Lambda}(\mathbf{x})$, we obtain

$$\frac{\hat{\Lambda}(\mathbf{x})}{g} = T\sum_\omega \left\{ \int \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\,\hat{\Lambda}(\mathbf{l})\,\hat{G}_{0\omega}(\mathbf{l}, \mathbf{x})\,d\mathbf{l} + \int \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\,\hat{Q}(\mathbf{l})\,\hat{G}_{0\omega}(\mathbf{l}, \mathbf{s})\,\hat{Q}(\mathbf{s})\,\hat{G}_{0\omega}(\mathbf{s}, \mathbf{x})\,d\mathbf{l}\,d\mathbf{s} \right\}. \quad (10)$$

It is also necessary to average the quantities contained in this equation over the positions of the impurity atoms. The technique for this averaging was developed in [5,6] and reduces briefly to the following.

Denoting scattering by impurity atoms on the diagrams for the Green's functions or for their products by means of a cross, we can show that each cross corresponds to a vertex $u(\mathbf{q})\exp(i\mathbf{q}\cdot\mathbf{x}_a)$ ($u(\mathbf{q})$—Fourier component of the potential of electron-impurity interaction, $\mathbf{x}_a$—position of the impurity atom). Averaging over $\mathbf{x}_a$, we obtain if $\mathbf{q} \neq 0$ a zero value if this diagram does not contain one more scattering at the point $\mathbf{x}_a$ with $\mathbf{q}' = -\mathbf{q}$. Otherwise the result of the averaging is equal to $n|u(\mathbf{q})|^2$, where n—concentration of the impurity atoms. The averaging of such a pair corresponds on the diagram to a dashed line joining two identical impurity atoms. It is shown in [5] that diagrams containing more than two scatterings from one impurity atom, or diagrams in which the dashed line encloses a vertex with momentum transfer $\sim p_0$, need not be taken into account, since their contribution is smaller than the remaining contributions by a factor $1/p_0 l \ll 1$.

It is possible to average $\hat{\Lambda}(\mathbf{x})$ and the products of the Green's functions in (10) separately, for whereas $\hat{\Lambda}(\mathbf{x})$ varies slowly, the Green's functions oscillate over distances $\sim 1/p_0$. Denoting the averaged products of the Green's function by $\hat{K}$ and $\hat{L}$:

$$\hat{K}_\omega(\mathbf{x}-\mathbf{l}, \mathbf{l}-\mathbf{x}') = \langle \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\,\hat{I}\hat{G}_{0\omega}(\mathbf{l}, \mathbf{x}')\rangle,$$

$$\hat{L}_\omega(\mathbf{x}-\mathbf{x}') = \left(\frac{ie}{m}\right)^2 \int \langle \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\,\hat{\sigma}\mathbf{n}_l\nabla_l \hat{G}_{0\omega}(\mathbf{l}, \mathbf{s}) \times \hat{\sigma}\mathbf{n}_s\nabla_s \hat{G}_{0\omega}(\mathbf{s}, \mathbf{x}')\rangle\,d\mathbf{s}\,d\mathbf{l} \quad (11)$$

and recognizing that the distance over which the integrand in the second term of the right side of (10) changes is $\sim l \ll \delta_0$, we can write the equation for $\hat{\Lambda}(\mathbf{x})$ in the form

$$\frac{\hat{\Lambda}(\mathbf{x})}{g} = T\sum_\omega \left\{ \int \Lambda(\mathbf{q})\,e^{i\mathbf{q}\mathbf{x}}\hat{K}_\omega(\mathbf{p}, \mathbf{p}-\mathbf{q})\frac{d\mathbf{p}\,d\mathbf{q}}{(2\pi)^6} + Q^2(\mathbf{x})\int \hat{L}_\omega(\mathbf{p})\frac{d\mathbf{p}}{(2\pi)^3} \right\}, \quad (12)$$

where $\hat{K}_\omega(\mathbf{p}_1, \mathbf{p}_2)$ and $\hat{L}_\omega(\mathbf{p})$ are the Fourier components of the functions $\hat{K}$ and $\hat{L}$:

$$\hat{K}_\omega(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^6}\int \hat{K}_\omega(\mathbf{p}_1, \mathbf{p}_2)\exp\{i\mathbf{p}_1\mathbf{x} + i\mathbf{p}_2\mathbf{x}'\}\,d\mathbf{p}_1\,d\mathbf{p}_2,$$

$$\hat{L}_\omega(\mathbf{x}) = \frac{1}{(2\pi)^3}\int \hat{L}_\omega(\mathbf{p})\,e^{i\mathbf{p}\mathbf{x}}\,d\mathbf{p}. \quad (13)$$

We now proceed directly to determine the functions $\hat{K}$ and $\hat{L}$.

