EQUATIONS OF MOTION FOR RADIATIVE OPERATORS

B. V. MEDVEDEV

Mathematics Institute of the Academy of Sciences, U.S.S.R.

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Equations of motion are established for the current-like operators which in their ensemble determine the scattering matrix and which have been introduced previously [1]. In the axiomatic construction of the scattering matrix, these equations (which involve functional derivatives) play the same role as the Hamiltonian equations in the Heisenberg picture.

1. INTRODUCTION

In our previous paper [1] 1) it has been shown that if one constructs the scattering matrix starting from the fundamental assumptions formulated by Bogolyubov, Polivanov, and the author [2] 2), the scattering matrix is completely determined by the (finite or infinite) sequence of operators $\Lambda_1, \ldots, \Lambda_\nu, \ldots$. Namely, the radiative operators $S^{(n)}$ are expressed in the form of a T-product with "counterterms" [I, (28)] in which the current-like operators $\Lambda_\nu(x_1, \ldots, x_\nu)$ have to satisfy the conditions of locality [I, (29)], hermiticity [I, (30)], symmetry [I, (31)] and spacelike (or local) commutativity [I, (32)], such that the first of these operators is identical to the current operator [I, (33)].

If these operators were to depend only on the coordinates which have been written out explicitly then, by virtue of Eqs. [I, (29)] and [I, (32)] they could be characterized as quasilocal operators in the sense of Bogolyubov and Shirkov [3]. However, all the operators j, $S^{(\nu)}$ and Λ_ν, besides the explicitly written coordinates $(x)_\nu$, also depend functionally on $\varphi(y)$. (We introduce an abbreviated notation: a subscript for any quantity in brackets denotes the repetition of l such quantities with the subscript increasing from left to right from 1 to l, e.g.:

$$(x)_l = x_1, \ldots, x_l, \qquad \psi(x)_l = \psi(x_1) \ldots \psi(x_l); \tag{1}$$

a supplementary arrow in the subscript will denote the reverse ordering, e.g.:

$$(\psi(x))_{\leftarrow n} = \psi(x_n) \ldots \psi(x_1).) \tag{2}$$

1)To be quoted in the following as I. We use below the notation of the paper I and the following papers, without explanation.

2)To be quoted in the following as PTDR.

The purpose of the present investigation is to clarify the character of this functional dependence.

2. THE EQUATIONS OF MOTION FOR THE OPERATORS Λ_ν

We consider the functional (variational) derivatives of the operators $S^{(n)}$ and Λ_ν. We start with the simplest operator $S^{(1)}$. The relation I, (22) yields for its functional derivative

$$\frac{\delta S^{(1)}(x)}{\delta\varphi(y)} = \begin{cases} [S^{(1)}(y),\ S^{(1)}(x)]_- & \text{for } y \gtrsim x \\ 0 & \text{for } y \lesssim x \end{cases}, \tag{3}$$

which can also be written as

$$\delta S^{(1)}(x) \,/\, \delta\varphi(y) = \vartheta(y-x)\,[S^{(1)}(y),\ S^{(1)}(x)]_-. \tag{4}$$

The second way of writing is somewhat symbolic, since the product of two generalized functions (the commutator and the ϑ-function) may introduce an ambiguity, which can be expressed by adding an operator of the form of a counterterm which does not vanish only for $x = y$.

It is essential that this will be a new counterterm for $S^{(1)}$, which is not included in the expression $S^{(1)}(x) = -i\Lambda_1(x)$ which gives this operator in terms of the current-like operators according to I, (28). However this term will not be quite arbitrary. Indeed, from the definition of the operators $S^{(n)}$ (cf. I) in terms of functional derivatives of the S-matrix, it is easy to derive, on the basis of unitarity,

$$\delta S^{(1)} \,/\, \delta\varphi(y) = S^{(2)}(x, y) - S^{(1)}(x)S^{(1)}(y), \tag{5}$$

and one can use for the operator $S^{(2)}$, in place of the causality requirement I, (21), the more comprehensive representation I, (28), which takes into account the counterterms. Then we obtain instead of (3)

$$\delta S^{(1)}(x)/\delta\varphi(y) = T[S^{(1)}(x)S^{(1)}(y)] - S^{(1)}(x)S^{(1)}(y) - \Lambda_2(x,y) = \vartheta(y-x)[S^{(1)}(y), S^{(1)}(x)] - \Lambda_2(x,y). \tag{4'}$$

We stress that in contradistinction to (4), the multiplication by the ϑ-function in (4′) does not lead to any new ambiguities—the second line should be understood only as an abbreviated notation of the first; as regards the indeterminacy in the T-product, from the very meaning of the derivation of the representation I,(28) itself, this indeterminacy in the symbol T(....) is in some way fixed and separated into the current-like operator Λ_2, which plays the role of a counterterm.

Since the operator Λ_1 differs from $S^{(1)}$ only by a factor, we now obtain for it the expression

$$\delta\Lambda_1(x)/\delta\varphi(y) = -iT[\Lambda_1(x)\Lambda_1(y)] + i\Lambda_1(x)\Lambda_1(y) + \Lambda_2(x,y) = -i\vartheta(y-x)[\Lambda_1(y), \Lambda_1(x)]_- + \Lambda_2(x,y), \tag{6}$$

which can be called an equation of motion not only because of the external analogy with the Heisenberg representation equations, but also because the functional dependence on the asymptotic fields serves here in fact as a description of the temporal evolution.

We go over now to the consideration of operators of higher order. Owing to the resultant combinatorial difficulties, this will be considerably more complicated. Therefore, for the sake of clarity of exposition we will recast the following discussions in the "mathematical" form of several consecutive lemmas.

