SOVIET PHYSICS JETP VOLUME 20, NUMBER 1 JANUARY, 1965

ENERGY SPECTRUM OF A BOSE GAS WITH WEAK ATTRACTION AT LARGE DISTANCES

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Submitted to JETP editor December 18, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **47**, 167-174 (July, 1964)

The graph technique developed by Belyaev is employed for calculating the ground-state energy and elementary-excitation spectrum of a Bose gas with negative scattering amplitudes at low momenta. An important point in the solution of the problem is that repulsion plays an important role at small distances and thus ensures the possibility of stable states at densities exceeding a certain critical value. The scattering amplitude for zero momenta is considered small so that the gas approximation is valid near the critical density. The final formulas resemble those derived by Huang, but the numerical coefficients are quite different.

A study of the model of a Bose gas with weak attraction at large distances is of certain interest. We note that, as is well known from experiment, the interaction potential of two helium atoms corresponds to attraction at large distances and repulsion at small distances between the atoms. Such a model should exhibit saturation, i.e., there should exist a critical density below which stable states are impossible.

A direct extension of the results of Bogolyubov[1] concerning the energy spectrum of a weakly nonideal Bose gas raises certain difficulties, for the excitation energy turns out to be imaginary in the lowest approximation. Huang[2] constructed some simplified model of a Bose gas with attraction and made an approximate calculation of the energy spectrum and of the ground-state energy. However, his method is exceedingly cumbersome and it is difficult to clarify the character of the approximations made. In the present paper we calculate, using a graph technique developed by Belyaev[3,4], the energy spectrum and the energy of the ground state for a model of a weak nonideal Bose gas with attraction close to that proposed by Huang.

We assume that the Bose particle pair-interaction potential energy $V(r) = V_1(r) + V_2(r)$ consists of a repulsion kernel $V_1(r) > 0$ of effective radius a, and a long-range attraction $V_2(r) < 0$ of radius $R \gg a$. Let us assume that the relation between these quantities is such that in the essential region we have $V_1 a^3 \sim V_2 R^3$, so that the corresponding Fourier components $U_1(\mathbf{k})$ and $U_2(\mathbf{k})$ have the same order of magnitude when $k < 1/R$. In addition, we assume that the Born parameter ξ_1 for the potential V_1 is of the order of unity, i.e., $\xi_1 = U_1 m/a \sim 1$, and then it follows that $\xi_2 = U_2 m/R \sim (U_1 m/a)(a/R) \ll 1$.

We shall show that in this case the scattering amplitude $f(\mathbf{k}, \mathbf{k}')$ for two particles scattered by each other and interacting via a potential $V(r)$ can be expressed simply in terms of the scattering amplitude of two particles with an interaction potential energy $V_1(r)$. The scattering amplitude can be represented by a Born series obtained by iterated solution of the integral equation for the scattering amplitude (see, for example [4,5])

$$\frac{4\pi}{m} f(\mathbf{k}, \mathbf{k}') = U(\mathbf{k} - \mathbf{k}') + \sum_{n=1}^{\infty} \frac{1}{(2\pi)^{3n}} \int d^3p_1 \ldots d^3p_n \times \frac{U(\mathbf{k}' - \mathbf{p}_1) \ldots U(\mathbf{p}_{n-1} - \mathbf{p}_n)\, U(\mathbf{p}_n - \mathbf{k})}{[k^2/m - p_1^2/m + i\delta] \ldots [k^2/m - p_n^2/m + i\delta]}$$

(we use a system of units in which $\hbar = 1$).

In the sum over n, the principal role is played by those regions of integration where $U \approx U_1$. Indeed, let us consider, for example, the integral over p_n. The relative contribution from the region where $|\mathbf{p}_n - \mathbf{k}| \sim 1/R$ will be of the order of $(U_1 m/a)(a/R) \ll 1$, and an analogous estimate will hold for the region $|\mathbf{p}_{n-1} - \mathbf{p}_n| \sim 1/R$. Similar reasoning holds for integration with respect to any p_i.