Figure 1 shows some diagrams obtained by averaging $\hat{K}$ (here $\mathbf{p}_1 - \mathbf{p}_2 = \mathbf{p}_1' - \mathbf{p}_2' = \mathbf{q}$). Summation of diagrams of the type of Fig. 1a leads to a replacement of the Green's functions of the pure superconductor by the corresponding functions of the alloy, which differ from the former by an exponential factor $\exp\{-|\mathbf{x}-\mathbf{x}'|/2l\}$. In the momentum representation this corresponds to replacing (ω, Δ) by $(\omega\eta_\omega, \Delta\eta_\omega)$, where

$$\eta_\omega = 1 + \frac{1}{2\tau\sqrt{\omega^2+\Delta^2}}, \qquad \frac{1}{\tau} = \frac{np_0 m}{(2\pi)^2}\int |u(\vartheta)|^2\,d\Omega_\vartheta.$$

Diagrams of the type of Fig. 1b must be taken into account, since the momentum transfer in the vertex is $q \ll p_0$. Summation of these diagrams leads to an integral equation for $\hat{K}$, the graphic form of which is given in Fig. 1c:

$$\hat{K}_\omega(\mathbf{p}_1, \mathbf{p}_2) = \hat{G}_\omega(\mathbf{p}_1)\hat{I}\hat{G}_\omega(\mathbf{p}_2) + \hat{G}_\omega(\mathbf{p}_1)\frac{n}{(2\pi)^3}\int |u(\mathbf{p}_1 - \mathbf{p}_1')|^2 \times \hat{K}_\omega(\mathbf{p}_1', \mathbf{p}_2')\,d\mathbf{p}_1'\hat{G}_\omega(\mathbf{p}_2). \quad (14)$$

It is more convenient, however, to solve the equation not for $\hat{K}$, but for the vertex function $\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2)$, connected with $\hat{K}$ by the relation

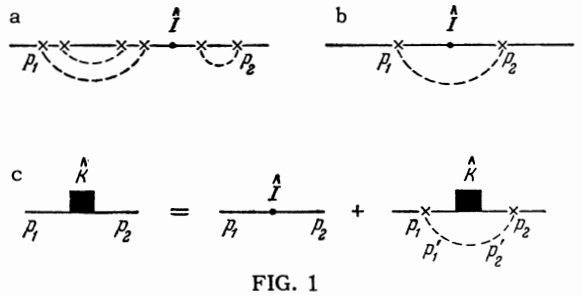

FIG. 1

$$\hat{K}_\omega(\mathbf{p}_1, \mathbf{p}_2) = \hat{G}_\omega(\mathbf{p}_1)\,\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2)\,\hat{G}_\omega(\mathbf{p}_2). \quad (15)$$

From (14) and (15) we obtain an equation for $\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2)$:

$$\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2) = I + \frac{n}{(2\pi)^3}\int |u(\mathbf{p}_1 - \mathbf{p}_1')|^2 \times \hat{G}_\omega(\mathbf{p}_1')\,\hat{\Pi}_\omega(\mathbf{p}_1', \mathbf{p}_2')\,\hat{G}_\omega(\mathbf{p}_2')\,d\mathbf{p}_1'. \quad (16)$$

Its solution is sought for small q, i.e., at large distances from the center of the filament, so that $\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2)$ can be expanded in powers of q accurate to terms quadratic in q inclusive:

$$\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2) = \hat{\Pi}_{0\omega}(\mathbf{p}_1) + \hat{\Pi}_{2\omega}(\mathbf{p}_1)(qv)^2.$$

If $p_1 \sim p_0$, the dependence of $\hat{\Pi}_{0\omega}$ and $\hat{\Pi}_{2\omega}$ on the momentum can henceforth be neglected.

Writing out (16) in terms of the components, we can deduce from the form of the resultant equations that each of the functions $\hat{\Pi}_{0\omega}$ and $\hat{\Pi}_{2\omega}$ will contain only two independent quantities

$$\hat{\Pi}_{0\omega} = \begin{pmatrix} \Pi_{0\omega}^{(1)} & \Pi_{0\omega}^{(2)} \\ -\Pi_{0\omega}^{(2)} & -\Pi_{0\omega}^{(1)} \end{pmatrix},$$

$$\hat{\Pi}_{2\omega} = \begin{pmatrix} \Pi_{2\omega}^{(1)} & \Pi_{2\omega}^{(2)} \\ -\Pi_{2\omega}^{(2)} & -\Pi_{2\omega}^{(1)} \end{pmatrix}.$$

An analogous statement holds for the structure of the vertex functions, which will be introduced below.

After integrating (16) over the angles, we arrive at the following equations:

$$\Pi_{0\omega}^{(1)} = \frac{1}{2\pi\tau}\int \{\Pi_{0\omega}^{(1)}[\mathfrak{G}_\omega(\mathbf{p})\,\mathfrak{G}_\omega(\mathbf{p}) + \mathfrak{F}_\omega(\mathbf{p})\,\mathfrak{F}_\omega(\mathbf{p})] + 2\Pi_{0\omega}^{(2)}\,\mathfrak{G}_\omega(\mathbf{p})\,\mathfrak{F}_\omega(\mathbf{p})\}\,d\xi,$$

$$\Pi_{0\omega}^{(2)} = -1 + \frac{1}{2\pi\tau}\int \{-2\Pi_{0\omega}^{(1)}\mathfrak{G}_\omega(\mathbf{p})\,\mathfrak{F}_\omega(\mathbf{p}) + \Pi_{0\omega}^{(2)}[\mathfrak{G}_\omega(\mathbf{p})\,\mathfrak{G}_{-\omega}(\mathbf{p}) - \mathfrak{F}_\omega(\mathbf{p})\,\mathfrak{F}_\omega(\mathbf{p})]\}\,d\xi \quad (17)$$

[$\xi = v(p - p_0)$], the solution of which is of the form

$$\Pi_{0\omega}^{(1)} = \frac{i\omega\Delta}{2\tau\varepsilon^3}, \quad \Pi_{0\omega}^{(2)} = -1 - \frac{1}{\pi\tau\varepsilon}\operatorname{arctg}\frac{\tilde{\omega}}{\varepsilon} + \frac{\Delta^2}{2\varepsilon^3\tau}, \quad (18)^*$$

where $\epsilon = \sqrt{\omega^2 + \Delta^2}$, and $\tilde{\omega}$—Debye frequency.