Lemma 1. The radiative operators $S^{(n)}$ satisfy the equations of motion

$$\frac{\delta S^{(n)}((x)_n)}{\delta\varphi(y)} = S^{(n+1)}((x)_n, y) - S^{(n)}((x)_n)S^{(1)}(y). \tag{7}$$

The proof can be obtained directly from the definition of the radiative operators.

We will now be interested in the equations for the current-like operators Λ_n. We first introduce the definition:

$$S^{(n)}((x)_n) = \bar{S}^{(n)}((x)_n) - i\Lambda_n((x)_n), \tag{8}$$

i.e., we separate from $S^{(n)}$ the "last counterterm" $i\Lambda_n$ and denote by $\bar{S}^{(n)}$ all the rest. Then we can, by regarding Λ_n as the difference between $S^{(n)}$ and $\bar{S}^{(n)}$, derive from Lemma 1 the corollary:

$$\frac{\delta\Lambda_n((x)_n)}{\delta\varphi(y)} = -iT[\Lambda_n((x)_n)\Lambda_1(y)] + i\Lambda_n((x)_n)\Lambda_1(y) + \Lambda_{n+1}((x)_n, y) - i\left[\frac{\delta\bar{S}^{(n)}((x)_n)}{\delta\varphi(y)} + \bar{S}^{(n)}((x)_n)S^{(1)}(y) - \bar{S}^{(n+1)}((x)_n, y) - iT(\Lambda_n((x)_n)S^{(1)}(y))\right]; \tag{9}$$

we have separated in the first line those terms which are analogous to the right hand side of (6), leaving the remainder of the terms in the second line, which necessitated also a splitting of the operator $S^{(n+1)}$ by means of (8).

The idea of the following reasoning consists in proving that the second line in (9) vanishes. We will use as a starting point the circumstance that it is sufficient for this purpose to learn how to compute the functional derivatives of T-products of the operators Λ, each of which is of order not higher than $(n-1)$, since only such operators occur in $S^{(n)}$. This was exactly the meaning of the definition (8). We shall carry out the proof inductively.

Hypothesis 1. The current-like operators Λ satisfy equations of motion of the form of the first line in (9).

By means of this hypothesis, which is a direct generalization of the equation of motion (6) to $\nu > 1$, we prove the following lemma.

Lemma 2. If hypothesis 1 is true for $\nu \leq n-1$, the functional derivative of the T-product of operators $\Lambda_{\nu_i}(\xi_i)$, where no ξ_i joins more than $n-1$ points x, has the following expression:

$$\frac{\delta T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)]}{\delta\varphi(y)} + T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)]S^{(1)}(y) = \sum_{\alpha=1}^{m} T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_{\alpha-1}}(\xi_{\alpha-1})\Lambda_{\nu_\alpha+1}(\xi_\alpha, y)\ldots\Lambda_{\nu_m}(\xi_m)] + T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)S^{(1)}(y)] \tag{10}$$

(for the sake of brevity ξ_α denotes the ensemble of ν_α coinciding points $x_{i+1}, \ldots, x_{i+\nu_\alpha}$).

For the proof we note first that the T-product can be written by means of symmetrizing operators in the form

$$T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)] = P(\xi_1, \ldots, \xi_m)\vartheta(\xi_1-\xi_2)\ldots\vartheta(\xi_{m-1}-\xi_m)\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m), \tag{11}$$

where the symmetrizing operator takes care of the summation over simultaneous permutations of the ξ_α and the corresponding ν_α. Applying the hypothesis 1 to each Λ_{ν_i} we get

$$\frac{\delta T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)]}{\delta\varphi(y)} = P(\xi_1, \ldots, \xi_m)\vartheta(\xi_1-\xi_2)\ldots\vartheta(\xi_{m-1}-\xi_m)$$

$$\times\left\{\sum_{\alpha=1}^{m}\vartheta(y-\xi_\alpha)\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_{\alpha-1}}(\xi_{\alpha-1})S^{(1)}(y)\Lambda_{\nu_\alpha}(\xi_\alpha)\right.$$
$$\ldots\Lambda_{\nu_m}(\xi_m)-\sum_{\alpha=1}^{m}\vartheta(y-\xi_\alpha)\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_\alpha}(\xi_\alpha)S^{(1)}(y)$$
$$\left.\ldots\Lambda_{\nu_m}(\xi_m)+\sum_{\alpha=1}^{m}\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_{\alpha+1}}(\xi_\alpha,y)\ldots\Lambda_{\nu_m}(\xi_m)\right\}.$$

Here the last line is simply the T-product of the first line in the right side of (10) and, adding to the first two lines the product $T[\Lambda\ldots\Lambda]S^{(1)}(y)$ and collecting together terms with the same order in both sums, we obtain from these two lines the T-product

$$T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)S^{(1)}(y)].$$

This proves Lemma 2.

Lemma 2 has taught us how to differentiate T-products. Let us now compute the functional derivative of $\bar{S}^{(n)}$. We will show that the following lemma holds:

Lemma 3. If hypothesis 1 is true for all $\nu \le n-1$, then the functional derivative of the operator $\bar{S}^{(n)}$ has the expression

$$\delta\bar{S}^{(n)}((x)_n)/\delta\varphi(y)+S^{(n)}((x)_n)S^{(1)}(y)$$
$$=\bar{S}^{(n+1)}((x)_n,y)+iT[\Lambda_n((x)_n)S^{(1)}(y)]. \tag{12}$$

For the proof we recall first that (12) is by definition a sum of T-products of operators Λ_ν of order not higher than $n-1$; therefore we can apply Lemma 2 to all T-products in the sum, and the whole problem is exhausted by combinatorics.