Since $U(k) \approx U_1(k)$ when $k \gg 1/R$, we can replace U by U_1 in the sum over n. From this we obtain approximately, neglecting the terms of order a/R,

$$4\pi m^{-1} f(\mathbf{k}, \mathbf{k}') = [U(\mathbf{k}' - \mathbf{k}) - U_1(\mathbf{k}' - \mathbf{k})] + 4\pi m^{-1} f_1(\mathbf{k}, \mathbf{k}'), \qquad (1)$$

where $f_1(\mathbf{k}, \mathbf{k}')$ —scattering amplitude for the potential V_1.

We assume, in addition, that the potential $U(k)$ is such that

$$4\pi m^{-1} f(0, 0) < 0, \text{ with } |f(0, 0)/f_1(0, 0)| = \beta \ll 1. \quad (2)$$

The quantity β characterizes the "weakness" of the attraction and is the small parameter of the problem. We note that the smallness of β does not signify smallness of U_2, and is connected with the almost total cancellation of the repulsion and attraction in the scattering amplitude at small k and k′. This assumption is essential to the calculation that follows, for in the opposite case ($\beta \sim 1$) there are no thermodynamically stable homogeneous states with low density. Therefore the model in question has limited applicability and cannot be used for the calculation of real He II.

Since the characteristic momentum for U_1 is 1/a, we have $f_1(\mathbf{k}, \mathbf{k}') \approx f_1(0, 0)$ for $k, k' \ll 1/a$. Therefore we have for $k \ll 1/a$

$$4\pi m^{-1} f(\mathbf{k}, \mathbf{k}) = [U(0) - U_1(0)] + 4\pi m^{-1} f_1(0, 0)$$
$$= -4\pi m^{-1} \beta f_0, \quad (3)$$

whereas for $1/R \ll |\mathbf{k} - \mathbf{k}'| \ll 1/a$ and $k \ll 1/a$ we have

$$4\pi m^{-1} f(\mathbf{k}, \mathbf{k}') = [U(\mathbf{k} - \mathbf{k}') - U_1(\mathbf{k} - \mathbf{k}')]$$
$$+ 4\pi m^{-1} f_1(0, 0) \approx 4\pi m^{-1} f_0$$

(we can neglect $U_2(k)$ when $k \gg 1/R$). We use the notation $f_0 \equiv f(0, 0) > 0$ throughout.

Thus, in the present model the scattering amplitude varies rapidly as $|\mathbf{k} - \mathbf{k}'|$ varies from zero to values $\sim 1/R$, and then remains constant to $k, k' \approx 1/a$ (see Fig. 1).

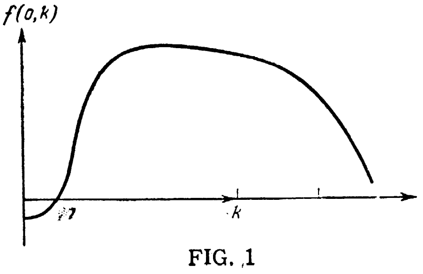

FIG. 1

The energy spectrum of the weakly excited states and of the ground-state energy can be obtained with the aid of the single-particle Green's function. To calculate the Green's function of the interacting Bose particles it is convenient to use the graph technique developed by Belyaev[3,4]. We introduce the Green's function of the super-condensed particles:

$$G'(\mathbf{x} - \mathbf{x}') = \langle T\psi'(\mathbf{x})\psi^{+\prime}(\mathbf{x})'\rangle,$$
$$\hat{G}(\mathbf{x} - \mathbf{x}') = \langle T\psi'(\mathbf{x})\psi'(\mathbf{x}')\rangle,$$
$$\check{G}(\mathbf{x} - \mathbf{x}') = \langle T\psi^{+\prime}(\mathbf{x})\psi^{+\prime}(\mathbf{x}')\rangle,$$

where T —symbol of the T-product and $\psi'(x)$, $\psi^{+\prime}(x)$ —operators of annihilation and creation of the super-condensed particles at the point x.