Calculation of $\hat{\Pi}_{2\omega}$ yields (we are interested in the component $\hat{\Pi}_{2\omega}^{(2)}$)

$$\Pi_{2\omega}^{(2)} = \frac{\tau_{tr}\omega^2}{12\tau\varepsilon^4}, \quad \frac{1}{\tau_{tr}} = \frac{nmp_0}{(2\pi)^2}\int |u(\vartheta)|^2(1 - \cos\vartheta)\,d\Omega_\vartheta. \quad (19)$$

The ratio of the part of $\hat{\Pi}_\omega(\mathbf{p}_1, \mathbf{p}_2)$ that is quadratic in q to $\hat{\Pi}_{0\omega}$ is of the order of $l\xi_0 q^2$, and is small in the region in which we seek the solution [$r \gg (l\xi_0)^{1/2}$ or $q \ll (l\xi_0)^{1/2}$], so that we shall henceforth neglect this part.

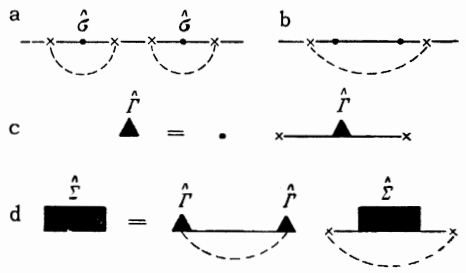

FIG. 2

Diagrams of the type of Fig. 2a, b appear in the calculation of $\hat{L}$. Diagrams of the type of Fig. 2a lead to a replacement of the simple vertex $-en\cdot p\sigma/m$ by $\hat{\Gamma}_\omega(p)$—the vertex function of first order in $\hat{\sigma}$, while diagrams of the type of Fig. 2b lead to a vertex function of second order in σ, i.e., $\hat{\Sigma}_\omega(p)$. $\hat{L}_\omega(p)$ is expressed with the aid of these functions in the following fashion:

$$\hat{L}_\omega(\mathbf{p}) = \hat{G}_\omega(\mathbf{p})\,\hat{\Gamma}_\omega(\mathbf{p})\,\hat{G}_\omega(\mathbf{p})\,\hat{\Gamma}_\omega(\mathbf{p})\,\hat{G}_\omega(\mathbf{p}) + \hat{G}_\omega(\mathbf{p})\,\hat{\Sigma}_\omega(\mathbf{p})\,\hat{G}_\omega(\mathbf{p}). \quad (20)$$

To find the values of $\hat{\Gamma}_\omega(p)$ and $\hat{\Sigma}_\omega(p)$ it is necessary to solve the corresponding equations, which can be readily obtained with the aid of the diagram technique (Figs. 2c, d). Analytic expressions for these equations and their solution are given in the appendix. The vertex function $\hat{\Sigma}_\omega(p)$ is connected with $\hat{L}_\omega(p)$ by the relation

$$\hat{\Sigma}_\omega(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p} - \mathbf{p}')|^2\,\hat{L}_\omega(\mathbf{p}')\,d\mathbf{p}'. \quad (21)$$

Using this relation, and also (16) and (15), we obtain the following formulas:

$$\frac{1}{(2\pi)^3}\int \hat{K}_\omega(\mathbf{p}, \mathbf{p})\,d\mathbf{p} = \frac{p_0 m\tau}{\pi}(\hat{\Pi}_{0\omega} - \hat{I}),$$

$$\frac{1}{(2\pi)^3}\int \hat{L}_\omega(\mathbf{p})\,d\mathbf{p} = \frac{p_0 m\tau}{\pi}\hat{\Sigma}_{0\omega}, \quad (22)$$

from which, substituting (18) and (7) in (12), we obtain

$$\frac{\Lambda(\mathbf{x})}{g} = \frac{\Lambda(\mathbf{x})}{g} - \frac{\Lambda(\mathbf{x})\,p_0 m\Delta^2}{2\pi}T\sum_\omega \frac{1}{\varepsilon^3} - \frac{p_0 m\Delta\,(eQv)^2}{6\pi}T\sum_\omega \frac{1}{\varepsilon^3\eta_{tr}}\left[1 - \frac{\Delta^2}{\varepsilon^2}\left(1 + \frac{1}{2\eta_{tr}}\right)\right], \quad (23)$$

and for a strongly contaminated alloy ($l \ll \xi_0$) we obtain finally

$$\Lambda(\mathbf{x}) = -2\tau_{tr}[eQ(\mathbf{x})v]^2 T\sum_\omega \frac{\omega^2}{\varepsilon^4} \Big/ 3\Delta T\sum_\omega \frac{1}{\varepsilon^3}. \quad (24)$$

In the region of distances $r \ll \delta_0$, we have

*arctg = $\tan^{-1}$.

$Q = 1/2er$ and by using this value we obtain an estimate $\Lambda(\mathbf{x}) \approx l\xi_0 r^{-2}\Delta$. It is therefore clear that $\Lambda(\mathbf{x})$ changes noticeably only over distances $\sim \sqrt{l\xi_0}$, as assumed beforehand.

