SOME PROPERTIES OF CASCADE DECAY

G. I. KOPYLOV

Joint Institute for Nuclear Research

Submitted to JETP editor July 24, 1963

J. Exptl. Theoret. (U.S.S.R.) **46**, 2063-2069 (June, 1964)

We consider the case when the resonant state decays in accordance with the scheme $A \to 1 + a \to 1 + 2 + 3$, with particle 3 being undetectable. It is shown that it is sufficient to know the effective-mass spectrum of the system $1 + 2$ in order to establish the presence or absence of the resonance A.

A method widely used at present to observe resonances in systems of fundamental particles is to detect the effective-mass spectrum peak of these systems. It is usually assumed that any peak in this spectrum in excess of the statistical background is proof of resonance, although this assumption is nowhere proved, nor are the reasons for the appearance of a peak alongside with the resonance clear. These causes can be both dynamic in origin (attempts are made to attribute to them the known ABC resonance) and kinematic in nature. A study of the causes of peaks would make it possible to identify the resonances with greater assurance.

In this paper we shall stop to discuss one clearly kinematic cause of peaks in the effective mass distribution. We shall show (Section 1) that if a system of three particles 1, 2, 3 with definite effective mass decays in cascade fashion, i.e., in accordance with the scheme

$$A \to 1 + \underset{2+3}{\overset{}{a}}\!\!\downarrow,$$

then under certain conditions a "false resonance" peak can appear in the distribution with respect to the effective mass m_{12} of the particles 1 and 2, which do not resonate with each other. Therefore the mere experimental observation of the peak in the m_{12} spectrum is insufficient to prove the presence of resonance in the system $1 + 2$: it is necessary to check at least on the absence of resonance in the system $2 + 3$.

Conversely, if it is known that multiple production is accompanied by the production of some particle a which subsequently decays into the particles 2 and 3, then the distribution with respect to the effective mass of the triad of particles 1, 2, and 3 can be obtained without knowing the momenta of all three particles. In Sec. 2 are given formulas which make it possible first, to obtain the effective mass spectrum of the triad $1 + 2 + 3$ from the distribution by the effective mass of only particles 1 and 2 (the resonance in the system $1 + 2 + 3$ is determined by the peak on this distribution); second, to determine the masses of the resonances A and a, in the presence of the distribution in m_{12} of a clearly pronounced peak or a plateau, and third, to obtain the spectrum of the effective masses of the system $1 + 3$. In Sec. 3 the results of Secs. 1 and 2 are generalized to the case when a search for resonance is being made in a system of more than three particles one of which is undetectable.

The derived formulas make it possible to extend the experimental capabilities of resonance detection. It turns out that there is no need to identify all the particles into which the resonance decays, provided that the cascade character of the decay is known. It thus becomes possible to search for resonance in systems of the type $\Lambda^0\pi^0 \to \Lambda^0\gamma\gamma$ (only one photon is detected), $\pi^{\pm}\omega^0 \to \pi^{\pm}\pi^+\pi^-\pi^0$, etc. The considerations advanced here were used in [1] to study the properties of the system $\Lambda^0\eta^0 \to \Lambda^0\gamma\gamma$, and in [2] they were used for the system

$$\rho^0\omega^0 \to \pi^+\pi^-\pi^0$$
$$\downarrow$$
$$\pi^+\pi^-$$

1. We shall show that when a certain system of particles decays into a particle 1 and a resonance a, which in turn decays into particles 2 and 3, the effective mass of particles 1 and 2 may have a narrow range of values, creating, if particle 3 is undetectable, the false impression that a resonance interaction exists between particles 1 and 2. To prove this we shall use the properties of the Dalitz diagram shown in Fig. 1 in a somewhat generalized form.

We consider a system in which three spinless particles 1, 2, and 3 have masses m_1, m_2, and m_3, energies ω_1, ω_2, and ω_3, and effective pair masses m_{12}, m_{23}, and m_{31}. The effective mass of the triad is M. The phase space of the states of these systems is represented by a Dalitz diagram with invariants $\Delta_i = m_{jk}^2 - (m_j + m_k)^2$ $(i \neq j \neq k)$, and with the property $\sum_i \Delta_i = \text{const}$ (Fig. 1; see, for example, [3]). The state of the three-particle system can be specified also in terms of non-invariant variables, namely the kinetic energies $t_i = \omega_i - m_i$ of the particles 1, 2, 3 in their c.m.s. (see, for example, [4]) or the kinetic energies $\bar{t}_j$, $\bar{t}_k$, and $\bar{T}_A$ of particles j and k and of the entire triad A in the rest system of the remaining particle i (provided only $m_i \neq 0$). In the former case the phase space is represented by the region Φ inside the equilateral triangle, while in the second case it is represented by regions Φ_i outside the triangle. In either case the distances of the points of the region from the sides of the triangle are equal to the kinetic energies, and the energy conservation law is automatically satisfied

