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NONLINEAR INTERACTION BETWEEN PLASMA WAVES

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Nonlinear interaction between plasma waves is considered. It is shown for the high-frequency electron Langmuir oscillations of an isotropic plasma that in the first nonvanishing approximation in powers of the ratio of Debye radius to wave length, $(r_D/\lambda)^2$, pumping occurs in the spectrum from the short to the long waves. Waves with parallel or perpendicular wave vectors do not interact in this case. Such waves are capable of interacting only in the next approximation in powers of $(r_D/\lambda)^2$. Energy damping of the Langmuir oscillations appears in the same approximation. Analysis of a system consisting of a plasma and slow beam of velocity less than that of the particle thermal velocity but greater than the ratio of the square of thermal velocity to phase velocity of the oscillations shows that if the temperature of the plasma at rest is lower than that of the beam temperature, pumping will occur from high frequencies to low frequencies and the oscillation energy decreases. Conversely, when the temperature of the plasma at rest is higher than that of the beam, spectral pumping proceeds from low to high frequencies. In this case the oscillation energy increases, that is, nonlinear buildup of oscillations occurs. In conclusion, nonlinear interaction between short wave ion-sound oscillations is considered both when spectral pumping is decisive and when nonlinear oscillation damping is the predominant effect.

1. The nonlinear theory of plasma waves has already been the subject of many papers. It behooves us therefore to indicate from the very outset the problems which we intend to discuss here. We shall treat the nonlinear interaction of Langmuir waves of an electron plasma, and the nonlinear interaction of short-wave ion waves of a non-isothermal electron-ion plasma. Our analysis differs from that of Sturrock[1] in the account of the thermal motion of the particles. Therefore the nonlinear interaction which we consider appears in a lower order of the nonlinearity. Further, unlike Kamaka et al. and also Akhiezer et al.[2] we consider problems in which the wave-decay conditions are not satisfied. Therefore the mechanism of nonlinear interaction turns out to be different in our case, viz., in our case we can speak of wave scattering by plasma particles. For a one-dimensional plasma model, a similar analysis for electronic high-frequency waves was presented by W. Drummond and Pines[3]. Our analysis pertains to the qualitatively different three-dimensional case, and also touches upon nonlinear interaction between oscillations in a plasma containing a beam, and nonlinear interaction between ion oscillations. An estimate of the nonlinear interaction for electron Langmuir oscillations was given by Karpman[4]. However, as follows from the solution obtained below for the explicit time dependence of the intensity of two nonlinearly interacting oscillations, such an estimate does not explain the physical meaning of the nonlinear relaxation.

Our analysis is based on the following nonlinear equation for the plasma intensity oscillations:

$$\frac{\partial w(\omega, \mathbf{k})}{\partial t}\left|\frac{\partial \varepsilon'(\omega, \mathbf{k})}{\partial \omega}\right| + 2\gamma(\omega, \mathbf{k})\, w(\omega, \mathbf{k})\left|\frac{\partial \varepsilon'(\omega, \mathbf{k})}{\partial \omega}\right|$$

$$- 2\pi\delta[\varepsilon'(\omega, \mathbf{k})]\sum_\alpha \frac{4\pi e_\alpha^2}{k^2}\int d\mathbf{p}_\alpha N_\alpha f_\alpha \delta(\omega - \mathbf{k}\mathbf{v}_\alpha)$$

$$= \operatorname{sgn}\omega\, w(\omega, \mathbf{k})\int\frac{d\mathbf{k}'}{(2\pi)^3}\int d\omega' w(\omega', \mathbf{k}')\left(\frac{4\pi}{kk'}\right)^2 2\operatorname{Im}\sum_\alpha e_\alpha^4 N_\alpha$$

$$\times\int d\mathbf{p}_\alpha \frac{1}{\omega + i0 - \mathbf{k}\mathbf{v}_\alpha}\left(\mathbf{k}'\frac{\partial}{\partial \mathbf{p}_\alpha}\right)\frac{1}{\omega - \omega' + i0 - (\mathbf{k} + \mathbf{k}', \mathbf{v}_\alpha)}$$

$$\times\left[\left(\mathbf{k}\frac{\partial}{\partial \mathbf{p}_\alpha}\right)\frac{1}{\omega' + i0 - \mathbf{k}'\mathbf{v}_\alpha}\left(\mathbf{k}'\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\right)\right.$$

$$\left.+\left(\mathbf{k}'\frac{\partial}{\partial \mathbf{p}_\alpha}\right)\frac{1}{\omega + i0 - \mathbf{k}\mathbf{v}_\alpha}\left(\mathbf{k}\frac{\partial f_\alpha}{\partial \mathbf{p}_\alpha}\right)\right]$$

$$+ 2\int\frac{d\mathbf{k}'d\mathbf{k}''}{(2\pi)^3}\int d\omega' d\omega''\delta(\omega + \omega' - \omega'')\,\delta(\mathbf{k} + \mathbf{k}' - \mathbf{k}'')$$

$$\times\left\{2w(\omega, \mathbf{k})\, w(\omega', \mathbf{k}')\operatorname{sgn}\omega\operatorname{Im}\left[\frac{1}{\varepsilon(\omega'' + i0, \mathbf{k}'')}\right.\right.$$

$$\times S(\omega + i0, \mathbf{k}, \omega' + i0, \mathbf{k}', -\omega'' - i0, -\mathbf{k}'')$$

$$\times S(\omega + i0,\ \mathbf{k},\ \omega' - i0,\ \mathbf{k}',\ -\omega'' - i0,\ -\mathbf{k}'')\Big]$$
$$+ \pi\delta[\varepsilon'(\omega,\ \mathbf{k})]\, w(\omega',\ \mathbf{k}')\, w(\omega'',\ \mathbf{k}'')$$
$$\times |S(\omega + i0,\ \mathbf{k},\ \omega' - i0,\ \mathbf{k}',\ -\omega'' - i0,\ -\mathbf{k}'')|^2\Big\}. \quad (1.1)$$