To calculate the energy of the filament we use the thermodynamic relation

$$\frac{\partial\Omega}{\partial g} = -\int \frac{|\Delta(\mathbf{x})|^2}{g^2} d\mathbf{x} \qquad (25)$$

or

$$\Omega = \Omega_n + \int d\mathbf{x} \int_0^g |\Delta(\mathbf{x})|^2 \, d\frac{1}{g}.$$

This formula contains the absolute value of the gap, i.e., its phase dependence drops out. We proceed from integration with respect to g to integration with respect to Δ, using the definition of Δ

$$\frac{\Delta}{g} = \frac{T}{(2\pi)^3} \sum_\omega \int \mathfrak{F}_\omega(\mathbf{p})\, d\mathbf{p}.$$

We have

$$d\frac{1}{g} = C(\Delta)\frac{d\Delta}{\Delta}, \qquad C(\Delta) = -\frac{p_0 m\Delta^2}{2\pi} T \sum_\omega \frac{1}{\varepsilon^3}. \qquad (26)$$

For the filament energy, defined as $\tilde{\epsilon} = F - F_{S0}$ (F_{S0}—free energy of the superconductor in the absence of the field), we obtain per unit length

$$\tilde{\varepsilon} = 2\pi \int_0^\Delta d\Delta \int_0^\infty r\, dr \frac{\Delta^2(\mathbf{x}) - \Delta^2}{\Delta} C(\Delta)\, d\Delta. \qquad (27)$$

In the integral (27), the integration region can be broken up into three parts $(0, \delta_0/\kappa)$, $(\delta_0/\kappa, \delta_0)$, and (δ_0, ∞). The greatest contribution to the integration is made by the second, logarithmic region. The essential distances are in this case $\delta_0/\kappa \ll r \ll \sigma_0$, with $\Delta(\mathbf{x}) = \Delta + \Lambda(\mathbf{x})$, $\Lambda(\mathbf{x}) \ll \Delta$, $Q = 1/2er$, and

$$\tilde{\varepsilon} = 4\pi \int_0^\Delta d\Delta \int_{\delta_0/\kappa}^{\delta_0} r\, dr \Lambda(r) C(\Delta). \qquad (28)$$

The contribution from the first region, as can be seen from (27), is of the order of $H_c^2\delta_0^2/\kappa^2$ and will not be taken into account in our approximation. The third region gives a small contribution, owing to the exponential decrease of Λ.

Using the expression (24) for Λ and the definition of the function $C(\Delta)$, we have

$$\Lambda(r) C(\Delta) = \frac{p_0 m \tau_{tr} \Delta (eQv)^2}{3\pi} T \sum_\omega \frac{\omega^2}{\varepsilon^4}$$

and we obtain after summation over ω:

$$\Lambda(r) C(\Delta) = \frac{\pi\sigma}{32e^2r^2} \frac{d}{d\Delta}\left\{\Delta \operatorname{th}\frac{\Delta}{2T}\right\}. \qquad (29)$$

Substituting (29) in (28) and introducing the temperature function $\kappa(T)$, defined by

$$\varkappa(T) = \sqrt{2}(2e) H_c \delta_0^2, \qquad \varkappa(T_c) = \varkappa, \qquad (30)$$

we obtain with logarithmic accuracy, after integrating (28),

$$\tilde{\varepsilon} = H_c^2 \delta_0^2 \ln\varkappa / 2\varkappa^2(T). \qquad (31)$$

Knowing the energy of the filament, we can write down the expression for the lower critical field. At the start of the transition, the phonons barely interact with one another, and their contribution to the free energy is equal to $n\tilde{\epsilon}$, where n—filament density. The free energy of a cylinder situated in a longitudinal field H is written in the form

$$F = F_{s0} + n\tilde{\varepsilon} - \frac{HB}{4\pi} + \frac{H^2}{8\pi}, \quad B = n\int \operatorname{rot}_z \mathbf{A}\, dS. \qquad (32)$$

So long as the field does not penetrate into the superconductor, the free energy of the cylinder is equal to $F = F_{S0} + H^2/8\pi$. We see therefore that the start of the transition is determined by the vanishing of the quantity $n\tilde{\epsilon} - HB/4\pi$, i.e., the lower critical field H_{c1} is equal to

$$H_{c1} = 4\pi n\tilde{\varepsilon} / B. \qquad (33)$$

On the other hand, by choosing the contour of integration in the integral for the magnetic induction B at infinity, where $Q = 0$ and $A = 1/2er$, we obtain

$$B = n \oint \mathbf{A}(\mathbf{l})\, d\mathbf{l} = \frac{\pi n}{e}.$$

We finally have for the lower critical field H_{c1}

$$H_{c1} = H_c \ln\varkappa / \sqrt{2}\varkappa(T). \qquad (34)$$

A plot of $\kappa(T)$ against the temperature is shown in Fig. 3. It must be noted that the curve showing the variation of the moment with the field will have the same singularity as in the paper of Abrikosov[1]. The result obtained for H_{c1} coincides in form exactly with the result of Abrikosov,

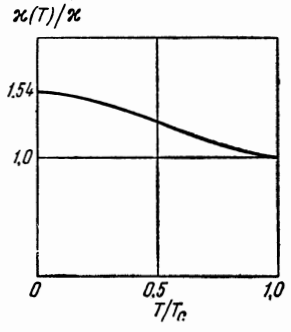

FIG. 3

differing from it only in the definition of the temperature dependent $\kappa(T)$.

Calculation of the next nonlogarithmic terms in (34) is quite difficult, since it calls for a solution of the theoretical equations in the region where the equations cease to be local.

