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ON THE STRUCTURE OF COLLECTIVE EXCITATIONS OF SPHERICAL NUCLEI

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It is shown that for an oscillator with anharmonicity of the x^{2n} type (of arbitrary strength) there exists a simple transformation which appreciably improves the convergence of the perturbation-theory series; the lower excited states in this case differ only slightly from those of an effective harmonic oscillator. This explains, in particular, the experimentally observed weak anharmonicity of the quadrupole oscillation spectrum of spherical even-even nuclei even though the interaction between the quadrupole phonons is sufficiently strong.

THERE exists a broad class of quantum-mechanical systems which can be represented in some approximation as an aggregate of oscillators with strong anharmonicity. In particular, the problem of analyzing such systems frequently arises in the investigation of quantized Bose fields (phonons and spin waves in a solid, meson field in the problem with the static nucleon, etc.). We are especially interested in quadrupole oscillations of spherical nuclei. As is well known[1], in an analysis of the lower excited state of a spherical nucleus the anharmonicity of the elementary excitations (quadrupole "phonons") is quite appreciable.

As a rule, in systems of this type it is sensible to use the following approach. Owing to the strong nonlinearity, the structure of both the ground and excited states of the system turns out to be complicated and essentially different from the structure of the state of the harmonic oscillator. The fluctuations in the number of quanta are quite large, so that it is impossible to use the ordinary form of perturbation theory. An important factor, however, is that the lower excited states differ little in their structure from the ground state (therefore, for example, the resulting observed anharmonicity in many spherical even-even nuclei is weak). It is thus natural to attempt to redefine the ground state of the system and the excitations in such a way as to take into account the principal effect of the nonlinearity—large fluctuations. One can hope here that the interaction between correctly chosen excitations turns out to be small, i.e., the convergence of the renormalized series of perturbation theory is greatly improved.

Let us illustrate the foregoing with a simple schematic example of a one-dimensional anharmonic oscillator, described by a Hamiltonian

$$H_n = \tfrac{1}{2}(p^2 + \omega_0^2 x^2) + \gamma x^{2n} \qquad (1)$$

($n \geq 2$ is an integer; we assume $\hbar = m = 1$). The "degree" of anharmonicity is characterized by a dimensionless parameter $\lambda = \gamma/\omega_0^{n+1}$, which is not assumed to be small. At large values of λ the correction to the average value $\overline{x^2}$, calculated from the wave functions of the harmonic oscillator with frequency ω_0, may also turn out to be large, i.e., the amplitude of the oscillations is large. Therefore in analogy with the classical problem concerning nonlinear oscillations[2] we can attempt to approximate the true potential by means of some effective parabola (which in principle is different for the different states), which must be chosen from a variational principle (for example, by minimizing the level energies).

On the other hand, in the second quantization representation, a large value of $\overline{x^2}$ denotes large fluctuations of the number of the "old" quanta in the ground state. In this connection it is necessary to redefine the vacuum and introduce new "true" quasiparticles. This is readily done with the aid of the Bogolyubov canonical transformation.

We proceed in standard fashion from the coordinate and momentum operators x and p to creation and annihilation operators a^+ and a, which satisfy the ordinary Bose commutation relations:

$$a = (2\omega_0)^{-1/2}(\omega_0 x + ip), \qquad a^+ = (2\omega_0)^{-1/2}(\omega_0 x - ip) \qquad (2).$$

Then the Hamiltonian assumes the form

$$H_n = \tfrac{1}{2}\omega_0 + \omega_0 a^+ a + 2^{-n}\omega_0 \lambda (a + a^+)^{2n}. \qquad (3)$$

We now carry out a canonical transformation to the new Bose amplitudes $(a, a^+) \rightarrow (b, b^+)$:

$$a = ub + vb^+, \qquad a^+ = ub^+ + vb, \qquad u^2 - v^2 = 1 \qquad (4)$$

(the unknown coefficients u and v can be regarded as real).