Here e_α —charge, N_α —number of particles, f_α —distribution function of the α species of particles normalized to unity, ϵ' and ϵ'' —respectively the real and imaginary parts of the dielectric constant of the collisionless plasma, $\gamma(\omega, \mathbf{k}) = \operatorname{sgn} \omega\epsilon''(\omega, \mathbf{k}) \times |\partial\epsilon'(\omega, \mathbf{k})/\partial\omega|^{-1}$ —plasma-oscillation damping decrement, and finally

$$S(\omega,\ \mathbf{k},\ \omega',\ \mathbf{k}',\ -\omega'',\ -\mathbf{k}'') = \frac{(2\pi)^{3/2}}{kk'k''} \sum_\alpha \frac{e_\alpha^3}{m_\alpha^2} N_\alpha \int d\mathbf{p}_\alpha f_\alpha$$
$$\times \Big\{ \frac{(k'')^4}{(\omega - \mathbf{k}\mathbf{v}_\alpha)(\omega' - \mathbf{k}'\mathbf{v}_\alpha)(\omega'' - \mathbf{k}''\mathbf{v}_\alpha)^2}$$
$$- \frac{(k')^4}{(\omega - \mathbf{k}\mathbf{v}_\alpha)(\omega' - \mathbf{k}'\mathbf{v}_\alpha)^2(\omega'' - \mathbf{k}''\mathbf{v}_\alpha)}$$
$$- \frac{k^4}{(\omega - \mathbf{k}\mathbf{v}_\alpha)^2(\omega' - \mathbf{k}'\mathbf{v}_\alpha)(\omega'' - \mathbf{k}''\mathbf{v}_\alpha)}$$
$$+ \frac{(kk'')^2}{(\omega - \mathbf{k}\mathbf{v}_\alpha)(\omega'' - \mathbf{k}''\mathbf{v}_\alpha)^2} + \frac{(kk')^2}{(\omega - \mathbf{k}\mathbf{v}_\alpha)^2(\omega' - \mathbf{k}'\mathbf{v}_\alpha)^2}$$
$$+ \frac{(k'k'')^2}{(\omega' - \mathbf{k}'\mathbf{v}_\alpha)^2(\omega'' - \mathbf{k}''\mathbf{v}_\alpha)^2}\Big\}.$$

In the last formula, the singularities of the integrand, with allowance for the corresponding infinitesimally small additions $\pm i0$, must be interpreted in the manner usually employed in the theory of Cauchy integrals.

Equation (1.1) was obtained by perturbation theory for simultaneous correlation functions by one of the authors (V.S.)[5], where a comparison was also made with the results of Kadomtsev and Petviashvili[6]. On the basis of the method of the last paper, an equation of the type (1.1) was obtained for the stationary case by Petviashvili[7]. We shall use below Eq. (1.1) to study the nonlinear interaction of electron Langmuir oscillations and nonlinear interaction of ion oscillations of a nonisothermal plasma.

We shall not consider the time variation of the particle distribution functions, assuming these functions to be Maxwellian with zero or finite drift. The justification for such an approximation is the fact that in the problems considered below the characteristic times of nonlinear relaxation of the oscillations are small compared with the relaxation times of the particle distributions, which can be regarded to be the same as in the absence of oscillations. The latter, for example in the problem of nonlinear interaction between electron Langmuir waves, is brought about by the fact that the presence of the oscillations greatly changes the relaxation of the fast epithermal particles and is inessential for the slow ones. However, it is precisely the latter which determine the magnitude of the nonlinear relaxation. In this section we shall consider the nonlinear interaction of Langmuir oscillations of an electron plasma with isotropic Maxwellian particle-velocity distribution. We shall therefore deal here with oscillations whose frequency is connected with the wave vector by the known relation

$$\omega^2 = \omega^2_L + 3(\varkappa T/m)k^2,$$

where $\omega_L = \sqrt{4\pi e^2 N/m}$ —Langmuir frequency, e —charge, m —mass, T —temperature, and N —electron concentration. Using the fact that the phase velocity of weakly-damped Langmuir oscillations greatly exceeds the thermal velocity of the electrons, we can write (1.1) in the form

$$\frac{\partial W(\mathbf{k})}{\partial t} + \sqrt{\frac{\pi}{2}} \frac{\omega_L}{(kr_D)^3} \exp\left\{-\frac{1}{2(kr_D)^2} - \frac{3}{2}\right\}\{W(\mathbf{k}) - \varkappa T\}$$
$$= -\frac{3}{2(2\pi)^{5/2}} \frac{\omega_L}{Nr_D^3} \frac{W(\mathbf{k})}{\varkappa T} r_D^6$$
$$\times \int d\mathbf{k}' W(\mathbf{k}') \frac{k^2 - k'^2}{|\mathbf{k} - \mathbf{k}'|^3} [\mathbf{k}\mathbf{k}']^2 \left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2. \quad (2.1)^*$$

Here

$$W(\mathbf{k}) = \int_0^\infty d\omega w(\omega,\ \mathbf{k}) \frac{\partial}{\partial\omega}(\omega\varepsilon')$$

—energy of the Langmuir oscillations, r_D —Debye radius of the electrons, and the integration in the right side of (2.1) is over the region of wave vectors smaller than r_D^{-1}. Equation (2.1) is meaningful only in such a region of wave vectors.