Thus, according to (8) (cf. I,(28)) and Lemma 2,

$$\frac{\delta\bar{S}^{(n)}((x)_n)}{\delta\varphi(y)}+\bar{S}^{(n)}((x)_n)S^{(1)}(y)$$
$$=\sum_{\substack{2\leqslant m\leqslant n\\ \nu_1+\ldots+\nu_m=n}}\frac{(-i)^m}{m!}P(x_1,\ldots,x_{\alpha_1}|\ldots|\ldots,x_n)$$
$$\times\sum_{\alpha=1}^{m}T[\Lambda_{\nu_1}(x_1,\ldots,x_{\nu_1})\ldots\Lambda_{\nu_\alpha+1}$$
$$\times(x_{\nu_1+\ldots+\nu_{\alpha-1}+1},\ldots,x_{\nu_1+\ldots+\nu_\alpha},y)\ldots\Lambda_{\nu_m}(\ldots,x_n)]$$
$$+\sum_{\substack{2\leqslant m\leqslant n\\ \nu_1+\ldots+\nu_m=n}}\frac{(-i)^{m+1}}{m!}P(x_1,\ldots,x_{\nu_1}|\ldots|\ldots,x_n)$$
$$\times T[\Lambda_{\nu_1}(x_1,\ldots,x_{\nu_1})\ \ldots\Lambda_{\nu_m}(\ldots,x_n)\Lambda_1(y)]. \tag{13}$$

We shall now transform the right side of (12). Using a more detailed notation for the sums which intervene in the definition of the operator $\bar{S}^{(n+1)}$ we can write:

$$\bar{S}^{(n+1)}((x)_n,y)=\sum_{m=2}^{n+1}\frac{(-i)^m}{m!}$$
$$\times\sum_{\substack{\nu_1+\ldots+\nu_m=n+1\\ \nu_j\geqslant 1}}P(x_1,\ldots,x_{\nu_1}|\ldots|x_{\nu_1+\ldots+\nu_{m-1}+1},\ldots,x_n,y)$$
$$\times T[\Lambda_{\nu_1}(x_1,\ldots,x_{\nu_1})\ldots\Lambda_{\nu_m}(x_{\nu_1+\ldots+\nu_{m-1}+1},\ldots,x_n,y)]. \tag{14}$$

The aim of the planned transformations is to exhibit explicitly all permutations of the variable y, leaving only the summations with respect to the permutations of $x_1,\ldots,x_n$. Obviously, the summations over permutations in (14) will be effected if we first add the results of substituting y for one of the variables

$$x_\lambda,\quad \nu_1+\ldots+\nu_{\alpha-1}<\lambda\leqslant\nu_1+\ldots+\nu_\alpha$$

of each group, and then sum over all permutations of the variables x.

Thus,

$$P(x_1,\ldots,x_{\nu_1}|\ldots|x_{\nu_2+\ldots+\nu_{m-1}+1},\ldots,x_n,y)$$
$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_m}(\ldots,x_n,y)]=\sum_{\alpha=1}^{m}P(x_1,\ldots,x_{\nu_1}|$$
$$\ldots|x_{\nu_1+\ldots+\nu_{\alpha-1}+1},\ldots,x_{\nu_1+\ldots+\nu_\alpha-1}|x_{\nu_1+\ldots+\nu_\alpha},$$
$$\ldots,x_{\nu_1+\ldots+\nu_{\alpha+1}-1}|\ldots|x_{\nu_1+\ldots+\nu_{m-1}},\ldots,x_n)$$
$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_\alpha}(x_{\nu_1+\ldots+\nu_{\alpha-1}+1},\ldots,x_{\nu_1+\ldots+\nu_\alpha-1},y)$$
$$\ldots\Lambda_{\nu_m}(x_{\nu_1+\ldots+\nu_{m-1}},\ldots,x_n)] \tag{15}$$

Let us now carry out in (14) the summation over the numbers ν_j for fixed m. Depending on the value of ν_α, we encounter two cases: the case A, when $\nu_\alpha \ge 2$ and the case B, when $\nu_\alpha = 1$.

In the case A, making the substitution

$$\nu_\alpha=\nu_\alpha'+1,\qquad \nu_j=\nu_j'\quad\text{for } j\neq\alpha,$$

we arrive at

$$\sum_{\substack{\nu_1'+\ldots+\nu_m'=n\\ \nu_j\geqslant 1}}\sum_{\alpha=1}^{m}P(x_1,\ldots,x_{\nu_1'}|\ldots|x_{\nu_1'+\ldots+\nu'_{\alpha-1}+1},\ldots,x_{\nu_1'+\ldots+\nu_\alpha'}|$$
$$\ldots|x_{\nu_1'+\ldots+\nu'_{m-1}+1},\ldots,x_n)\ T[\Lambda_{\nu_1'}\ldots\Lambda_{\nu_\alpha'+1}$$
$$\times(x_{\nu_1'+\ldots+\nu'_{\alpha-1}+1},\ldots,x_{\nu_1'+\ldots+\nu_\alpha'},y)\ldots\Lambda_{\nu_m'}]$$

(in the following we omit the primes). In the case B the α-th group will in general not participate in the permutations; as a result both the number of groups and the sum over all ν decrease by one unit and we obtain the contribution

$$m\sum_{\substack{\nu_1+\ldots+\nu_{m-1}=n\\ \nu_j\geqslant 1}}P(x_1,\ldots,x_{\nu_1}|\ldots|x_{\nu_1+\ldots+\nu_{m-2}+1},\ldots,x_n)$$
$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_{m-1}}\Lambda_1(y)]$$

(the factor m is due to the fact that the case B is realized once for each of the $1 \le \alpha \le m$).