According to [3] (see also [5]), we obtain for the Fourier transforms of the introduced Green's functions

$$G'(p) = \frac{p^0 + \varepsilon_0(\mathbf{p}) + S(p) - A(p) - \mu}{[p^0 - A(p)]^2 - [\varepsilon_0(\mathbf{p}) + S(p) - \mu]^2 + \Sigma_{02}^2(p)},$$

$$\hat{G}(p) = \check{G}(p)$$
$$= \frac{\Sigma_{02}(p)}{[p^0 - A(p)]^2 - [\varepsilon_0(\mathbf{p}) + S(p) - \mu]^2 + \Sigma_{02}^2(p)}.$$

Here

$$S(p) = {}^1/_2(\Sigma_{11}(p) + \Sigma_{11}(-p)), \quad A(p) = {}^1/_2(\Sigma_{11}(p)$$
$$- \Sigma_{11}(-p)), \quad \varepsilon_0(\mathbf{p}) = p^2/2m. \quad (5)$$

The chemical potential μ is expressed in terms of the mass operators Σ_{11} and Σ_{02} by the relation

$$\mu = \Sigma_{11}(0) - \Sigma_{02}(0), \quad (6)$$

first derived by Hugengoltz and Pines[6].

The mass operators $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ are sums of Feynman diagrams, in which the straight lines correspond to the Green's functions of the free particles and the broken lines to the condensed particles, while the wavy lines correspond to the Fourier component of the potential. The general rules for the construction of the graphs and for writing down the corresponding analytical expressions can be found in [3,5]. In the general case it is impossible to sum the graphs. The effective calculation of the mass operators with a Born parameter $\xi \sim 1$ can be carried out only for the case of low density n[4], when $nf_0^3 \sim na^3 \ll 1$. We shall likewise assume the density to be small, so that $nf_0^3 \sim \beta^2$, and we shall calculate Σ_{11} and Σ_{02} in the lowest order in β.

The corresponding calculations were made by Belyaev[4], but our case is different in that the dependence of the scattering amplitude on the momenta cannot be neglected, since this amplitude varies rapidly in the interval $0 \le k \lesssim 1/R$, as can be seen from (3) and (4). In addition, the order of magnitude of the Green's function changes at low values of the momentum. This, however, is not an essential factor in the estimate of the contributions of different graphs, since Σ_{11} and Σ_{02} contain only integrals of the Green's functions, and the order of magnitude of the integrals remains the same as before, owing to the smallness of the corresponding momentum region. In the second-order approximation, the corresponding contribution to the integrals turns out to be small because in addition the scattering amplitude is also small in this region (see below).

The applicability of the gas expansion can be proved also by using the initial formulation of the diagram technique, without resorting to the scattering amplitude. Since U(k) has in the present model the same order of magnitude for repulsion and for attraction, we obtain, by equating U(k) to a positive constant U_0 for $k < 1/a$ and to zero for $k > 1/a (R > a)$, the same estimate as in the case of pure repulsion for any graph. It follows therefore that any loop in Σ_{ik} with more than two straight lines introduces a small parameter (for $n_0a^3 \sim \beta^2$) i.e., the gas approximation can be used (see [5]).

Thus, following [4], we can confine ourselves to graphs of the first and second orders in $n_0f_0^3$, where n_0 —density of the number of particles with zero momentum. The first-order graphs have the form shown in Fig. 2a, where the square denotes the ladder vertex function $\Gamma^0(p_1, p_2; p_3, p_4)$, which coincides in first approximation with $f((p - p_2)/2, (p_3 - p_4)/2)$, and the broken lines correspond to the creation or annihilation of particles in the condensate.