In writing down (2.1) no account was taken of the change in the intensity of the oscillations, due to the absorption of the Langmuir oscillations by the electrons which are produced when the electrons collide with the ions. The effective collision frequency, which characterizes the damping time of the oscillations due to such collisions, is determined, as is well known, by the formula

$$\nu_{\text{eff}} = \frac{L}{3(2\pi)^{3/2}} \frac{\omega_L}{Nr_D^3} \frac{1}{Ne^2} \Sigma N_i e_i^2,$$

where the summation is over all species of ions, and $L = \ln(r_{\max}/r_{\min})$ —Coulomb logarithm [1)]. Comparison of such an expression for the effective collision frequency with the right side of (2.1) shows that the nonlinear action may turn out to be more significant than the collisions of the plasma particles with one another, only if

$$W \gg \varkappa TL. \quad (2.2)$$

*$[\mathbf{kk}'] = \mathbf{k} \times \mathbf{k}'$.

[1)]For a classical plasma $r_{\max}/r_{\min} \approx 4\pi r_D^3 N$.

We shall assume henceforth that this inequality is satisfied. Therefore, in particular, we can also neglect the inhomogeneous term of the left side of Eq. (2.1).

As is well known, linear dissipation of Langmuir oscillations, due to the inverse Cerenkov effect, is quite small in the long-wave limit. We therefore neglect the linear dissipation in (2.1), assuming the oscillation wavelengths to be long

$$\lambda_{\text{cr}}^{-} = r_D\sqrt{\ln[Nr_D^3(\varkappa T / W)]}.$$

After stipulating this, we can write in place of (2.1) the following approximate expression

$$\frac{\partial W(\mathbf{k})}{\partial t} = -\frac{3}{2(2\pi)^{5/2}}\frac{\omega_L}{Nr_D^3}\frac{W(\mathbf{k})}{\varkappa T}r_D^6$$
$$\times\int d\mathbf{k}'W(\mathbf{k}')\frac{k^2-k'^2}{|\mathbf{k}-\mathbf{k}'|^3}[\mathbf{k}\mathbf{k}']^2\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2. \quad (2.3)$$

It is easy to see that the quantity conserved under this condition is $W_0 = \int d\mathbf{k}W(\mathbf{k})$ (in such an approximation the energy of the longitudinal oscillations is conserved). Therefore the damping of the oscillations in one wavelength region should be accompanied by an increase in the intensity of oscillation in another region. Such a statement corresponds to that made by Drummond and Pines[3], who discussed the interaction between Langmuir oscillations in the one-dimensional model. We note that, as can be seen from (2.3), in our analysis waves with parallel wave vectors (and also those with mutually-perpendicular ones) do not interact. In other words, the right sides of (2.1) and (2.3) vanish in the one-dimensional case.

Let us consider a simple example of a nonlinear interaction between two wave packets of width small enough to be neglected. Then

$$W(\mathbf{k}, t) = W_1(t)\delta(\mathbf{k}-\mathbf{k}_1) + W_2(t)\delta(\mathbf{k}-\mathbf{k}_2). \quad (2.4)$$

Equation (2.3) reduces here to a system of two equations

$$\frac{dW_1}{dt} = -a_{12}W_1W_2, \quad \frac{dW_2}{dt} = +a_{12}W_2W_1, \quad (2.5)$$

where

$$a_{12} = \frac{3}{2(2\pi)^{5/2}}\frac{\omega_L}{Nr_D^3}\frac{r_D^6}{\varkappa T}\frac{k_1^2-k_2^2}{|\mathbf{k}_1-\mathbf{k}_2|^3}[\mathbf{k}_1\mathbf{k}_2]^2\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right)^2. \quad (2.6)$$

The solution of the system (2.5) can be written in the form

$$W_1(t) + W_2(t) = W_1(t_0) + W_2(t_0) = W_0,$$
$$\frac{W_1(t)}{W_2(t)} = \frac{W_1(t_0)}{W_2(t_0)}\exp\{-a_{12}W_0(t-t_0)\},$$
$$\frac{W_1(t)}{W_1(t_0)} = \frac{W_0}{W_1(t_0) + W_2(t_0)\exp(a_{12}W_0(t-t_0))}. \quad (2.7)$$

Inasmuch as a_{12} is positive when $k_0 > k_2$ we can state, in accordance with (2.7) that a transformation of the short wave oscillations into long wave oscillations takes place in the nonlinear interaction between the two waves. Such a transformation differs greatly from the damping of the oscillations due to the inverse Cerenkov effect, or due to the collisions of the plasma particles, where the energy transfer from the oscillations to the particles causes a decrease in the oscillation amplitude.