$$\sum_i t_i = T \quad \text{or} \quad \bar{t}_j + \bar{t}_k - \bar{T}_A = T \equiv M - \sum_i m_i.$$

The kinetic energies t_i and $\bar{t}_i$, $\bar{T}_A$ are linearly related[1)]:

$$\bar{T}_A = \frac{M}{m_i} t_i, \quad \bar{t}_j = \frac{M}{m_i}(t_{k_{max}} - t_k), \quad \bar{t}_k = \frac{M}{m_i}(t_{j_{max}} - t_j).$$

The form of the regions Φ and Φ_i is therefore identical, and an area element on any diagram is proportional to the phase volume element

$$S_3 = \int \prod_1^3 \frac{d\mathbf{p}_i}{2\omega_i} \delta^4 \left(\sum_1^3 p_i - P \right)$$
$$= \pi^2 \iint_{\Phi} dt_i dt_j = \left(\frac{\pi m_i}{M} \right)^2 \iint_{\Phi_i} d\bar{t}_j d\bar{t}_k$$
$$= \left(\frac{\pi m_i}{M} \right)^2 \iint_{\Phi_i} d\bar{T}_A d\bar{t}_k = \left(\frac{\pi}{2M} \right)^2 \iint_{\Phi_{\text{inv}}} d\Delta_i d\Delta_j. \tag{1}$$

Let us examine in greater detail the diagram in terms of the invariance Δ_i. We write out in the integral over the region Φ_{inv} the integration limits (see, for example, [5]):

$$S_3 = \left(\frac{\pi}{2M} \right)^2 \int_{(m_2+m_3)^2}^{(M-m_1)^2} ds_{23} \int_{F^-(s_{23}, M)}^{F^+(s_{23}, M)} ds_{12} f(s_{12}, s_{23}, M), \tag{2}$$

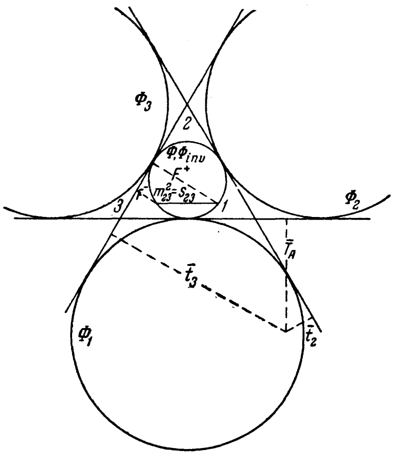

FIG. 1

where f is the density of the point on the diagram, and

$$s_{ij} = m_{ij}^2,$$
$$\omega_1^* = \frac{M^2 - s_{23} - m_1^2}{2\sqrt{s_{23}}}, \tag{3}$$
$$\omega_3^* = \frac{s_{23} + m_3^2 - m_2^2}{2\sqrt{s_{23}}},$$
$$p_i^{2*} = \omega_i^{*2} - m_i^2,$$
$$F^{\pm}(s_{23}, M) = M^2 + m_2^2 - s_{23} - 2\omega_1^*\omega_3^* \pm 2p_1^*p_3^*. \tag{4}$$

Let now the particles 2 and 3 result from the decay of a bound state with mass m_{23}.

The phase space of the states with m_{23} = const is the segment AB. The points along the segment correspond to different values of $m_{12}^2 = s_{12}$ or to different cosines of the angle of emission of particle 2 in the rest system of the pair 2 + 3 [6]. It follows therefore that if the probabilities of the states are described by the covariant model of multiple production, or if the state with given value of m_{23} decays isotropically (f = const), then in accordance with (2) the distribution in m_{12}^2 is rectangular with height $(\pi/2m)^2$ and limits $F^{\pm}(s_{23}, M)$, while the distribution in m_{12} has a trapezoidal form with a maximum at $F^+(s_{23}, M)$. When the chord AB is sufficiently close to the boundary of the Dalitz "ellipse," for example when the decay energy of the system 2 + 3 is small, the distribution in m_{12} can have a narrow peak at $s_{12} \approx F^{\pm}(s_{23}, M)$. At high decay energies the peak shifts to $F^+(s_{23}, M)$. In both cases this peak can imitate a resonance in the system 1 + 2, although in fact it is particles 2 and 3 which resonate. This is still another cause of false resonances, in addition to those analyzed in [7].