Comparison of (34) with experiment is difficult, owing to the lack of measurements for superconducting alloys with sufficiently large κ. As shown in Fig. 4, the temperature variation of H_{c1} is in qualitative agreement with the available data[7,8]. From the quantitative point of view, on the other hand, formula (34), when applied to alloys with small κ, yields, as in [1], too low a value for the lower critical field.

2. DEPENDENCE OF THE DEPTH OF PENETRATION ON THE FIELD

We proceed to calculate the corrections to the depth of penetration of a weak constant magnetic field into a superconducting alloy. The field is chosen smaller than the lower critical field, i.e., no filaments are produced. It is easy to see that in order to calculate the correction quadratic in the field to the depth of penetration, it is necessary to know the current density accurate to terms cubic in $\mathbf{A}$ inclusive. Using the connection between the current density and the Green's function (3), we have

$$\mathbf{j}(\mathbf{x}) = \frac{2e}{m(2\pi)^3} T \sum_{\omega} \int \mathbf{p}\mathfrak{G}_{\omega}(\mathbf{p}, \mathbf{x})\, d\mathbf{p} - \frac{e^2}{m}\mathbf{A}(\mathbf{x})N, \quad (35)$$

where $\mathfrak{G}_{\omega}(\mathbf{p}, \mathbf{x})$—Fourier component of the Green's function averaged over the positions of the atoms:

$$\mathfrak{G}_{\omega}(\mathbf{x}, \mathbf{x}') = \frac{1}{(2\pi)^3}\int \mathfrak{G}_{\omega}(\mathbf{p}, \mathbf{x})\, e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}')}\, d\mathbf{p}.$$

We expand $\mathfrak{G}_{\omega}(\mathbf{p}, \mathbf{x})$ up to terms cubic in $\mathbf{A}$

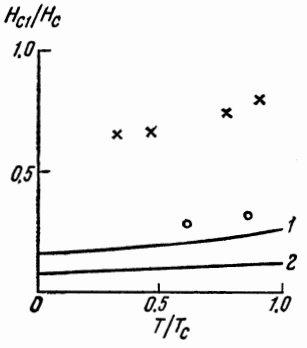

FIG. 4. Variation of the field H_{c1} with the temperature. Alloy In + 2.5% Bi, $\kappa = 1.24$: × — experiment, curve 2 — theory; alloy $Ta_{36}Nb_{64}$, $\kappa = 3.02$: O — experiment, curve 1 — theory.

inclusive. In calculating the current density it will be convenient to use a matrix notation. Expressing the averaged products of the Green's functions by means of the relations

$$\hat{R}_{\omega}(\mathbf{x}-\mathbf{x}') = \frac{ie}{m}\int \langle \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\, \hat{I}\hat{G}_{0\omega}(\mathbf{l}, \mathbf{s})\, \hat{\sigma}\mathbf{n}_s \nabla_s \hat{G}_{0\omega}(\mathbf{s}, \mathbf{x}')\, d\mathbf{l}\, d\mathbf{s}\rangle,$$

$$\hat{S}_{\omega}(\mathbf{x}-\mathbf{x}') = \left(\frac{ie}{m}\right)^3 \int \langle \hat{G}_{0\omega}(\mathbf{x}, \mathbf{l})\, \hat{\sigma}\mathbf{n}_l \nabla_l \hat{G}_{0\omega}(\mathbf{l}, \mathbf{s})\, \hat{\sigma}\mathbf{n}_s \nabla_s \hat{G}_{0\omega}(\mathbf{s}, \mathbf{t})$$
$$\times\, \hat{\sigma}\mathbf{n}_t \nabla_t \hat{G}_{0\omega}(\mathbf{t}, \mathbf{x}')\rangle\, d\mathbf{s}\, d\mathbf{l}\, d\mathbf{t}, \quad (36)$$

we transform to Fourier components $\hat{R}_{\omega}(\mathbf{p})$ and $\hat{S}_{\omega}(\mathbf{p})$:

$$\hat{R}_{\omega}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int \hat{R}_{\omega}(\mathbf{p})\, e^{i\mathbf{p}\mathbf{x}}\, d\mathbf{p},$$

$$\hat{S}_{\omega}(\mathbf{x}) = \frac{1}{(2\pi)^3}\int \hat{S}_{\omega}(\mathbf{p})\, e^{i\mathbf{p}\mathbf{x}}\, d\mathbf{p}. \quad (37)$$

Separating from the current density the part linear in $\mathbf{A}(\mathbf{x})$, obtained in [5], we get

$$\mathbf{j}(\mathbf{x}) = \mathbf{j}_0(\mathbf{x}) + \frac{2e}{m(2\pi)^3} T \sum_{\omega}\left\{\Lambda(\mathbf{x})\, A(\mathbf{x}) \int \mathbf{p} R_{\omega}^{(1)}(\mathbf{p})\, d\mathbf{p}\right.$$
$$\left. + A^3(\mathbf{x}) \int \mathbf{p} S_{\omega}^{(1)}(\mathbf{p})\, d\mathbf{p}\right\}. \quad (38)$$

Here $R_{\omega}^{(1)}$ and $S_{\omega}^{(1)}$—components of the matrix quantities $\hat{R}_{\omega}$ and $\hat{S}_{\omega}$, corresponding to the corrections to $\mathfrak{G}_{\omega}(\mathbf{p}, \mathbf{x})$:

In (38) we made use of the fact that $\Lambda(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ vary significantly over distances $\sim\delta_0$ in a weak field, whereas the largest region of variation of the corresponding kernels in (38) is $\sim\delta_0/\kappa \ll \delta_0$. Because of this, the electrodynamics of alloys is local in a weak field.