Thus,

$$\sum_{\substack{\nu_1+\ldots+\nu_m=n+1\\ \nu_j\geqslant 1}} P(x_1,\ldots,x_{\nu_1}|\ldots|\ldots,x_n,y)$$

$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_m}(\ldots,x_n,y)]$$

$$=\sum_{\substack{\nu_1+\ldots+\nu_{m-1}=n\\ \nu_j\geqslant 1}} P(x_1,\ldots,x_{\nu_1}|\ldots|x_{\nu_1+\ldots+\nu_{m-2}+1},\ldots,x_n)$$

$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_m}\Lambda_1(y)]$$

$$+\sum_{\substack{\nu_1+\ldots+\nu_m=n\\ \nu_j\geqslant 1}}\sum_{\alpha=1}^{m} P(x_1,\ldots,x_{\nu_1}|\ldots|\ldots,x_n)$$

$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_\alpha+1}(\xi_\alpha,y)\ldots\Lambda_{\nu_m}]. \quad (16)$$

Therefore, performing the remaining summation with respect to m, we obtain

$$\overline{S}^{(n+1)}(x_1,\ldots,x_n,y)$$

$$=\sum_{m=2}^{n+1}\frac{(-i)^m}{m!}m$$

$$\times\sum_{\substack{\nu_1+\ldots+\nu_{m-1}=n\\ \nu_j\geqslant 1}} P(x_1,\ldots,x_{\nu_1}|\ldots|x_{\nu_1+\ldots+\nu_{m-2}+1},\ldots,x_n)$$

$$\times T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_{m-1}}\Lambda_1(y)]$$

$$+\sum_{m=2}^{n+1}\frac{(-i)^m}{m!}\sum_{\substack{\nu_1+\ldots+\nu_m=n\\ \nu_j\geqslant 1}} P(x_1,\ldots,x_{\nu_1}|\ldots|\ldots,x_n)$$

$$\times\sum_{\alpha=1}^{m} T[\Lambda_{\nu_1}\ldots\Lambda_{\nu_\alpha+1}(\xi_\alpha,y)\ldots\Lambda_{\nu_m}].$$

We note now that in the second sum the summation goes in fact not up to $n+1$, but only to n, since for $m = n+1$ all $\nu_j = 1$ and case A is not realized. In the first sum it is convenient to make the substitution $m = m'+1$ and to separate the term with $m' = 1$ (we omit the primes in the following). We obtain finally

$$\overline{S}^{(n+1)}(x_1,\ldots,x_n,y)=\sum_{\substack{2\leqslant m\leqslant n\\ \nu_1+\cdots+\nu_m=n\\ \nu_j\geqslant 1}}\frac{(-i)^m}{m!}P(x_1,\ldots,x_{\nu_1}|$$

$$\ldots|x_{\nu_1+\cdots+\nu_{m-1}+1},\ldots,x_n)\,T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_m}(\xi_m)\Lambda_1(y)]$$

$$+\sum_{\substack{2\leqslant m\leqslant n\\ \nu_1+\cdots+\nu_m=n\\ \nu_j\geqslant 1}}\frac{(-i)^m}{m!}P(x_1,\ldots|\ldots|\ldots,x_n)$$

$$\times\sum_{\alpha=1}^{m} T[\Lambda_{\nu_1}(\xi_1)\ldots\Lambda_{\nu_\alpha+1}(\xi_\alpha,y)\ldots\Lambda_{\nu_m}(\xi_m)]$$

$$-T[\Lambda_n((x)_n)\Lambda_1(y)]. \quad (17)$$

Carrying over the last term to the left we see that the right side of (17) then coincides with the right side of (13). This concludes the proof of Lemma 3, since according to I,(33) $\Lambda_1(y) = iS^{(1)}(y)$.

We can now conclude the planned induction and prove the following theorem:

Theorem 1. The current-like operators Λ_ν satisfy (for any ν) the equations of motion assumed in hypothesis 1:

$$\delta\Lambda_\nu(x_1,\ldots,x_\nu)/\delta\varphi(y)=-iT[\Lambda_\nu(\xi_\nu)\,\Lambda_1(y)]$$

$$+i\Lambda_\nu(\xi_\nu)\Lambda_1(y)+\Lambda_{\nu+1}(\xi_\nu,y)$$

$$=-i\vartheta(y-\xi_\nu)[\Lambda_1(y),\ \Lambda_\nu(\xi_\nu)]_-+\Lambda_{\nu+1}(\xi_\nu,y). \quad (18)$$

For the proof we assume that Theorem 1 is true for any $\nu \leq n-1$; we then find that Lemma 3 will be true for $\nu = n$, and therefore the second line in (9) will vanish. Thus from the corollary of Lemma 1 follows the validity of Theorem 1 for $\nu = n$. Since for $\nu = 1$ the Theorem 1 is true [Eq. (6)], it will be true by induction for any ν.