2. Assume that it is known that the system A, the mass of which is not fixed, decays into parti-

---

1)It is easy to prove that in no other reference frame is a linear connection possible.

cles 1, 2, and 3 via an intermediate state $(2 + 3)$ with mass m_{23}, and that the decay $a \to 2 + 3$ is isotropic. Assume also that the distribution density $w(s_{12})$ with respect to the effective mass m_{12} has been measured. We shall show that: 1) the mass m_{23} of particle a can be determined from the position of the peak in $w(s_{12})$, and 2) the distribution with respect to the effective mass of the system A = (123) and the system (13) can be determined from $w(s_{12})$. This theorem can serve as a basis for searches of resonances in the system (123) or (13) if the efficiency for registration of particle 3 is insufficiently high.

To prove this, let us consider the variation of the limit of $F^{\pm}(s_{23}, M)$ as M increases from the smallest value $m_{23} + m_1$. When $M = m_{23} + m_1$, we have

$$F^+ = F^- = \bar{s}_{12} = m_{23}m_1 + m_1^2 + m_2^2 + \frac{m_1}{m_{23}}(m_2^2 - m_3^2). \quad (5)$$

Further, by investigating the functions $dF^{\pm}/dM^2$, we can readily find that $dF^+/dM^2 > 0$ for all M, while $dF^-/dM^2 \lesseqgtr 0$ for

$$M^2 \lesseqgtr M_{cr}^2 \equiv m_1 m_2 + m_{23}^2 + \frac{m_1}{m_2}(m_{23}^2 - m_3^2 + m_1 m_2),$$

and at $M = M_{cr}$ the function F^- reaches the smallest possible limit $(m_1 + m_2)^2$. When $m \neq 0$ the derivative $dF^2/dM^2 < 0$, for all M, i.e., $M_{cr} = \infty$ (Fig. 2).

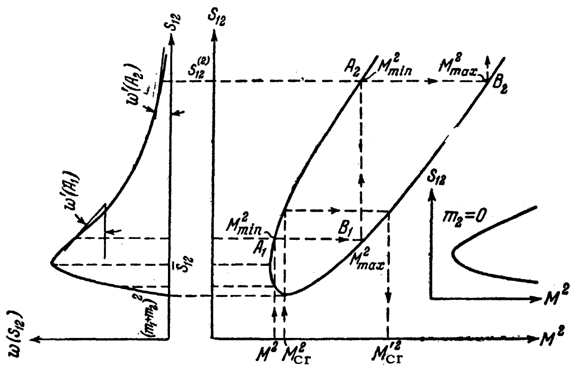

FIG. 2

Thus, as M increases from $m_{23} + m_1$, any value of s_{12} larger than $\bar{s}_{12}$ turns out to be at first, for some $M = M_{min}$, the right end point of the spectrum of m_{12}^2, while for another value $M = M_{max}$ it turns out to be the left end point. In exactly the same way, if $m_2 \neq 0$, each $s_{12} < \bar{s}_{12}$ belongs to the rectangular m_{12}^2 spectrum first for some $M = M_{min}$, and finally for $M = M_{max}$. In order to find these values of M, it is simpler, instead of going through the cumbersome direct solution of (2), to find the extremum of the invariant $(p_1 + p_2 + p_3)^2$ at $(p_2 + p_3)^2 = m_{23}^2$:

$$M_{\substack{max\\min}}^2 = m_{23}^2 + m_{12}^2 - m_2^2 + 2(\omega_1\omega_3 \pm p_1 p_3) \qquad (m_2 \neq 0);$$

$$\omega_1 = \frac{m_{12}^2 - m_1^2 - m_2^2}{2m_2}, \qquad \omega_3 = \frac{m_{23}^2 - m_2^2 - m_3^2}{2m_2}, \quad (6)$$

$$p_i^2 = \omega_i^2 - m_i^2. \quad (7)$$

When $m_2 = 0$ we have

$$M_{max}^2 = 0,$$

$$M_{min}^2 = (m_{12}^2 m_{23}^2 - m_1^2 m_3^2)\left(\frac{1}{m_{12}^2 - m_1^2} + \frac{1}{m_{23}^2 - m_3^2}\right). \quad (8)$$