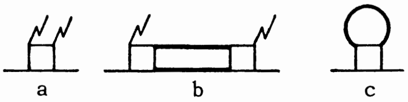

FIG. 2

The second order in Σ is described by graphs consisting of two small squares connected by two lines corresponding to all possible total Green's functions (5) (Fig. 2b), and also graphs consisting of one square with a loop, which also corresponds to some total Green's function (Fig. 2c). It is necessary to subtract from these graphs the aggregate of graphs of the same type, but now contained in the first-order graphs. The corresponding analytic expression for Σ_{02} is of the form (see [4])

$$\Sigma_{02}(p) = n_0\Gamma^0(p, -p; 0, 0) + 4in_0\int \widetilde{\Gamma}^0(p, q; p+q, 0)$$
$$\times \widetilde{\Gamma}^0(p+q, -p; 0, q)\, G'(p+q)\, G'(q)\frac{d^4q}{(2\pi)^4} + 4in_0$$
$$\times \int \widetilde{\Gamma}^0(p, 0; p-q, q)\, \widetilde{\Gamma}^0(q, -p; q-p, 0)\, G'(q)\, \check{G}(p-q)$$
$$\times \frac{d^4q}{(2\pi)^4} + 4in_0\int \widetilde{\Gamma}^0(p, q; 0, p+q)$$
$$\times\ \widetilde{\Gamma}^0(-p, 0; q, -p-q)\, G'(q)\, \check{G}(p+q)\frac{d^4q}{(2\pi)^4}$$
$$+ 4in_0\int \widetilde{\Gamma}^0(p, -q; 0, p-q)\, \widetilde{\Gamma}^0(q, -p; 0, -p+q)$$
$$\times \hat{G}(q)\, \check{G}(p-q)\frac{d^4q}{(2\pi)^4} + 2in_0\int \Gamma^0(p, 0; p-q, q)$$
$$\times\ \widetilde{\Gamma}^0(-p, 0; -q, -p+q)\, \check{G}(p-q)\, \check{G}(q)\frac{d^4q}{(2\pi)^4}$$
$$+ i\int \Gamma^0(p, -p; q, -q)\,[\check{G}(q) - n_0\Gamma^0(q, -q; 0, 0)$$
$$\times G_0(-q)\, G_0(q)]\frac{d^4q}{(2\pi)^4}, \tag{7}$$

$$\widetilde{\Gamma}^0(p_1, p_2; p_3, p_4) = {}^1/_2\,[\Gamma^0(p_1, p_2; p_3, p_4) + \Gamma^0(p_1, p_2; p_4, p_3)],$$
$$G_0(q) = (q^0 - \mathbf{q}^2/2m + i\delta)^{-1}.$$

The expression for Σ_{11} is written in the form

$$\Sigma_{11}(p) = 2n_0\widetilde{\Gamma}^0(p, 0; p, 0) + 4in_0\int \widetilde{\Gamma}^0(p, 0; p-q, q)$$
$$\times \widetilde{\Gamma}^0(q, p-q; p, 0)[G'(q)\, G'(p-q) - G_0(q)\, G_0(p-q)]$$
$$\times \frac{d^4q}{(2\pi)^4} + 4in_0\int \widetilde{\Gamma}^0(p, q; p+q, 0)\, \widetilde{\Gamma}^0(p+q, 0; p, q)$$
$$\times\quad G'(p+q)\, G'(q)\frac{d^4q}{(2\pi)^4} + 4in_0\int \widetilde{\Gamma}^0(p, 0; q, p-q)$$
$$\times\quad \widetilde{\Gamma}^0(q, 0; p, q-p)\, G'(q)\, \check{G}(p-q)\frac{d^4q}{(2\pi)^4} + 4in_0$$
$$\times \int \widetilde{\Gamma}^0(p, -q; 0, p-q)\, \widetilde{\Gamma}^0(q, p-q; 0, p)\, \hat{G}(q)\, G'(p+q)$$
$$\times \frac{d^4q}{(2\pi)^4} + 4in_0\int \widetilde{\Gamma}^0(p, -q; 0, p-q)\, \widetilde{\Gamma}^0(q, 0; q-p, p)$$
$$\times\ \hat{G}(q)\, \check{G}(p-q)\frac{d^4q}{(2\pi)^4} + 2in_0\int \widetilde{\Gamma}^0(p, q; p, q)\, G'(q)\frac{d^4q}{(2\pi)^4}. \tag{8}$$