As applied to the time evolution of the wave packet of finite width, we must speak, in accordance with the foregoing, not of damping of the wave packet, but of a change in its shape, due to a change in its spectrum. A simple characteristic of such a change in the waveform is the time of spectral transformation $\tau_S = (a_{12}W_0)^{-1}$ for the case of two waves whose wave vectors differ little ($\mathbf{k}_1 = \mathbf{k}_0$, $\mathbf{k}_2 = \mathbf{k}_0 + \Delta$). Then

$$\tau_s = \frac{(2\pi)^{5/2}}{3}\frac{1}{\omega_L}\frac{N\varkappa T}{W_0}\frac{1}{(k_0r_D)^3}\frac{1}{\sin^2\theta\,|\cos\theta + \Delta/2k_0|}, \quad (2.8)$$

where θ —angle between vectors $\mathbf{k}_0$ and Δ.

For a plasma oscillation distribution which is independent of direction, we can rewrite (2.3) after integrating over the angles, in the form

$$\frac{\partial W(k)}{\partial t} = -\frac{2}{35(2\pi)^{5/2}}\frac{\omega_L}{Nr_D^3}\frac{W(k)\,r_D^6}{\varkappa T\;k^3}\left\{\int_0^k dk'k'^4(7k^4+5k^2k'^2\right.$$
$$-12k'^4)W(k') - k^5\int_k^{k_{max}}\frac{dk'}{k'}(7k'^4+5k'^2k^2$$
$$\left.-12k^4)W(k')\right\}. \quad (2.9)$$

As applied to a nonlinear interaction of two waves

$$W(k, t) = (1/4\pi k^2)[W_1(t)\delta(k-k_1) + W_2(t)\delta(k-k_2)],$$

(2.9) leads to a solution of the type (2.7), the only difference being that a_{12} must be taken to mean

$$\frac{1}{35(2\pi)^{5/2}}\frac{\omega_L}{Nr_D^3}\frac{1}{\varkappa T}r_D^6\frac{k_2^2}{k_1^3}(k_1^2-k_2^2)(7k_1^2+12k_2^2),$$
$$k_1 > k_2. \quad (2.10)$$

The order of magnitude of the time of the spectral redistribution can be characterized in this case by the expression[2)]

$$\tau_s \approx \frac{L}{\nu_{\text{eff}}}\frac{\varkappa T}{(W_0/k^3)}\frac{1}{r_D^6k^5\Delta k}, \quad (2.11)$$

[2)]Comparison of formulas (2.8) and (2.11) with formula (4.5) of the paper by Karpman[4] for the relaxation time of the Langmuir oscillations, written in the same order of the expansion in powers of the Debye radius, shows that the quadratic dependence on the width of the packet, obtained from qualitative considerations in[4], is not exact.

where k —quantity on the order of k_1 and k_2, and $\Delta k = |k_1 - k_2|$. For example, for the case when $(kr_D)^2 \sim 0.1$, $L \sim 10$, and $\Delta k \sim k$, we obtain $\tau_S \sim 10^4 \nu_{\text{eff}}^{-1}(\varkappa T k^3/W_0)$. Therefore for $W_0/k^3 \sim 10^5 \times \varkappa T$, which is apparently a perfectly realistic value, the time of the spectral distribution turns out to be one order of magnitude smaller than the time of oscillation damping due to the Coulomb collisions of the plasma particles.

The absence of a nonlinear interaction between the waves with parallel or mutually-perpendicular vectors, and also the conservation of the energy of the oscillations under nonlinear interaction, characterize the foregoing first nonvanishing approximation in the expansion in powers of the square of the ratio of the Debye radius to the wavelength. Even the first correction to the right side of (2.1) [or (2.3)] leads both to interaction between waves with parallel (and mutually perpendicular) wave vectors, and to nonconservation of the oscillation energy. It is precisely these effects that are described by the following addition to the right side of (2.1):

$$-\frac{3}{2}\frac{1}{(2\pi)^{5/2}}\frac{\omega_L}{Nr_D^3}\frac{W(\mathbf{k})}{\varkappa T}r_D^6\int d\mathbf{k}'W(\mathbf{k}')\frac{k^2-k'^2}{|\mathbf{k}-\mathbf{k}'|}\times\left\{\frac{3}{4}r_D^2(k^2-k'^2)\left(\frac{\mathbf{kk}'}{kk'}\right)^2\frac{[\mathbf{kk}']^2}{|\mathbf{k}-\mathbf{k}'|^2}+\frac{1}{4}r_D^2\frac{[\mathbf{kk}']^4+9(\mathbf{kk}')^4}{k^2k'^2}\right\}. \quad (2.12)$$

That part of (2.12) which is symmetrical with respect to $\mathbf{k}$ and $\mathbf{k}'$ determines the change in the oscillation energy, whereas the interaction between waves having parallel (or mutually-perpendicular) wave vectors is determined by the antisymmetrical part. If, as in (2.3), we confine ourselves only to nonlinear wave interaction, then, with account of the addition (2.12), the kinetic equation for the wave has again an integral. Now, however, the conserved quantity is not the oscillation energy. It is easy to verify that, at the accuracy which we have assumed for the power expansion of the square of the ratio of the Debye radius to the wavelength of the oscillation, we can speak of conservation of the following integral:

$$\int d\mathbf{k}\,\frac{W(\mathbf{k},t)}{\omega(\mathbf{k})}. \quad (2.13)$$

In quantum language, such a conservation law signifies the invariance of the number $n(\mathbf{k},t)$ of the oscillation quanta $[W(\mathbf{k},t) \equiv n(\mathbf{k},t)\hbar\omega(\mathbf{k})]$.