We now assume that the spectrum of the squares of the effective masses of the system $(1 + 2 + 3)$ has a density $\Phi(M^2)$ between the limits $(m_{23} + m_1)^2$ and infinity. We multiply (2) by $\Phi(M^2)$ and integrate with respect to M^2. Rearranging the order of integration, we obtain a formula for the distribution density in m_{12}^2 for m_{23}^2 under the conditions

$$\frac{d^2N}{ds_{12}ds_{23}} \equiv w(s_{12}) = \int_{M^2_{min}}^{M^2_{max}} dM^2 \frac{\pi^2}{4M^2}\Phi(M^2). \quad (9)$$

It follows from (9) that if the spectrum of $\Phi(M^2)$ does not contain values $M^2 > M_{cr}^2$, then no matter what the form of this spectrum, the distribution in m_{12}^2 has a maximum—peak or plateau—at $s_{12} = \bar{s}_{12}$. From the position of the peak we can obtain, using (5), the value of m_{23}—mass of the resonant state $a \to 2 + 3$. If a plateau is observed, i.e., the spectrum $\Phi(M^2)$ does not contain a mass smaller than some value M_0, then from the abscissas of the ends of the plateau we can simultaneously determine m_{23} and M_0. Let ω_1^a and p_1^a be the energy and momentum [formula (7)] corresponding to the left end of the plateau, and let ω_1^b and p_1^b correspond to the right end.

If we assume that $M_0 < M_{cr}$, then

$$m_{23} = U^{-1/2}[m_2(\omega_1^b + \omega_1^a) + R], \quad (10)$$

$$M_0^2 = m_1^2 + m_2^2 + m_3^2 + U^{-1}\{2m_2^2(p_1^a + p_1^b)^2 + m_2(\omega_1^a + \omega_1^b)U + [2m_2(\omega_1^b + \omega_1^a) + U]R\}, \quad (11)$$

$$U = 2(\omega_1^a\omega_1^b - p_1^b p_1^a + m_1^2),$$

$$R^2 = m_2^2(p_1^b + p_1^a)^2 + m_3^2 U. \quad (12)$$

If we assume that $M_0 > M_{cr}$, then

$$m_{23} = U^{-1/2}[m_2(p_1^b + p_1^a) + R], \quad (10')$$

$$M_0^2 = m_1^2 + m_2^2 + m_3^2 + U^{-1}\{2m_2^2(\omega_1^b - \omega_1^a)^2 + m_2(\omega_1^b + \omega_1^a)U + 2[p_1^a(\omega_1^b + m_2) + p_1^b(\omega_1^a + m_2)]R\}, \quad (11')$$

$$U = 2(\omega_1^b \omega_1^a + p_1^b p_1^a - m_1^2),$$
$$R_2 = m_2^2(\omega_1^b - \omega_1^a)^2 + m_3^2 U. \qquad (12')$$

To choose the true solution among these one must involve additional considerations concerning the expected values of m_{23} and M_0.

When $m_2 = 0$ formulas (10)—(12) cannot be used. We have in their place

$$m_{23} = {}^1\!/_2 \sqrt{q}\,(1 + \sqrt{1 + 4m_3^2/q}), \qquad (10'')$$

$$M_0^2 = m_3^2 + m_1^2 + \frac{s_{12}^{(a)} s_{12}^{(b)} - m_1^4}{2m_1^2}\left(1 + \sqrt{1 + \frac{4m_3^2}{q}}\right), \qquad (11'')$$

$$q = (s_{12}^{(a)} - m_1^2)(s_{12}^{(b)} - m_1^2)\, m_1^{-2}, \qquad (12'')$$

where $s_{12}^{(a)}$, $s_{12}^{(b)}$—abscissas of the ends of the plateau on the $w(s_{12})$ curve. In particular, when $m_2 = m_3 = 0$ we have

$$m_{23} = q^{1/2}, \qquad M_0^2 = s_{12}^{(a)} s_{12}^{(b)} m_1^{-2}. \qquad (13)$$

If it is known that contributions to the $w(s_{12})$ spectrum are made only by values $M < M_{cr}$, or only $M' > M > M_{cr}$ (Fig. 2), then by drawing several horizontal sections through the $w(s_{12})$ spectrum and calculating m_{23} from the corresponding formula we can obtain a more accurate value of m_{23} (use will be made not only of the region of the peak but of the entire statistics of the measurements of m_{12}). On the other hand, if it is known that the $w(s_{12})$ curve is strictly rectangular, then it is possible to obtain the masses of the particles A and a with the aid of formulas (10)—(12) (see Sec. 1).