The quantity $\Gamma^0(p_1, p_2; p_3, p_4)$ is expressed in terms of the scattering amplitude [4]:

$$\Gamma^0(p_1, p_2; p_3, p_4) = \frac{4\pi}{m} f(\mathbf{k}, \mathbf{k}') + \left(\frac{4\pi}{m}\right)^2 \int f(\mathbf{k}', \mathbf{q})\, f^*(\mathbf{k}, \mathbf{q})$$
$$\times \left[\frac{1}{\varkappa_0^2/m - q^2/m + i\delta} + \frac{1}{q^2/m - k^2/m - i\delta}\right] d^3q;$$
$$\varkappa_0^2 = P^0 - \mathbf{P}^2/4m + 2\mu, \quad \mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2, \quad 2\mathbf{k} = \mathbf{p}_3 - \mathbf{p}_4,$$
$$2\mathbf{k}' = \mathbf{p}_1 - \mathbf{p}_2,$$
$$P^0 = p_1^0 + p_2^0. \tag{9}$$

A circumstance of importance for what follows is that in the region $p \gg 1/R$ the principal role is played by the first-approximation terms, since $\Gamma^0(p, 0; 0, p)$ is in this region of the order of f_0, and the second-approximation correction contains an extra factor β compared with these terms. However, when $p \lesssim 1/R$ the first- and second-approximation terms have the same order of magnitude, owing to the smallness of $\Gamma^0 \sim \beta f_0$ in this region, while allowance for the more complicated graphs yields as before a small correction to the graphs of Fig. 2.

In calculating the second-approximation corrections we can neglect everywhere the integral terms in formula (9), and also the quantity $\Gamma^0(p, 0; p, 0) \approx f(p/2, p/2)$, since they contain an extra power of β. In addition, the regions in which the Green's functions G', $\hat{G}$, and $\check{G}$ are not described by formulas (5) with first-approximation Σ_{11} and Σ_{02} are small, and in these regions the quantities Γ^0 under the integral signs in (7) and (8) are also small (we have in mind small momenta $p \ll 1/R$). Therefore we can use, with the assumed accuracy, Green's functions with lower-approximation Σ_{11} and Σ_{02} in the integral terms of (7) and (8); these Green's functions, with allowance for (3), (4), and (9), can be written, with the same accuracy, in the form

$$\Sigma_{11}(p) = n_0\Gamma^0(p, 0; p, 0) + n_0\Gamma^0(p, 0; 0, p) \simeq 4\pi m^{-1}n_0 f(p, 0),$$

$$\Sigma_{02}(p) = n_0\Gamma^0(p, -p; 0, 0) \approx 4\pi m^{-1} f(p, 0). \qquad (10)$$

In addition, we can neglect the chemical potential μ, since it is of the order of $4\pi m^{-1} n_0\beta f_0$, in accordance with (6), (7), and (8). Using (5), we thus obtain for the Green's functions themselves, in the region $1/a \gg p \gtrsim 1/R$, the expressions

$$G'(p) = \frac{p^0 + \varepsilon_0(\mathbf{p}) + 4\pi m^{-1} n_0 f(\mathbf{p}, 0)}{2\varepsilon(\mathbf{p})} \times \left(\frac{1}{p^0 - \varepsilon(\mathbf{p}) + i\delta} - \frac{1}{p^0 + \varepsilon(\mathbf{p}) - i\delta}\right), \qquad (11)$$

$$\hat{G}(p) = \check{G}(p) = \frac{4\pi m^{-1} n_0 f(\mathbf{p}, 0)}{2\varepsilon(\mathbf{p})} \times \left(\frac{1}{p^0 - \varepsilon(\mathbf{p}) + i\delta} - \frac{1}{p^0 + \varepsilon(\mathbf{p}) - i\delta}\right);$$

$$\varepsilon(\mathbf{p}) = [8\pi m^{-1} n_0 f(\mathbf{p}, 0) + \varepsilon_0^2(\mathbf{p})]^{1/2}. \qquad (12)$$

We shall henceforth confine ourselves to the calculation of $\Sigma_{02}(0)$ and $\Sigma_{11}(0)$, for to this end it is sufficient to find the spectrum at small values of p, and also to calculate the chemical potential and the ground-state energy.