For the distribution of the isotropic plasma oscillations, an account of the nonlinear dissipation leads to the occurrence of a factor $1 + \frac{3}{4} r_D^2(k^2 - k'^2)$ in the kernel of the integral operator of (2.9). Then, for example, in the problem of the interaction between two waves, we obtain again a solution of the form (27) corresponding to a spectral transformation with practically the same characteristic time as given by (2.10). Therefore the intensity of the short wave oscillation again tends to zero with increasing time, and the intensity of the long wave oscillation tends to a finite limit. However, because the number of quanta is conserved in accordance with (2.13), the oscillation energy decreases with increase in wavelength. In the problem of interaction between the two waves, the energy lost by the oscillations tends asymptotically to the value

$$\Delta W = \hbar(\omega_1-\omega_2)n(\mathbf{k}_1,0) \cong {}^3/_2(k_1^2-k_2^2)r_D^2W_1(t_0). \quad (2.14)$$

It is evident therefore that the nonlinear damping of the oscillations is weak, since the fraction of the energy transferred by the oscillations to the particles is small for the case which we are considering, where the wavelengths exceed the Debye radius.

To conclude this section let us compare our results with those by others. First, since the oscillations with parallel wave vectors do not interact in the first approximation in powers of $(kr_D)^2$, an appreciable difference is observed between our three-dimensional case and the one-dimensional case considered previously [3]. Second, the solution for the time attenuation of the intensity of the Langmuir oscillations, written down by Karpman [4] on the basis of qualitative considerations, is not confirmed by our exact analysis. The reason is that the fact that a spectral redistribution takes place has not been brought out in [4]; to the contrary, it is assumed there that the principal manifestation of the interaction between the oscillations is attenuation.

3. Let us turn now to examine the interaction between electron oscillations in a system of two interpenetrating plasmas. The ion contribution can in this case again be neglected, and we can use for the electrons a distribution function in the form

$$f(\mathbf{p}) = \frac{N_1}{(2\pi m\varkappa T_1)^{3/2}}\exp\left\{-\frac{(\mathbf{p}-m\mathbf{U}_1)^2}{2m\varkappa T_1}\right\} + \frac{N_2}{(2\pi m\varkappa T_2)^{3/2}}\exp\left\{-\frac{(\mathbf{p}-m\mathbf{U}_2)^2}{2m\varkappa T_2}\right\}.$$

Assuming the directional velocities of the electrons to be small compared with the thermal velocities, and assuming the phase velocities of the oscillations to be accordingly large, we can write the following relations for the dependence of the oscillation frequency on the wave vector:

$$\omega = \omega_L + \mathbf{kU} + {}^3/_2(\varkappa T / m\omega_L),$$

where

$$\mathbf{U} = (N_1 / N)\mathbf{U}_1 + (N_2 / N)\mathbf{U}_2,$$

$$\varkappa T = (N_1 / N)\varkappa T_1 + (N_2 / N)\varkappa T_2,$$

$$N = N_1 + N_2.$$

With the aid of (1.1) and the expressions written out here for the electron distribution function and for the frequency, we can obtain a relatively simple expression describing the nonlinear interaction of oscillations and serving as an analog of (2.3):

$$\frac{\partial W(\mathbf{k})}{\partial t} = -\frac{1}{(2\pi)^{5/2}}\frac{W(\mathbf{k})}{N^3 m\omega_L}\int d\mathbf{k}' W(\mathbf{k}')\frac{[\mathbf{kk}']^2}{|\mathbf{k}-\mathbf{k}'|^3}\left(\frac{\mathbf{kk}'}{kk'}\right)^2$$
$$\times\left\{N_1 N_2\left(\frac{1}{v_{T_1}}-\frac{1}{v_{T_2}}\right)(\mathbf{k}-\mathbf{k}',\ \mathbf{U}_2-\mathbf{U}_1)\right.$$
$$\left.+\frac{3}{2}\frac{k^2-k'^2}{\omega_L}\frac{N\varkappa T}{m}\left(\frac{N_1}{v_{T_1}}+\frac{N_2}{v_{T_2}}\right)\right\}, \qquad (3.1)$$

where $v_{T_j} = (\kappa T_j / m_j)^{1/2}$ —thermal velocity. The greatest difference between (3.1) and (2.3) occurs when the directional velocity of the electrons greatly exceeds the ratio of the square of the thermal velocity of the particles to the phase velocity of the oscillations. We can then neglect the last term in the curly bracket of the right side of (3.1). For the nonlinear interaction between two oscillations we can use, for example, formulas (2.4), (2.5), and (2.7) with a_{12} replaced by

$$-\frac{1}{(2\pi)^{5/2}}\frac{\omega_L}{N r_D{}^3}\frac{N_1 N_2}{\varkappa T N^2}\left(\frac{1}{v_{T_1}}-\frac{1}{v_{T_2}}\right)$$
$$\times(\mathbf{k}_1-\mathbf{k}_2,\ \mathbf{U}_1-\mathbf{U}_2)\, r^5{}_D\frac{[\mathbf{k}_1\mathbf{k}_2]^2}{|\mathbf{k}_1-\mathbf{k}_2|^3}\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1 k_2}\right)^2. \qquad (3.2)$$