3. SOLVABILITY CONDITIONS

Thus, we have established that the current-like operators $\Lambda_1,\ldots,\Lambda_\nu,\ldots$ which determine the scattering matrix via the expansion I,(28), must satisfy, beside the conditions I,(29)—(32), also the equations of motion (18). Therefore, in distinction from the apparently analogous situation when the scattering matrix is constructed in the form of a functional expansion in terms of the "intensity of switching-on the interaction" $g(x)$ [3], these operators cannot be given in an entirely arbitrary manner, taking into account only the conditions I,(29)—(32), since the current-like operators of different orders are coupled with each other. We can even take, as we shall do in the present section, a different point of view, namely that the equations of motions (18) are recurrence formulas which determine each operator $\Lambda_{\nu+1}$ in terms of the preceding one.

At first sight it might seem that the problem of arbitrariness in these operators is exhausted by these requirements: one could prescribe an arbitrary hermitian (I,(30)) and space-like-commutative (I,(32)) current $j(x)$ which could serve as the operator $\Lambda_1(x)$; then one could determine successively all other Λ by means of the equations of motion (18), and this, in turn, will determine via I,(28) all the coefficient-functions of the scattering matrix. However, in reality, as in the integration of the ordinary equations of motion, the situation is not so simple. Each of the operators Λ_ν must obey a set of conditions I,(29)—(32) "of its own," which, as is easy to see, do not follow automatically, if one accepts (18) as the definition of higher current-like operators in terms of the lower ones. Therefore the conditions I,(29)—(32) impose for each ν new restrictions, which must also be applied to the operator Λ_1, in order to follow the point of view developed here.

We start this program with the construction of the operator Λ_2 and with the derivation of the supplementary conditions which impose on the operator Λ_1 the requirement that Λ_2 satisfy the conditions I, (29)—(32). According to (18),

$$\Lambda_2(x, y) = \delta\Lambda_1(x) / \delta\varphi(y) + iT[\Lambda_1(x)\Lambda_1(y)] - i\Lambda_1(x)\Lambda_1(y). \tag{19}$$

It can be seen directly from here that in order that I, (29) be satisfied for $\nu = 2$, it is in any event necessary that the condition

$$\delta\Lambda_1(x) / \delta\varphi(y) = 0 \text{ for } y \lesssim x \tag{20}$$

hold, for otherwise Λ_2 would not vanish for such values of the arguments. Thus, it is not sufficient to impose on Λ only the Lehmann condition[4] of space-like commutativity; it is necessary that it satisfy the Bogolyubov causality condition[3] (as we shall see, it is exactly the Lehmann condition which need not be imposed as a special requirement).

Without touching for the moment upon the problem of the vanishing of $\Lambda_2(x, y)$ for $y > x$, let us see how to check that the symmetry condition I, (31) is satisfied. Writing (19) with x and y interchanged and subtracting the new equation from the original one, we should obtain zero if symmetry is satisfied. Thus, in order to satisfy I, (31) for $\nu = 2$ it is necessary that the following condition hold:

$$\delta\Lambda_1(x) / \delta\varphi(y) - \delta\Lambda_1(y) / \delta\varphi(x) - i[\Lambda_1(x), \Lambda_1(y)] = 0, \tag{21}$$

which will be called in what follows the solvability (integrability) condition. It is easy to see now, that it is sufficient that the causality condition (20) and the solvability condition (21) be satisfied for the operator Λ_1, in order that the operator Λ_2 satisfy conditions I, (29) and I, (31), and the operator Λ_1 satisfy the Lehmann condition of space-like commutativity (I, (32)).

Thus, in order that all conditions I, (29)—(32) be satisfied up to second order in ν, it is necessary and sufficient that the operator $\Lambda_1(x)$ satisfy, beside the condition I, (30), also the requirements of causality (20) and solvability (21)[3)].

Let us now consider the situation for higher orders, i.e., let us see, whether no new "solvability conditions" for Λ_1 appear because of the requirement of symmetry of higher operators Λ_ν, $\nu > 2$ (there is no reason to fear that the other conditions I, (29)—(31) are violated in higher orders). We note first that owing to the solvability condition the operator Λ_2 can be written clearly in the symmetric form:

$$\Lambda_2(x, y) = \frac{1}{2}\left(\frac{\delta\Lambda_1(x)}{\delta\varphi(y)} + \frac{\delta\Lambda_1(y)}{\delta\varphi(x)}\right) + iT[\Lambda_1(x)\Lambda_1(y)] - \frac{i}{2}[\Lambda_1(x), \Lambda_1(y)]_+. \tag{22}$$

The following rather cumbersome calculations consist in forming the difference of operators Λ_3 with permuted arguments, on the basis of (22) and (18). In order to prove that this difference vanishes one has to get rid consecutively of functional differentiations, making use of the solvability condition and of Lemma 2 for the functional derivatives of T-products. As a result only products of operators having all possible orders are left over, and such that each term appears twice, with opposite signs.

Thus, from the requirements imposed previously on Λ_1 it follows automatically that

$$\Lambda_3(x_1, x_2, y) = \Lambda_3(x_2, y, x_1), \tag{23}$$

i.e., in third order there appear no new solvability conditions. We will not generalize this proof to higher orders, which would involve only the use of cumbersome combinatorics, but would not yield anything new.

Remembering now that the operator Λ_1 is simply the current j, we can formulate the result we have obtained in the following manner: the scattering matrix is completely determined by prescribing the current operator $j(x)$, which is hermitian, and satisfies the conditions of causality (20) and solvability (21); if these conditions are satisfied (together with the condition of relativistic covariance, which can be formulated for the current in an obvious manner) the current operator can be chosen completely arbitrary, and to each choice of current operator there will correspond an appropriate scattering matrix, satisfying all the requirements of PTDR, Sec. 2.