Carrying out the integration with respect to dq^0 in (7), and making use of (11), (12), (9), and the smallness of $f(\mathbf{q}, \mathbf{q})$ in accordance with (3), we obtain

$$\Sigma_{20}(0) = \frac{4\pi}{m} n_0 f(0, 0) + n_0^2 \left(\frac{4\pi}{m}\right)^3 \int \frac{f^3(\mathbf{q}, 0)\,\varepsilon_0(\mathbf{q})}{2\varepsilon^3(\mathbf{q})} \frac{d^3q}{(2\pi)^3} + \left(\frac{4\pi}{m}\right)^2 n_0 \int f^2(\mathbf{q}, 0) \frac{\varepsilon(\mathbf{q}) - \varepsilon_0(\mathbf{q})}{2\varepsilon_0(\mathbf{q})\,\varepsilon(\mathbf{q})} \frac{d^3q}{(2\pi^3)}. \qquad (13)$$

We calculate analogously the quantity $\Sigma_{11}(0)$:

$$\Sigma_{11}(0) = \frac{8\pi}{m} n_0 f(0, 0) + n_0 \left(\frac{4\pi}{m}\right)^2 \int \frac{f^2(\mathbf{q}, 0)(\varepsilon(\mathbf{q}) - \varepsilon_0(\mathbf{q}))}{\varepsilon(\mathbf{q})\,\varepsilon_0(\mathbf{q})} \times \frac{d^3q}{(2\pi)^3} + \frac{4\pi}{m} \int f(\mathbf{q}, 0) \frac{\varepsilon_0(\mathbf{q}) - n_0 f(\mathbf{q}, 0) - \varepsilon(\mathbf{q})}{2\varepsilon(\mathbf{q})} \frac{d^3q}{(2\pi)^3} + n_0 \left(\frac{4\pi}{m}\right)^3 \int \frac{f^3(\mathbf{q}, 0)\,\varepsilon_0(\mathbf{q})}{2\varepsilon^3(\mathbf{q})} \frac{d^3q}{(2\pi)^3}. \qquad (14)$$

As is well known, for small p the Green's functions take the form (see [5]):

$$G'(p) = \frac{\Sigma_{20}(0)}{B[(p^0)^2 - c^2p^2]}, \quad \hat{C}(p) = \check{G}(p) = -\frac{\Sigma_{20}(0)}{B[(p^0)^2 - c^2p^2]}; \qquad (15)$$

$$B = \left(1 - \left.\frac{\partial\Sigma_{11}}{\partial p^0}\right|_0\right)^2 - \left.\frac{\partial^2\Sigma_{11}}{\partial (p^0)^2}\right|_0 \Sigma_{20}(0) + \frac{1}{2}\left.\frac{\partial^2\Sigma_{20}{}^2}{\partial (p^0)^2}\right|_0,$$

$$Bc^2 = 2\Sigma_{20}(0)\left\{\frac{1}{2m} + \left.\frac{\partial(\Sigma_{11} - \Sigma_{20})}{\partial p^2}\right|_0\right\}.$$

In our case

$$2m\partial(\Sigma_{11} - \Sigma_{20})/\partial p^2|_0 \sim \beta, \qquad \partial\Sigma_{11}/\partial p^0 \approx \beta,$$

and therefore, with the assumed accuracy,

$$c^2 = m^{-1}\Sigma_{20}(0). \qquad (16)$$

The energy of the elementary excitations is a positive real quantity, and this leads to the requirement $c^2 > 0$ (this condition coincides with the condition for the thermodynamic stability of the system), thus defining a certain critical density below which there are no stable states. According to (16) and (13), the region of stable states is determined by the inequality

$$-f(0, 0) \leqslant n_0 \left(\frac{4\pi}{m}\right)^2 \int \frac{f^3(\mathbf{q}, 0)\,\varepsilon_0(\mathbf{q})}{2\varepsilon^3(\mathbf{q})} \frac{d^3q}{(2\pi)^3} + \frac{4\pi}{m} \int f^2(\mathbf{q}, 0) \frac{\varepsilon(\mathbf{q}) - \varepsilon_0(\mathbf{q})}{2\varepsilon_0(\mathbf{q})\,\varepsilon(\mathbf{q})} \frac{d^3q}{(2\pi)^3}.$$