An important factor here is the dependence on the ratio of the electron flux temperatures. Assuming, for example, that $U_1 = 0$ and that the temperature of the plasma at rest is higher than the beam temperature ($T_1 > T_2$) we obtain, as can be readily seen, pumping from the oscillation with the small wave-vector projection on the beam velocity direction U_2 to the oscillations with the larger value or, what is the same, from lower to higher frequencies. Indeed, in this case a_{12} is positive if $k_1 \cdot U_2 < k_2 \cdot U_2$. To the contrary, if the temperature of the plasma at rest is lower than the beam temperature, then the pumping is from the larger values of the projection of the wave vector on the beam velocity direction to the lower values. In other words, in this case the pumping is from the higher frequencies to the lower ones.

Two such cases differ in fact even more. To demonstrate this, we must use a more accurate equation than (3.1). Assuming that the directional velocity of the particles is much lower than the thermal velocity and is at the same time much larger than the ratio of the square of the thermal velocity of the particles to the phase velocity of the oscillations, we can make Eq. (3.1) more precise by obtaining a correction that leads to violation of the oscillation-energy conservation. Such a correction corresponds to the occurrence in the integrand of the right side of (3.1) of a factor

$$1 + {}^1/_2(\mathbf{k}-\mathbf{k}',\ \mathbf{U}). \qquad (3.3)$$

Taking such a correction into account, the conserved quantity is no longer the excitation energy, but the integral (2.13). In other words, we can speak of conservation of the number of quanta. In this case the expression

$$\Delta W \doteq (\omega_1-\omega_2) W_1(t_0) / \omega_1 \approx W_1(t_0)(\mathbf{U}, \mathbf{k}_1-\mathbf{k}_2) / \omega_L \qquad (3.4)$$

holds true for the difference in the energies of the two oscillations at the initial instant of time and at an infinite instant, when the oscillation with the wave vector $\mathbf{k}_2$ has attenuated and only the oscillation with the wave vector $\mathbf{k}_1$ remains.

Let us consider the change in the oscillation energy (3.4) as applied to the examples selected in the present section, involving a plasma at rest and a beam ($U_1 = 0$, $U_2 \neq 0$). In the case when the temperature of the plasma at rest is lower than the beam temperature and the pumping is from the higher frequencies to the lower ones, a decrease in the oscillation energy takes place in accordance with (3.4). To the contrary, if the temperature of the plasma at rest is higher than the beam temperature and the pumping goes from the lower frequencies to the higher ones, then the oscillation energy increases. Thus, in the latter case we have not only pumping of the energy within the spectrum, but also an increase in the oscillation energy, i.e., a nonlinear buildup of plasma oscillations takes place.

The example considered here, nonlinear buildup of oscillations by a slow beam, corresponds to the possibility in principle, indicated by one of the authors[8], of buildup in the case of wave scattering. This example demonstrates the essential difference between nonlinear and linear buildup of oscillations. Namely, whereas in the linear theory no limitations arise on the amplitude (or on the energy) of the growing oscillations, in the present example of nonlinear buildup the oscillation energy increases by a finite amount.

4. We shall devote the last section to the interaction between ion-sound waves, which, as is well known, is possible when the electron temperature T_e is much higher than the ion temperature T_i. In this case the phase velocity of the ion-sound oscillations is much smaller than the thermal velocity of the electrons ($v_{T_e} = \sqrt{\kappa T_e/m}$) and is at the same time much larger than the thermal velocity of the ions ($v_{T_i} = \sqrt{\kappa T_i/M}$). Let us confine ourselves to the case of short waves, when the wavelength of the oscillations is smaller than the Debye radius of the electrons (r_{D_e}). We then have for the dependence of the ion-sound frequency on the wave number

$$\omega^2 = \frac{\omega_{L_i}^2}{1+(kr_{D_e})^{-2}} \simeq \omega_{L_i}^2 - \frac{\omega_{L_i}^2}{k^2 r_{D_e}^2}, \qquad (4.1)$$

where $\omega_{L_i} = [4\pi e_i^2 N_i/M]^{1/2}$ —Langmuir frequency of the ion oscillations.