Expanding $\hat{R}_{\omega}(\mathbf{p})$ and $\hat{S}_{\omega}(\mathbf{p})$ in Legendre polynomials of $\mathbf{p}\cdot\mathbf{n}/p_0$ and substituting in (38), we find that contributions are made to the current density only by the first harmonic:

$$\mathbf{j}(\mathbf{x}) = \mathbf{j}_0(\mathbf{x}) + \frac{4ep_0^2}{3(2\pi)^2} T \sum_{\omega}\left\{\Lambda(\mathbf{x})\, \mathbf{A}(\mathbf{x}) \int R_{\omega 1}^{(1)}(\mathbf{p})\, d\xi\right.$$
$$\left. + \mathbf{A}(\mathbf{x})\, [A(\mathbf{x})]^2 \int S_{\omega 1}^{(1)}(\mathbf{p})\, d\xi\right\},$$

$$\hat{R}_{\omega}(\mathbf{p}) = \hat{R}_{\omega 0}(\mathbf{p}) + \hat{R}_{\omega 1}(\mathbf{p})\, P_1\left(\frac{\mathbf{n}\mathbf{p}}{p_0}\right) + \ldots,$$

$$\hat{S}_{\omega}(\mathbf{p}) = \hat{S}_{\omega 0}(\mathbf{p}) + \hat{S}_{\omega 1}(\mathbf{p})\, P_1\left(\frac{\mathbf{n}\mathbf{p}}{p_0}\right) + \ldots. \quad (39)$$

The averaging over the positions of the impurity atoms leads to equations for $\hat{R}_{\omega}(\mathbf{p})$ and $\hat{S}_{\omega}(\mathbf{p})$ in the graphic form shown in Fig. 5. As before, it is more convenient to solve the equations for the vertex functions defined by the relations

$$\hat{Z}_{\omega}(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \hat{R}_{\omega}(\mathbf{p}')\, d\mathbf{p}', \quad (40)$$

$$\hat{\Xi}_\omega(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \hat{S}_\omega(\mathbf{p}')\,d\mathbf{p}'. \qquad (41)$$

The equation for $\hat{Z}_\omega(\mathbf{p})$ is obtained from (40) and from the corresponding equation for $\hat{R}_\omega(\mathbf{p})$:

$$\hat{Z}_\omega(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \{\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Pi}_\omega\hat{G}_\omega(\mathbf{p}')$$
$$+ \hat{G}_\omega(\mathbf{p}')\hat{\Pi}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\}\,d\mathbf{p}'$$
$$+ \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \hat{G}_\omega(\mathbf{p}')\hat{Z}_\omega(\mathbf{p}')\hat{G}_\omega(\mathbf{p}')\,d\mathbf{p}'. \qquad (42)$$

Equation (42) is solved, as before, by expanding all the quantities in Legendre polynomials. As a result we have for the first harmonic of the expansion of $\hat{Z}_{\omega 1}$:

$$\hat{Z}_{\omega 1} = \begin{pmatrix} Z_{\omega 1}^{(1)} & Z_{\omega 1}^{(2)} \\ -Z_{\omega 1}^{(2)} & -Z_{\omega 1}^{(1)} \end{pmatrix}, \qquad Z_{\omega 1}^{(1)} = -\frac{2\tau_{tr} ev\omega^2}{\tau_1 \varepsilon^4}. \qquad (43)$$

The contribution of this term to the current density is equal to

$$-\frac{4}{3\pi} p_0 m \Lambda(\mathbf{x})\mathbf{A}(\mathbf{x})\Delta\tau_{tr}(ev)^2 T\sum_\omega \frac{\omega^2}{\varepsilon^4}. \qquad (44)$$

Inasmuch as the equations used to find $\Lambda(\mathbf{x})$ are derived from the corresponding equations for the lower critical field by the substitution $\mathbf{Q} \to \mathbf{A}$, we must take for $\Lambda(\mathbf{x})$ the previously obtained expression with the corresponding suitable substitution $\mathbf{Q} \to \mathbf{A}$.

It now remains to find the contribution from $\hat{S}_\omega(\mathbf{p})$. The equation for $\hat{\Xi}$ is of the form

$$\hat{\Xi}_\omega(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2$$
$$\times \{\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')$$
$$+ \hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Sigma}_\omega\hat{G}_\omega(\mathbf{p}')$$
$$+ \hat{G}_\omega(\mathbf{p}')\hat{\Sigma}_\omega\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega\hat{G}_\omega(\mathbf{p}')\}\,d\mathbf{p}'$$
$$+ \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \hat{G}_\omega(\mathbf{p}')\hat{\Xi}_\omega(\mathbf{p}')\hat{G}_\omega(\mathbf{p}')\,d\mathbf{p}'. \qquad (45)$$

It is solved in analogy with the preceding ones. We note merely that, as can be readily seen, the first harmonic of the expansion of $\hat{\Xi}$ in Legendre polynomials of $\mathbf{p}\cdot\mathbf{n}/p_0$ receives contributions from the zeroth and second harmonics of $\hat{\Sigma}_\omega$ in the second and third terms of the free part of (45).

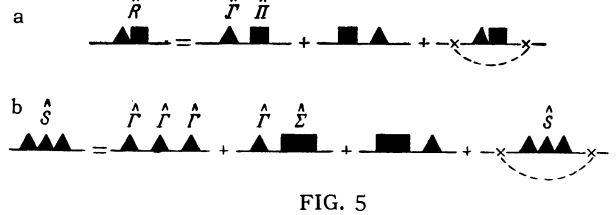

FIG. 5

However, as shown by an estimate, it is necessary to take into account in (45) only the terms with $\hat{\Sigma}_{\omega 0}$. The discarded quantities are small in a ratio $l/\xi_0 \ll 1$.