One should not, however, overestimate the value of this result; it is not much simpler to find a current satisfying the conditions (20) and (21), than to find an S-matrix satisfying the PTDR requirements. These conditions are operator equations and in going over to matrix elements they become an infinite set of coupled equations, that has been studied by Polivanov and the author[5,6]. Our result indicates the equivalence of such a system with

[3)]Obviously, if we were not constructing the scattering matrix out of the operators Λ, but conversely, the operators Λ in terms of a known scattering matrix satisfying all the conditions in PTDR, Sec. 2, then the solvability condition (21) would be automatically satisfied (the causality condition (20) differs from the one used in PTDR only by a factor i).

those systems which are obtained directly from the S-matrix. Our result occupies here the same place as the well known theorem of perturbation theory [3] which states the equivalence of requirements imposed on the scattering matrix or on the Hamiltonian.

4. SPINOR RADIATIVE OPERATORS

If the theory under consideration involves not only a boson field $\varphi(x)$, but also fermion fields, say the fields $\psi(x)$ and $\bar{\psi}(x)$, then determination of the scattering matrix and of the radiative operators is subject to an additional complication due to the noncommutativity of the spinor field operators and of the functional derivatives with respect to these: one must take care everywhere to maintain the order of factors and to take into account the sign changes introduced by changes in this order.

It will now be convenient to write the expansion of the S-matrix in terms of normal products (I, (10)):

$$S = \sum_{\lambda,\mu,\nu}^{\infty} \frac{(-i)^{\lambda+\mu+\nu}}{\lambda!\,\mu!\,\nu!} \int (dx)_\lambda (dy)_\mu (dz)_\nu \, \Phi^{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu)$$
$$\times :(\psi(z))_{\leftarrow\nu}(\varphi(y))_{\leftarrow\mu}(\bar{\psi}(x))_{\leftarrow\lambda}:, \tag{24}$$

and to define the radiative operator of order $l+m+n$ by the equality:

$$S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n) = \frac{\delta^{l+m+n}S}{(\delta\bar{\psi}(x))_l(\delta\varphi(y))_m(\delta\psi(z))_n} S^{+}. \tag{25}$$

It can be seen from (25) that $S^{(\cdots)}(\ldots)$ is symmetric with respect to the arguments of the middle group, but is antisymmetric with respect to the arguments of each of the extreme groups. A permutation of arguments between the groups is obviously meaningless. The same symmetry properties are of course assumed also for the functions $\Phi^{\lambda\mu\nu}$ in (24). The choice of opposite ordering of the arguments of (24) and (25) will save us a large number of sign changes.

Thus the contents of Lemma 1 in I remains unchanged:

$$\Phi^{lmn}((x)_l;\ (y)_m;\ (z)_n)$$
$$= i^{l+m+n}\langle 0|S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)|0\rangle. \tag{26}$$

In order to formulate the causality requirement, we have to impose now not only one, but a whole series of conditions:

$$\frac{\delta S^{(100)}(x';\ -;\ -)}{\delta\bar{\psi}(x)} = 0 \ \text{ for } \ x \lesssim x';$$

$$\frac{\delta S^{(100)}(x';\ -;\ -)}{\delta\varphi(y)} = 0 \ \text{ for } \ y \lesssim x';\ \ldots;$$

$$\frac{\delta S^{(001)}(-;\ -;\ z')}{\delta\psi(z)} = 0 \ \text{ for } \ z \lesssim z'. \tag{27}$$

Lemmas 5 and 6 of I, which follow from these, are also unchanged, and contain only functional derivatives with respect to arbitrary fields.

A less trivial result will be obtained in carrying out the functional differentiations in the left side of Lemma 5 in I: there will appear some additional sign changes which are maintained in the analogues of the equations of motion (7):

$$\frac{\delta S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)}{\delta\bar{\psi}(\xi)} = (-1)^l S^{(l+1\,m\,n)}((x)_l,\ \xi;\ (y)_m;\ (z)_n)$$
$$-(-1)^{l+n}S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)\,S^{(100)}(\xi;\ -;\ -); \tag{28}$$

the second of these equations remains unchanged:

$$\frac{\delta S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)}{\delta\varphi(\eta)} = S^{(l\,m+1\,n)}((x)_l;\ (y)_m,\ \eta;\ (z)_n)$$
$$-S^{(lmn)}((x_l);\ (y)_m;\ (z)_n)\,S^{(010)}(-;\ \eta;\ -), \tag{29}$$

and the third one takes on the form

$$\frac{\delta S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)}{\delta\psi(\zeta)}$$
$$= (-1)^{l+n}S^{(l\,m\,n+1)}((x)_l;\ (y)_m;\ (z)_n,\ \zeta)$$
$$-(-1)^{l+n}S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n)\,S^{(001)}(-;\ -;\ \zeta). \tag{30}$$

We recall that according to Eq. (25) and the definitions of currents adopted in PTDR[4)],

$$S^{(100)}(\xi;\ -;\ -) = i\iota(\xi), \qquad S^{(010)}(-;\ \eta;\ -) = -ij(\eta),$$
$$S^{(001)}(-;\ -;\ \zeta) = -i\iota(\zeta). \tag{31}$$