Using the premises of the preceding paragraph, we obtain from (1.1) the following approximate equation, which describes the nonlinear interaction of short-wave ion sound:

$$\frac{\partial W(\mathbf{k})}{\partial t} + \sqrt{\frac{\pi}{2}}\frac{\omega_{L_i}^2}{k^3}\left[\frac{W(\mathbf{k})-\varkappa T_e}{r^2_{D_e}v_{T_e}} + \frac{W(\mathbf{k})-\varkappa T_i}{r_{D_i}^2 v_{T_i}} \times\exp\left(-\frac{1}{2k^2 r_{D_i}^2}\right)\right] = -\frac{1}{(2\pi)^{3/2}}\frac{e_i^2}{M^2}\frac{W(\mathbf{k})}{\omega_{L_i}^2 r_{D_e}^2 v_{T_i}}$$
$$\times\int d\mathbf{k}' W(\mathbf{k}')\left(\frac{[\mathbf{k}\mathbf{k}']}{kk'}\right)^2\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2\frac{k^2-k'^2}{|\mathbf{k}-\mathbf{k}'|^3} \times\exp\left[-\frac{(k^2-k'^2)^2}{(\mathbf{k}-\mathbf{k}')^2 8k^4k'^4 r_{D_e}^4 r_{D_i}^2}\right]$$
$$-\frac{4}{9(2\pi)^{3/2}}\frac{e_i^2}{M^2}\frac{W(\mathbf{k})}{\omega_{L_i}^2 r_{D_e}^2 v_{T_e}}\int d\mathbf{k}' W(\mathbf{k}')\frac{k^2k'^2}{|\mathbf{k}+\mathbf{k}'|^5} \times\left[2+\frac{3}{2}\mathbf{k}\mathbf{k}'\left(\frac{1}{k^2}+\frac{1}{k'^2}\right)+\left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2\right]^2$$
$$\times\left\{1+\frac{r_{D_e}^2 v_{T_e}}{r_{D_i}^2 v_{T_i}}\exp\left[-\frac{2}{r_{D_i}^2|\mathbf{k}+\mathbf{k}'|^2}\right]\right\}. \qquad (4.2)$$

Being interested in the nonlinear interaction of oscillations of large amplitude, we can neglect the terms in the left side of (4.2) which do not contain the derivative with respect to the time. Then, for example, for the problem of nonlinear interaction of two oscillations, when $W(\mathbf{k}, t)$ is of the form (2.4), we obtain the system of two equations

$$dW_1/dt = -W_1W_2(a_{12}+b_{12}),$$
$$dW_2/dt = -W_1W_2(-a_{12}+b_{12}), \qquad (4.3)$$

where

$$a_{12} = \frac{1}{(2\pi)^{3/2}}\frac{e_i^2}{M^2}\frac{1}{\omega_{L_i}^2 r_{D_e}^2 v_{T_i}}\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right)^2\left(\frac{[\mathbf{k}_1\mathbf{k}_2]}{k_1k_2}\right)^2\frac{k_1^2-k_2^2}{|\mathbf{k}_1-\mathbf{k}_2|^3} \times\exp\left\{-\frac{1}{8r_{D_i}^2 r_{D_e}^4 k_1^4 k_2^4}\frac{(k_1^2-k_2^2)^2}{|\mathbf{k}_1-\mathbf{k}_2|^2}\right\}, \qquad (4.4)$$

$$b_{12} = \frac{4}{9(2\pi)^{3/2}}\frac{e_i^2}{M^2}\frac{1}{\omega_{L_i}^2 r_{D_e}^2 v_{T_e}}\frac{k_1^2k_2^2}{|\mathbf{k}_1+\mathbf{k}_2|^5} \times\left[2+\frac{3}{2}\mathbf{k}_1\mathbf{k}_2\left(\frac{1}{k_1^2}+\frac{1}{k_2^2}\right)+\left(\frac{\mathbf{k}_1\mathbf{k}_2}{k_1k_2}\right)^2\right]^2$$
$$\times\left\{1+\frac{r_{D_e}^2 v_{T_e}}{r_{D_i}^2 v_{T_i}}\exp\left[-\frac{2}{r_{D_i}^2|\mathbf{k}_1+\mathbf{k}_2|^2}\right]\right\}. \qquad (4.5)$$

Under conditions when

$$(k_1^2-k_2^2) \gg (\mathbf{k}_1-\mathbf{k}_2)^2(r_{D_i} r^2_{D_e} k_1^2 k_2^2)^2,$$

we can neglect the coefficient a_{12} in (4.3). The amplitudes of both oscillations then decrease, in other words, the oscillations are damped. The solution of the system (4.3) can then be written in the form

$$W_1(t) - W_1(t_0) = W_2(t) - W_2(t_0), \qquad (4.6)$$

$$\frac{W_1(t)}{W_2(t)} = \frac{W_1(t_0)}{W_2(t_0)}\exp\{(t-t_0)b_{12}[W_1(t_0)-W_2(t_0)]\}. \qquad (4.7)$$

Thus, although the intensities of both oscillations decrease, their rates of decrease are different, inasmuch as the intensity ratio varies in accordance with (4.7) exponentially with the time, provided only the intensities of the oscillations are not equal at the initial instant.

It is obvious that according to (4.6) and (4.7) the time variation of the oscillation intensity has the form[3)]

$$\frac{W_1(t)}{W_1(t_0)} = \frac{W_1(t_0)-W_2(t_0)}{W_1(t_0)-W_2(t_0)\exp\{-(t-t_0)b_{12}[W_1(t_0)-W_2(t_0)]\}}. \qquad (4.8)$$

It follows therefore that if the oscillation intensity W_1 exceeds W_2 at the initial instant of time, then the amplitude of the less intense oscillation tends to zero with increasing time, while the intensity of the second oscillation tends to a value equal to the difference between the initial values of the intensities of the oscillations. The characteristic time of such a nonlinear relaxation of the oscillations has an order of magnitude

[3)]If the intensities of the oscillations are the same at the initial instant of time, then, in accordance with (4.6), they remain the same also in the succeeding instants of time, varying like $W_1(t)/W_2(t) = [1 + W_1(t_0)b_{12}(t - t_0)]^{-1}$.

$$\sim 30 \frac{M}{m} \frac{L}{\nu_{\text{eff}}} \frac{1}{(kr_{D_e})^2} \frac{\varkappa T_e k^3}{|W_1(0) - W_2(0)|} .$$

Therefore, for example, for $L \sim 10$, $(kr_{D_e})^2 \sim 0.1$, and $|W_1(0) - W_2(0)| \sim 10^5 \kappa T_e k^3$ the time of this relaxation is approximately two orders of magnitude smaller than the electron mean free time due to the Coulomb collisions.