Since the first part is positive and increases with increasing n_0, such a region actually exists. It is easy to see that the order of magnitude of the critical density n_{0cr} will be $n_{0cr}f_0^3 \sim (f(0, 0)/f_0)^2 \ll 1$, i.e., in this case we actually operate in the region of applicability of the gas approximation. Formulas (13), (14), and (6) enable us to calculate the chemical potential:

$$\mu(n_0) = n_0 \frac{4\pi}{m} f(0, 0) + n_0 \left(\frac{4\pi}{m}\right)^2 \int \frac{f^2(\mathbf{q}, 0)}{2\varepsilon(\mathbf{q})\,\varepsilon_0(\mathbf{q})} (\varepsilon(\mathbf{q}) - \varepsilon_0(\mathbf{q})) \frac{d^3q}{(2\pi)^3} + \frac{4\pi}{m} \int f(\mathbf{q}, 0) \times \frac{\varepsilon_0(\mathbf{q}) - n_0 f(\mathbf{q}, 0) - \varepsilon(\mathbf{q})}{2\varepsilon(\mathbf{q})} \frac{d^3q}{(2\pi)^3}. \qquad (17)$$

In order to find the ground-state energy it is necessary to know the chemical potential as a function of the total number of particles. In this case the particle density n_0 in the condensate coincides in the lowest approximation in β with the total particle density N/V, and owing to the smallness of $f(0, 0)$ this approximation can be used in all the terms that enter into μ. Thus, the ground-state energy is determined by integrating (17):

$$E - E_{\mathrm{cr}} = \int\limits_{N_{\mathrm{cr}}}^{N} \mu(N/V)\, dN, \tag{18}$$

where N_{cr} —total number of particles in the volume V at the critical density, E_{cr} —ground state energy at $N = N_{\mathrm{cr}}$.

The integration of (13), (14), (17), and (18) cannot be carried out without specifying the concrete form of the scattering amplitude $f(\mathbf{q}, 0)$. At sufficiently large R, however, it is possible to calculate the integrals in explicit form. Indeed, the essential integration region, for example in (13), depends both on $f(\mathbf{q}, 0)$ and on the second factor, which decreases like $1/q^4$ when $q^2 \gg n_0 f_0$, making the region $q \sim 1/a$ unimportant, and remains approximately constant for $q^2 \ll n_0 f_0$.

If $n_0 f_0 \gg 1/R^2$ or, what is the same, $a/R \ll \beta$, then the region with $q \lesssim 1/R$ in which $f(\mathbf{q}, 0)$ varies rapidly (see Fig. 1) can be neglected, and we can put $f(\mathbf{q}, 0) = f_0$ in all the integrals with $f(\mathbf{q}, 0)$. The integrations in (13), (17), and (18) can be readily carried out, so that we obtain

$$c^2 = \frac{4\pi}{m^2} \frac{N}{V} \left[f(0, 0) + \frac{16}{\sqrt{\pi}} f_0 \left(f_0^3 \frac{N}{V} \right)^{1/2} \right], \tag{19}$$

$$\mu = \frac{4\pi}{m} \left[\frac{N}{V} f(0, 0) + \frac{32}{3\sqrt{\pi}} \frac{N}{V} f_0 \left(f_0^3 \frac{N}{V} \right)^{1/2} \right], \tag{20}$$

$$E = E_{\mathrm{cr}} - \frac{4\pi}{m} \left[\frac{N_{\mathrm{cr}}^2}{2V} f(0, 0) + \frac{64}{15\sqrt{\pi}} \frac{N_{\mathrm{cr}}^2}{V} f_0 \left(\frac{N_{\mathrm{cr}}}{V} f_0^3 \right)^{1/2} \right] + \frac{4\pi}{m} \left[\frac{N^2}{2V} f(0, 0) + \frac{64}{15\sqrt{\pi}} \frac{N^2}{V} f_0 \left(\frac{N}{V} f_0^3 \right)^{1/2} \right]. \tag{21}$$