The equations of motion (28)–(30) allow one to construct a proof for a lemma of the type of Lemma 7 in I. We will not do this however, since it is clear that the differences from the pure boson case can appear only through signs, which can be reestablished if one takes care which derivatives have been permuted when going from the left to the right side. Therefore the integral causality conditions will read

$$S^{(lmn)}((x)_l;\ (y)_m;\ (z)_n) = (-1)^{(l-\lambda)\nu}S^{(\lambda\mu\nu)}((x)_\lambda;\ (y)_\mu;\ (z)_\nu)$$
$$\times S^{(l-\lambda\ m-\mu\ n-\nu)}(x_{\lambda+1},\ \ldots,\ x_l;\ (y)_{m-\mu};\ (z)_{n-\nu}),$$

if

$$\{(x)_\lambda;\ (y)_\mu;\ (z)_\nu\} \gtrsim \{(x)_{l-\lambda},\ (y)_{m-\mu},\ (z)_{n-\nu}\}. \tag{32}$$

In order to derive representations of the type of I, (28) for the scattering matrix in terms of current-like operators Λ, one also needs the uni-

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[4)]For typographical convenience we will represent here the Georgian letter "in" used in PTDR by the Greek "iota."

tarity condition in a form similar to I, (27). In order to formulate this requirement we must introduce first conjugation rules for radiative operators $S^{(lmn)}$. Naturally, it is not convenient to use for this ordinary hermitean conjugation, since it is necessary that under such a conjugation the form of the condition I, (27) be conserved. It turns out that such a generalized conjugation, which we will call in the following "taking the adjoint," is conveniently defined by the equation

$$S^{\neq(lmn)}((x)_l;\ (y)_m;\ (z)_n) = (-1)^{n+l}[\gamma^0]_l[S^{(nml)}((z)_{\leftarrow n};\ (y)_{\leftarrow m};(x)_{\leftarrow l}[^+[\gamma^0]_n. \quad (33)$$

It is easy to see that this definition is chosen so as to compensate for the non-hermitian character of the operators $\psi(x)$ and $\bar{\psi}(x)$. The only change in the unitarity condition will then be the appearance of sign-factors $(-1)^{(l-\lambda)\nu}$ of the same type as in (32). It is clear that one must impose on the current-like operators, in place of the hermiticity requirement which held in the boson case, a condition of self adjointness; then adding to $S^{(lmn)}$, with known lower order $S^{(\cdots)}$, of a term $-i\Lambda_{lmn}(\ldots)$ will not destroy unitarity in the order lmn.

Thus we introduce the sequence of operators

$$\Lambda_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu);\quad \lambda,\ \mu,\ \nu = 0,\ 1,\ldots, \qquad (34)$$

with the conditions of:

a) locality

$$\Lambda_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu) = 0, \qquad (35a)$$

excluding the case $x_1 = \cdots = x_\lambda = y_1 = \cdots = z_\nu$;

b) self-adjointness:

$$\Lambda^{\neq}{}_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu) = \Lambda_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu); \qquad (35b)$$

c) symmetry:

$$\Lambda_{\lambda\mu\nu}(x_{\alpha_1};\ldots,\ x_{\alpha_\lambda};\ y_{\beta_1},\ldots,\ y_{\beta_\mu};\ z_{\gamma_1},\ldots,\ z_{\gamma_\nu}) = (\pm 1)\Lambda_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu), \qquad (35c)$$

according to whether the permutations $(\alpha_1,\ldots,\alpha_\lambda)$ and $(\gamma_1,\ldots,\gamma_\nu)$ are even or odd;

d) space-like commutativity (local commutativity):

$$[\Lambda_{\lambda\mu\nu}((x)_\lambda;\ (y)_\mu;\ (z)_\nu),\ \Lambda_{\lambda'\mu'\nu'}((x')_{\lambda'};\ (y')_{\mu'};(z')_{\nu'})]_{\mp} = 0, \quad (35d)$$

for

$$\{x_1,\ldots,\ z_\nu\} \sim \{x_1',\ldots,\ z_{\nu'}'\},$$

with

$$\Lambda_{100}(x) = -\iota(x),\qquad \Lambda_{010}(y) = j(y),$$
$$\Lambda_{001}(\zeta) = \bar{\iota}(\zeta). \qquad (36)$$

Then, on the basis of the same arguments as in the Bose-case, the following representation holds for the radiative operators $S^{(lmn)}$

$$\begin{aligned}S^{(lmn)}((x)_l;(y)_m;(z)_n) &= (-i)^{l+m+n}T\\ &\times[\Lambda_{100}(x_1)\ldots\Lambda_{100}(x_l)\Lambda_{010}((y)_m)\Lambda_{001}((z)_n)]\\ &+\sum_N\frac{(-i)^N}{N!}P(x_1,\ldots,x_{\lambda_1}|\ldots|\ldots,x_l)\\ &\times P(y_1,\ldots,y_{\mu_1}|\ldots|\ldots,y_m)\\ &\times P(z_1,\ldots,z_{\nu_1}|\ldots|\ldots,z_n)(-1)^KT\\ &\times[\Lambda_{\lambda_1\mu_1\nu_1}((x)_{\lambda_1};(y)_{\mu_1},(z)_{\nu_1}) \quad -i\Lambda_{lmn}((x)_l;(y)_m;(z)_n).\end{aligned} \qquad (37)$$

Here

$$\begin{aligned}K = \nu_1(\lambda_2+\ldots+\lambda_N)+\nu_2(\lambda_3+\ldots+\lambda_N)\\ +\ldots+\nu_{N-2}(\lambda_{N-1}+\lambda_N)+\nu_{N-1}\lambda_N,\end{aligned} \qquad (38)$$

and the sum N is calculated over all the ways of breaking up the three natural numbers (l, m, n) into a set of N groups of three natural numbers $(\lambda_j, \mu_j, \nu_j)$, including zero, $j = l,\ldots, N$, such that

$$\sum_{j=1}^N\lambda_j = l,\qquad \sum_{j=1}^N\mu_j = m,\qquad \sum_{j=1}^N\nu_j = n,$$
$$\lambda_j+\mu_j+\nu_j > 0,\qquad 2\leqslant N\leqslant l+m+n-1. \qquad (39)$$