Solving (45), we obtain for the first harmonic $\hat{\Xi}_{\omega 1}$:

$$\hat{\Xi}_{\omega 1} = \begin{pmatrix} \Xi_{\omega 1}^{(1)} & \Xi_{\omega 1}^{(2)} \\ -\Xi_{\omega 1}^{(2)} & -\Xi_{\omega 1}^{(1)} \end{pmatrix}, \qquad \Xi_{\omega 1}^{(1)} = \frac{4(ev)^3\tau_{tr}^2\Delta^2\omega^2}{3\tau_1\varepsilon^5}, \qquad (46)$$

and the corresponding contribution to the current density is

$$\frac{8p_0 m}{9\pi}(ev)^4\Delta^2\tau_{tr}^2\mathbf{A}(\mathbf{x})[A(\mathbf{x})]^2 T\sum_\omega \frac{\omega^2}{\varepsilon^5}. \qquad (47)$$

Gathering together the obtained expressions, we obtain ultimately for the current density

$$\mathbf{j}(\mathbf{x}) = -\frac{\mathbf{A}(\mathbf{x})}{4\pi\delta_0^2}\left[1 - \frac{4[A(\mathbf{x})]^2}{H_c\delta_0^2} f(T)\right], \qquad (48)$$

where $f(T)$—dimensionless temperature function:

$$f(T) = \frac{4}{\Delta^2 \operatorname{th}^2(\Delta/2T)} T\sum_{\omega>0}\left\{2\varepsilon - 2\omega - \frac{\Delta^2}{\varepsilon}\right\}$$
$$\times\left\{\left(T\sum_\omega \frac{\omega^2}{\varepsilon^4}\right)^2 \Big/ T\sum_\omega \frac{1}{\varepsilon^3} + \Delta^2 T\sum_\omega \frac{\omega^2}{\varepsilon^5}\right\}.$$

In writing down the current density in the form (48), we have used the following expression for the thermodynamic critical field

$$H_c^2 = 8p_0 mT\sum_{\omega>0}\left\{2\varepsilon - 2\omega - \frac{\Delta^2}{\varepsilon}\right\}.$$

Let us consider the case of a superconducting half-space occupying the region $z > 0$, and let us direct the external field H_0 along the y axis and the vector potential $\mathbf{A}$ along the x axis. Then, substituting (48) in Maxwell's equation, we obtain

$$\frac{d^2A}{dz^2} = \frac{A}{\delta_0^2}\left[1 - \frac{4A^2}{H_c^2\delta_0^2} f(T)\right]. \qquad (49)$$

Solving (49) with boundary conditions $H(z=0) = H_0$ and $A(\infty) = 0$, we obtain for A, accurate to terms cubic in H_0,

$$A(z) = -H_0\delta_0 e^{-z/\delta_0} + \frac{H_0\delta_0}{2}\left(\frac{H_0}{H_c}\right)^2 f(T)[e^{-3z/\delta_0} - 3e^{-z/\delta_0}]. \qquad (50)$$

The depth of penetration is determined from the formula

$$\delta = \frac{1}{H_0}\int_0^\infty H dz = -\frac{A(0)}{H_0},$$

from which, using (50), we obtain

$$\delta = \delta_0[1 + (H_0/H_c)^2 f(T)]. \qquad (51)$$

The function $f(T)$ changes little with tempera-

ture and assumes values $f(T_c) = 0.125$ and $f(0) = 0.096$ on the boundaries of the temperature interval. As $T \to T_c$ formula (51) goes over into the Ginzburg-Landau theory result.

In conclusion I am grateful to L. P. Gor'kov for valuable advice and numerous discussions of the work.

APPENDIX

The equations for the vertex functions $\hat{\Gamma}_\omega$ and $\hat{\Sigma}_\omega$ are written analytically in the following fashion:

$$\hat{\Gamma}_\omega(\mathbf{p}) = -\frac{e}{m}\mathbf{n}\mathbf{p}\hat{\sigma} + \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \times \hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega(\mathbf{p}')\hat{G}_\omega(\mathbf{p}')\,d\mathbf{p}', \tag{A.1}$$

$$\hat{\Sigma}_\omega(\mathbf{p}) = \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega(\mathbf{p}')\hat{G}_\omega(\mathbf{p}')\hat{\Gamma}_\omega(\mathbf{p}') \times \hat{G}_\omega(\mathbf{p}')\,d\mathbf{p}' + \frac{n}{(2\pi)^3}\int |u(\mathbf{p}-\mathbf{p}')|^2 \times \hat{G}_\omega(\mathbf{p}')\hat{\Sigma}_\omega(\mathbf{p}')\hat{G}_\omega(\mathbf{p}')\,d\mathbf{p}'. \tag{A.2}$$

$\hat{\Gamma}_\omega(\mathbf{p})$ and $\hat{\Sigma}_\omega(\mathbf{p})$ depend essentially on the angle $\mathbf{p}\cdot\mathbf{n}/p_0$. After expanding these quantities in Legendre polynomials

$$\hat{\Gamma}_\omega(\mathbf{p}) = \hat{\Gamma}_{\omega 0} + \hat{\Gamma}_{\omega 1}P_1\left(\frac{\mathbf{np}}{p_0}\right) + \cdots,$$

$$\hat{\Sigma}_\omega(\mathbf{p}) = \hat{\Sigma}_{\omega 0} + \hat{\Sigma}_{\omega 1}P_1\left(\frac{\mathbf{np}}{p_0}\right) + \hat{\Sigma}_{\omega 2}P_2\left(\frac{\mathbf{np}}{p_0}\right) + \cdots,$$

we substitute these expansions in (A.1) and (A.2).