From (19) we see that the dimensionless critical density is

$$N_{\mathrm{cr}} f_0^3 / V = (f(0, 0)\sqrt{\pi} / 16 f_0)^2 \ll 1,$$

in accordance with the assumption that the density is low.

From (21) we can determine the pressure

$$p = -\frac{\partial E}{\partial V} = \frac{4\pi}{m} \left[\frac{N^2 f(0, 0)}{2V^2} + \frac{32}{5\sqrt{\pi}} \frac{N^2}{V^2} f_0 \left(\frac{N}{V} f_0^3 \right)^{1/2} \right]. \tag{22}$$

As should be the case, the velocity of sound c determined from the spectrum agrees with that determined thermodynamically. We note that at the critical density, the pressure p is negative, indicating that this state is metastable. The pressure becomes positive when

$$\frac{N}{V} = f_0^{-3} \left(\frac{5\sqrt{\pi}}{64} \frac{f(0, 0)}{f_0} \right)^2 = \frac{25}{16} \frac{N_{\mathrm{cr}}}{V}.$$

If terms of order $1/(n_0 f_0)^{1/2} R$ are neglected, the quantities $\Sigma_{11}(p)$ and $\Sigma_{02}(p)$ can be approximately described by the formulas

$$\Sigma_{02}(p) = \frac{4\pi N}{mV} f(\mathbf{p}, 0) + \frac{64}{m} f_0 \frac{N}{V} \left(\pi f_0^3 \frac{N}{V} \right)^{1/2},$$

$$\Sigma_{11}(p) = \frac{4\pi}{m} \frac{N}{V} f(0, 0) + \frac{4\pi}{m} \frac{N}{V} f(\mathbf{p}, 0) + \frac{320}{3m} \frac{N}{V} f_0 \left(\pi f_0^3 \frac{N}{V} \right)^{1/2}$$

in the entire momentum interval, since the rapid variation occurs only in the scattering amplitude, i.e., in the terms of lowest order in the density. The corresponding formulas for the spectrum will describe a gradual transition from $\epsilon = cp$ for $p \ll 1/R$ and for a value of c calculated from (19), to formula (12) for $\epsilon(p)$ for $p \gg 1/R$.

The obtained formulas (19)—(21) are similar to the results of Huang[2], but the numerical coefficients in the corresponding formulas differ greatly.

Thus, the presence of even weak attraction leads to the "sticking" of Bose particles, so that no stable homogeneous states exist at densities below critical. There exists also a metastable region of densities with negative pressure, which is stable against infinitesimally small density perturbations but is unstable relative to the occurrence of voids of finite radius. A phase transition should exist in our model at a finite temperature. The energy spectrum of the system consists of a straight-line (phonon) section for $k \ll 1/R$ with a sound velocity on the order of $(\beta f_0 N/m^2 V)^{1/2}$ and a gradual transition to another almost linear section for $1/R \ll k \ll 1/a$ with a larger slope, on the order of $(f_0 N/m^2 V)^{1/2}$, which goes over into the free-particle spectrum.

In conclusion we note that the relative ease with which the long-range and the short-range forces can be taken into account in the Bose-gas problem is apparently connected with the fact that the so-called plasma approximation in the Bose gas is taken into account already in a lower perturbation-theory order in Σ, since any closed ring made up of the zero Green's function causes the corresponding diagram to vanish.

The author is grateful to N. N. Bogolyubov and S. V. Tyablikov for a discussion and for interest in the work.

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Translated by J. G. Adashko
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