So far we have spoken of the solution of (4.3) under conditions when $b_{12} \gg a_{12}$. In the opposite case Eqs. (4.3) reduce to Eqs. (2.5), and the solutions are determined by formulas (2.7) in which a_{12} is given by (4.4). In such an approximation we can again speak only of pumping from the short-wave oscillations to the long-wave ones within the spectrum. The order of magnitude of the corresponding time of the spectral pumping is determined by the formula

$$\tau_s \sim \frac{L}{\nu_{\text{eff}}} \left[\frac{MT_i}{mT_e}\right]^{1/2} \frac{\varkappa T_e}{[W(0)/k^3]} .$$

As in the estimates of the relaxation time, in the cases considered above we can readily see a wide range of plasma parameters for which such a relaxation time is sufficiently small.

APPENDIX

In Sec. 2 we have ignored completely the possible influence of the ions when considering the nonlinear interaction between electron Langmuir oscillations. We present below the results of an analysis of the opposite case, when the influence of the ions is decisive for the nonlinear interaction of the electron Langmuir oscillations. Namely, in the region of large wavelengths, when

$$(\omega - \omega')^2 \equiv \frac{9}{4} v_{T_e}{}^2 r_{D_e}{}^2 (k^2 - k'^2)^2 \ll v_{T_i}{}^2 (\mathbf{k} - \mathbf{k}')^2 \ln \left[\frac{e_i^2}{e^2} \frac{M}{m} \frac{T_e^3}{T_i^3}\right], \tag{A.1}$$

the most important factor becomes the induced scattering of the oscillations by the ions. In the language of quantum-mechanical perturbation theory, we can interpret the resultant interaction in the following fashion. The plasmons (high-frequency Langmuir oscillations) interact with an electron, which emits a virtual low-frequency oscillation. The latter is absorbed by the ion. The corresponding diagrams of such a matrix element and those which differ chronologically constitute a triangle made of the electron propagation lines, with the free lines of the incident and scattered oscillations going to two vertices, while the third vertex is joined by the low-frequency oscillation line with the ion propagation line.

If inequality (A.1) is satisfied, we obtain from (1.1) in lieu of (2.3)

$$\frac{\partial W(\mathbf{k})}{dt} = -\frac{3}{8(2\pi)^{5/2}} \frac{W(\mathbf{k})}{\varkappa T_e} \frac{\omega_{L_e}}{N r_{D_e}^3} \frac{v_{T_e}}{v_{T_i}} r_{D_e}^4 \times \int d\mathbf{k}' W(\mathbf{k}') \frac{k^2 - k'^2}{|\mathbf{k} - \mathbf{k}'|} \left(\frac{\mathbf{k}\mathbf{k}'}{kk'}\right)^2 \frac{(r_{D_i}/r_{D_e})^2}{[F + (r_{D_i}/r_{D_e})^2]^2} \times \exp\left\{-\frac{9}{8} \frac{v_{T_e}^2}{v_{T_i}^2} r_{D_e}^2 \frac{(k^2 - k'^2)^2}{(\mathbf{k} - \mathbf{k}')^2}\right\}. \tag{A.2}$$

Here

$$F = 0, \quad \text{if} \quad 1 \gg \frac{v_{T_i}^2 (\mathbf{k} - \mathbf{k}')^2}{(\omega - \omega')^2} \gg \left\{\ln \frac{e_i^2}{e^2} \frac{M}{m} \frac{T_e^3}{T_i^3}\right\}^{-1},$$

$$F = 1, \quad \text{if} \quad v_{T_i}^2 (\mathbf{k} - \mathbf{k}')^2 \gg (\omega - \omega')^2. \tag{A.3}$$

Inequality (A.1) can be readily satisfied at long wavelengths, when

$$k^2 r_{D_e}^2 \ll \frac{mT_i}{MT_e} \ln \left[\frac{e^2}{e_i^2} \frac{M}{m} \frac{T_e^3}{T_i^3}\right]$$

In the region of such wavelengths, as follows from (A.2), the spectral redistribution goes from the short waves to the long ones, and the characteristic time of such a redistribution is of the order of magnitude

$$\tau_s \sim \frac{L}{\nu_{\text{eff}}} \frac{\varkappa T_e}{W} \frac{1}{r_{D_e}^4 k^3 \Delta k} \frac{v_{T_i}}{v_{T_e}} .$$

A comparison of the latter expression with (2.11) shows that in the region of long waves the spectral redistribution occurs more rapidly than could be predicted by a theory that neglects the interaction between the electron oscillations and the ions.

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31