Using the theorem for the addition of Legendre polynomials

$$P_n(z) = P_n(z_1)P_n(z_2) + 2\sum_{m=1}^{n}(-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(z_1)P_n^m(z_2)\cos\vartheta m,$$

where $z = \mathbf{p}\cdot\mathbf{p}'/p_0^2$, $z_1 = \mathbf{p}\cdot\mathbf{n}/p_0$, and $z_2 = \mathbf{p}'\cdot\mathbf{n}/p_0$ are connected by the relation $z = z_1z_2 + (1-z_1^2)^{1/2}(1-z_2^2)^{1/2}\cos\vartheta$, and integrating over the angles, we find that the only nonvanishing harmonics of the expansions are the first in $\hat{\Gamma}_\omega$ and the zeroth and second in $\hat{\Sigma}_\omega$.

For these quantities we have a system of linear equations

$$\hat{\Gamma}_{\omega 1} = -ev\hat{\sigma} + \frac{1}{2\pi\tau_1}\int \hat{G}_\omega(p)\hat{\Gamma}_{\omega 1}\hat{G}_\omega(p)\,d\xi, \tag{A.3}$$

$$\hat{\Sigma}_{\omega 0} = \frac{1}{2\pi\tau}\int \hat{G}_\omega(p)\hat{\Gamma}_{\omega 1}\hat{G}_\omega(p)\hat{\Gamma}_{\omega 1}\hat{G}_\omega(p)\,d\xi + \frac{1}{2\pi\tau}\int \hat{G}_\omega(p)\hat{\Sigma}_{\omega 0}\hat{G}_\omega(p)\,d\xi, \tag{A.4}$$

$$\hat{\Sigma}_{\omega 2} = \frac{1}{3\pi\tau_2}\int \hat{G}_\omega(p)\hat{\Gamma}_{\omega 1}\hat{G}_\omega(p)\hat{\Gamma}_{\omega 1}\hat{G}_\omega(p)\,d\xi + \frac{1}{2\pi\tau_2}\int \hat{G}_\omega(p)\hat{\Sigma}_{\omega 2}\hat{G}_\omega(p)\,d\xi, \tag{A.5}$$

which have the following solutions

$$\hat{\Gamma}_{\omega 1} = \begin{pmatrix} \Gamma_{\omega 1}^{(1)} & \Gamma_{\omega 1}^{(2)} \\ -\Gamma_{\omega 1}^{(2)} & -\Gamma_{\omega 1}^{(1)} \end{pmatrix},$$

$$\Gamma_{\omega 1}^{(1)} = -ev\left(1+\frac{\Delta^2}{2\varepsilon^3\eta_{tr}\tau_1}\right), \qquad \Gamma_{\omega 1}^{(2)} = -\frac{evi\omega\Delta}{2\varepsilon^3\eta_{tr}\tau_1}; \tag{A.6}$$

$$\hat{\Sigma}_{\omega 0} = \begin{pmatrix} \Sigma_{\omega 0}^{(1)} & \Sigma_{\omega 0}^{(2)} \\ -\Sigma_{\omega 0}^{(2)} & -\Sigma_{\omega 0}^{(1)} \end{pmatrix}, \qquad \hat{\Sigma}_{\omega 2} = \begin{pmatrix} \Sigma_{\omega 2}^{(1)} & \Sigma_{\omega 2}^{(2)} \\ -\Sigma_{\omega 2}^{(2)} & -\Sigma_{\omega 2}^{(1)} \end{pmatrix},$$

$$\Sigma_{\omega 0}^{(1)} = -\frac{(ev)^2 i\omega\Delta^2}{6\eta_{tr}\tau\varepsilon^5}\left[1+\frac{1}{2\eta_{tr}}\right],$$

$$\Sigma_{\omega 0}^{(2)} = \frac{(ev)^2\Delta}{6\eta_{tr}\tau\varepsilon^3}\left[1-\frac{\Delta^2}{\varepsilon^2}\left(1+\frac{1}{2\eta_{tr}}\right)\right]; \tag{A.7}$$

$$\Sigma_{\omega 2}^{(1)} = -\frac{(ev)^2 i\omega\Delta^2}{6\tau_2\eta_2\eta_1\varepsilon^5}\left[2+\frac{\eta_2}{\eta_1}\right],$$

$$\Sigma_{\omega 2}^{(2)} = \frac{(ev)^2\Delta}{3\tau_2\eta_1\eta_2\varepsilon^3}\left[1-\frac{\Delta^2}{\varepsilon^2}\left(1+\frac{\eta_2}{\eta_1}\right)\right], \tag{A.8}$$

with

$$\frac{1}{\tau_l} = \frac{nmp_0}{(2\pi)^2}\int |u(\vartheta)|^2 P_l(\cos\vartheta)\,d\Omega_\vartheta, \quad l = 1, 2;$$

$$\eta_l = 1 + 1/2\,\varepsilon\tau_l, \qquad \eta_{tr} = 1 + 1/2\,\varepsilon\tau_{tr}.$$

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