The following reasoning is carried out exactly as in the boson case: one can define "incomplete radiative operators" $\bar{S}^{(lmn)}$ and prove for these, starting from (28)—(30) equations of the type of the corollary to Lemma 1, one can formulate hypothesis 1 and then consecutively prove Lemmas 2 and 3. All these points do not suffer any modifications of principle, the only difference being the sign-factors which we already know, and more complicated combinatorics. Therefore we will not repeat here all this reasoning and will formulate the result directly in the form of the following theorem:

Theorem 2. In the case where spinor fields are present, the current-like operators Λ_{lmn} (for any l, m, n) satisfy the equations of motion:

$$\begin{aligned}\frac{\delta\Lambda_{lmn}((x)_l;\ (y)_m;\ (z)_n)}{\delta\bar{\psi}(\xi)} &= (-1)^{l+n}\{T[\Lambda_{lmn}(\ldots)S^{(100)}(\xi;\ -;\ -)]\\ &-\Lambda_{lmn}(\ldots)S^{(100)}(\xi;\ -;-)\}+(-1)^l\Lambda_{l+1\,m\,n}((x)_l,\\ &\xi;\ (y)_m;\ (z)_n),\end{aligned} \qquad (40a)$$

$$\begin{aligned}\frac{\delta\Lambda_{lmn}((x)_l;\ (y)_m;\ (z)_n)}{\delta\varphi(\eta)} &= T[\Lambda_{lmn}(\ldots)S^{(010)}(-;\ \eta;\ -)]\\ &-\Lambda_{lmn}(\ldots)S^{(010)}(-;\ \eta;\ -)+\Lambda_{l\,m+1\,n}((x)_l;\ (y)_m,\ \eta;\ (z)_n),\end{aligned} \qquad (40b)$$

$$\frac{\delta\Lambda_{lmn}((x)_l;\ (y)_m;\ (z)_n)}{\delta\psi(\zeta)} = (-1)^{l+n}\{T[\Lambda_{lmn}(\ldots)S^{(001)}(-;\ -;\ \zeta)] - \Lambda_{lmn}(\ldots)S^{(001)}(-;\ -;\ \zeta) + \Lambda_{l\,m\,n+1}((x)_l;\ (y)_m;\ (z)_n,\ \zeta)\}. \quad (40c)$$

In particular, the most important equations, those for the three currents, are:

$$\frac{\delta\iota_\alpha(x)}{\delta\psi_\lambda(\zeta)} = -i\vartheta(\zeta - x)\{\iota_\alpha(x),\ \bar{\iota}_\lambda(\zeta)\}_+ + \Lambda_{101/\alpha\gamma}(x;\ -;\ \zeta), \quad (41a)$$

$$\frac{\delta j(y)}{\delta\varphi(\eta)} = i\vartheta(\eta - y)[j(y),\ j(\eta)]_- + \Lambda_{020}(-;\ y,\ \eta;\ -), \quad (41b)$$

$$\frac{\delta\bar{\iota}_\gamma(z)}{\delta\bar{\psi}_\lambda(\xi)} = i\vartheta(\xi - z)\{\iota_\lambda(\xi),\ \bar{\iota}_\gamma(z)\}_+ + \Lambda_{101/\lambda\beta}(\xi;\ -;\ z), \quad (41c)$$

where the spinor indices have been explicitly written.

It is not difficult to generalize the results of Sec. 3, also. The only difference will consist in the fact, that in order to fix the theory it will be necessary to specify a current for each field. The different currents can be specified independently, as long as the "crossed" solvability conditions are satisfied.

The main result of the present paper are the equations of motion (18) [or (40)]. Since in the approach which had been chosen in PTDR for the construction of the theory the time evolution of the system can be expressed only in terms of functional differentiations (we have no other way of distinguishing different points in space time), these equations are called upon to play the role of the Heisenberg equations of motion and therefore possess a fundamental significance.

One can look for further applications of these equations in different ways. If one takes the point of view developed in Sec. 3, these equations can serve for an ordering of the introduction of counterterms in the system of previously considered equations [5,6], which is essential, since the constants which intervene in the individual equations of this system are not independent. In some problems, however, it is more convenient not to eliminate the higher-order operators, but rather to operate with the whole ensemble, taking specifically into account the fact that, as it seems, in a renormalizable theory their number has to be finite, on the basis of considerations of growth of the matrix elements. This treatment seems indicated in cases when one distinguishes between matrix elements of current-like operators on and off the energy shell, since it allows to characterize very conveniently the arbitrariness involved in extensions off the energy shell.

The most radical approach would be, of course, an attempt to free these equations of nonlinear terms by means of a canonical transformation carried out by means of an operator of the type of a "halved" S-matrix $S(\infty, \sigma(x))$. Indeed, neglecting complications due to indeterminacies with coinciding arguments (divergences), this procedure permits to solve the transformed equations of motion and to "prove" the theorem of equivalence of the theory based on the axioms of PTDR, with the usual theory, which starts from a Lagrangian (cf. [7], where this result is obtained in a similar approximation from the axioms of [4]). A more detailed consideration, which will be carried out elsewhere, shows however, that taking into account the counterterms such a program seems impossible to carry through, which is connected with the difficulties in the definition of the "halved" scattering matrix (cf. [8]), and there remains only the possibility to work with the Eqs. (18) in their original representation.

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