Once m_{23} is determined, Eq. (9) can be solved with respect to $\Phi(M^2)$. If $m_2 = 0$, then, by differentiating (9), we obtain

$$\frac{\pi^2}{4M^2}\Phi(M^2) = -dw(s_{12}(M^2))/dM^2. \qquad (14)$$

The choice of either $F^+(s_{23}, M)$ or $F^-(s_{23}, M)$ for the function $s_{12}(M^2)$ leads to the same function $\Phi(M^2)$. In practice it is more convenient to use the right slope of the peak

$$\frac{\pi^2}{4M^2}\Phi(M^2) = -\frac{dw(F^+(M^2))}{dF^+(M^2)}\,\frac{(m_{23} - \omega_3^*)p_1^* + \omega_1^* p_3^*}{m_{23} p_1^*}. \qquad (15)$$

If $m_2 \neq 0$, then we rewrite Eq. (9) in such a way that M^2_{min} for $s_{12}^{(2)}$ coincides with M^2_{max} from (9) (Fig. 2). Adding the obtained equality to (9) we obtain

$$w(s_{12}^{(1)}) + w(s_{12}^{(2)}) = \int\limits_{M_{min}^2(s_{12}^{(1)})}^{M_{max}^2(s_{12}^{(2)})} dM^2 \Phi(M^2)\frac{\pi^2}{4M^2}. \qquad (16)$$

We repeat this procedure until we get $\Phi(M^2) = 0$ on the upper integration limit. We then have after integration

$$\frac{\pi^2}{4M^2}\Phi(M^2) = -\frac{dw(s_{12}^{(1)})}{dM^2} - \frac{dw(s_{12}^{(2)})}{dM^2} - \ldots, \qquad (17)$$

where $s_{12}^{(1)}$ is calculated from one of the formulas (14), depending on which slope of the $w(s_{12})$ is used, i.e., $s_{12}^{(1)}$ is the ordinate of the point A_1 with abscissa M^2, after which we calculate from (6) the abscissa of the point B_1, which upon substitution in (4) yields $s_{12}^{(2)}$—the ordinate of the point A_2 etc. It is more convenient to carry out the calculation in accordance with (17) graphically, using the formula

$$\frac{\pi^2}{4M^2}\Phi(M^2) = -\sum_i w'(A_i)\, a_i, \qquad (18)$$

where $w'(A_i)$—slopes of the tangents to the spectrum m_2^2 at the points A_i, determined from the rule indicated in the diagram; $a_i = ds_{12}(A_i)/dM^2$—ratio of the increments of the ordinates of the points A_i of the small increment of M^2, which can also be readily determined graphically.

After determining the spectrum of M^2, we can obtain the spectrum of m_{23}^2 by a formula analogous to (9):

$$w(s_{23}) = \int\limits_{\widetilde{M}^2_{min}}^{\widetilde{M}^2_{max}} dM^2 \Phi(M^2)\frac{\pi^2}{4M^2}, \qquad (19)$$

where

$$\widetilde{M}_{\substack{max\\min}}^2 = m_{23}^2 + m_{13}^2 - m_3^2 + 2(\tilde{\omega}_1\tilde{\omega}_2 \pm \tilde{p}_1\tilde{p}_2), \qquad (20)$$

$$\tilde{\omega}_1 = \frac{m_{13}^2 - m_1^2 - m_3^2}{2m_3},$$

$$\tilde{\omega}_2 = \frac{m_{23}^2 - m_2^2 - m_3^2}{2m_3}, \quad \tilde{p}_i^2 = \tilde{\omega}_i^2 - m_i^2. \qquad (21)$$

Substituting (17), we obtain

$$w(s_{13}) = \sum_i w(s_{12}^{(i)}(\widetilde{M}_{min}^2)) - \sum_i w(s_{12}^{(i)}(M_{max}^2)), \qquad (22)$$

i.e., the spectrum $w(m_{13}^2)$ is the sum of the ordinates of the spectrum of m_{12}^2, taken at the points which depend on m_{13}^2 in accordance with formulas (20) and then in accordance with Fig. 2.

In particular, if $m_2 = 0$ and $m_3 \neq 0$, then

$$w(s_{13}) = w(s_{12}(\widetilde{M}_{min}^2)) - w(s_{12}(\widetilde{M}_{max}^2)). \qquad (23)$$

These properties of the effective-mass spectrum are very close to the known singularities of the energy spectrum of the products of two-particle decay in the laboratory frame [8,9]. However, they cannot be derived from each other.

With the aid of the formulas of this section we can seek resonances in the systems $K^0\pi^0$, $\Lambda^0\pi^0$, and in general YX, where X is an isotropically decaying particle with small lifetime ($\leq 10^{-15}$ sec) having only one decay product that can be detected.

3. The theorems of Secs. 1 and 2 pertained to two-particle cascade decays of the type $A \to 1 + a \to 1 + 2 + 3$. The generalization to many-particle decays can be carried out in elementary fashion. Let either the first or the second decay in the cascade be a many-particle decay:

$$A \to 1 + 1' + 1'' + \ldots + a \to 1 + 1' + 1'' + \ldots + 2 + 3,$$

or

$$A \to 1 + a \to 1 + 2 + 2' + 2'' + \ldots + 3.$$

In either case, let all the end products of the decay except particle 3 be observable and let it be known that the angular distribution of the particle 3 is isotropic in the rest system of a.

We determine in the first case the events with nearly equal values of the effective mass of the system $1 + 1' + 1'' \ldots$ (in the second case—the system $2 + 2' + 2'' \ldots$). We then obtain in both cases two-particle cascade decays with artificially chosen "particles" with "masses" $m_1 \pm \Delta m_1$ (first case) or $m_2 \pm \Delta m_2$ (second case). All the derivations of Secs. 1 and 2 are already applicable to such decays. Let it be necessary, for example, to obtain the distribution $\Psi(m^2)$ with respect to the effective mass of the system A; then the calculation by means of the formulas of Sec. 2 must be carried out for each interval Δm_i in the region of possible values of m_1; thus, the entire statistical material of the observations comes into play; in the final statistical reduction it is necessary to take into account the fact that the form of the $\Psi(M^2)$ curve should be the same for all intervals of m_1.

We can analogously generalize the results of[9] to many-particle decay. From this follows, for example, the possibility of reconstituting the energy spectrum of K^0 from measurements in the laboratory system of the momenta of π^+ and π^- in the decay $K^0 \to \pi^+\pi^-\pi^0$ without knowing the direction of K^0 (from the spectrum of the number $\omega_{\pi^+} + \omega_{\pi^-}$ of events with $m_{\pi^+\pi^-} = \text{const}$).

The foregoing generalization has not yet been encountered in any experiment.

[1] Wang, Kim, Kladnitskaya, Kopylov, Kuznetsov, Nguyen, Mel'nikova, and Sokolova, Preprint Joint Inst. Nuc. Res. R-1615, 1964.

[2] Belyakov, Veksler, Viryasov, Kladnitskaya, Kopylov, Mihul, Penev, Sokolova, and Solov'ev, JETP 1967 (1964). This issue p. 1326.

[3] G. Puppi, Proc. 1962 Intern. Conf. on High Energy Physics at CERN, Geneva, 1962, p. 713. W. H. Barkas and A. H. Rosenfeld. UCRL-8030, 1963.

[4] M. Gell-Mann and A. H. Rosenfeld, Ann. Rev. Nucl. Sci. **7**, 407 (1957); L. B. Okun', Slaboe vzaimodeĭstvie elementarnykh chastits (Weak Interaction of Elementary Particles), Fizmatgiz, 1963.

[5] G. Pinski, Nuovo cimento **24**, 719 (1962).

[6] P. Eberhardt and M. Pripstein, Phys. Rev. Lett. **10**, 351 (1963).

[7] G. I. Kopylov and V. E. Komolova, Nucl. Phys. **47**, 33 (1963). G. I. Kopylov, Preprint Joint Inst. Nuc. Res. R-1079.

[8] Carlson, Hooper, and King, Phil. Mag. **41**, 701 (1957).

[9] G. I. Kopylov, JETP **33**, 430 (1957), Soviet Phys. JETP **6**, 335 (1958).

Translated by J. G. Adashko
308