Using the customary notation, we write the transformed Hamiltonian in the form

$$H_n = U_n + H_{11} + H_{20} + H_n', \tag{5}$$

where

$$U_n = \tfrac{1}{2}\omega_0\,[1 + 2v^2 + (2n-1)!!\,2^{1-n}\,(u+v)^{2n}\,\lambda] \tag{6}$$

is the part which contains no operators,

$$H_{11} = \omega_0\,[u^2 + v^2 + 2^{1-n}n\,(2n-1)!!\,(u+v)^{2n}\,\lambda]\,b^+b,$$

$$H_{20} = \frac{\omega_0}{2}\,[2uv + 2^{1-n}\,n\,(2n-1)!!\,(u+v)^{2n}\lambda]\,(bb + b^+b^+),$$

$$H_n' = 2^{-n}\omega_0\,(u+v)^{2n}\lambda N\left\{\sum_{m=2}^{n} c_m\,(b+b^+)^{2m}\right\}, \tag{7}$$

N—symbol for the normal product of operators, c_m—integer factors which depend on n; $c_n = 1$.

We choose the transformation coefficients (4) from the conditions for the diagonalization of that part of the Hamiltonian (5) which is quadratic in the operators b and b^+, i.e., we put $H_{20} = 0$. Then the coefficient of b^+b in H_{11} gives the quasiparticle energy, i.e., a new frequency ω of the effective oscillator. We introduce the unknown quantity $y = (u+v)^2$ and obtain equations for y and ω:

$$y^{-n-1}\,(1-y^2) = 2^{2-n}n\,(2n-1)!!\,\lambda, \tag{8}$$

$$\omega = \omega_0/y, \tag{9}$$

i.e., y yields directly the frequency renormalization.

We can arrive at Eqs. (8) and (9) also by using a direct variational method: if we replace ω_0 in (2) by the unknown frequency ω, calculate the energy of the ground state $U_n = U_n(\omega)$ with the aid of Hamiltonian (1) and the wave functions of the harmonic oscillator of frequency ω, and then find the maximum $U_n(\omega)$, then we obtain exactly the same equations, as expected.

Inasmuch as the ground state of a Bose system with attraction is unstable, let us consider Eq. (8) for the case of repulsion ($\lambda > 0$). Then the inequality $0 \le y \le 1$ holds true, i.e., $\omega > \omega_0$—the excitations become more stable. Equation (8) has a unique solution, and with increasing λ the value of y decreases in proportion to $\lambda^{-1/(n+1)}$, so that if we decrease ω_0 and leave γ unchanged, then the renormalized frequency will tend to a constant value:

$$\omega \to \left[\frac{n\,(2n-1)!!}{2^{n-2}}\,\gamma\right]^{1/(n+1)}.$$

Introducing in place of λ the renormalized parameter of interaction $\tilde{\lambda} = \gamma/\omega^{n+1}$, we write the final form of the Hamiltonian

$$H_n = U_n + \omega b^+b + \frac{\omega}{2^n}\tilde{\lambda}N\left\{\sum_{m=2}^{n} c_m\,(b+b^+)^{2m}\right\}, \tag{10}$$

$$U_n = \tfrac{1}{4}\omega\,[1 + y^2 + 2^{2-n}\,(2n-1)!!\,\tilde{\lambda}]. \tag{6'}$$

The entire meaning of the change of variables lies in the fact that according to (8) and (9), regardless of the value of the "bare" interaction λ, the renormalized interaction $\tilde{\lambda}$ is bounded from above:

$$\tilde{\lambda} = \lambda y^{n+1} = (1-y^2)\,\frac{2^{n-2}}{n\,(2n-1)!!} \leqslant \frac{2^{n-2}}{n\,(2n-1)!!} \leqslant \frac{1}{6}.$$

Thus, the Hamiltonian (10) can already be studied by perturbation theory, using as the point of departure the states of the harmonic oscillator with frequency ω, the corrections to the energies of the ground and first-excited (single-quantum) states being only of second order in $\tilde{\lambda}$. For example, the exact numerical calculation[3] for the potential γx^4 yields for the ratio of energy of the first level to the energy of the ground state a value 3.58. In our case this quantity (in the corresponding limiting case $\lambda = \gamma/\omega_0^3 \to \infty$, $\tilde{\lambda} \to 1/6$) is equal to

$$(\omega + U_2)/U_2 = [4 + (2 - 3\tilde{\lambda})]/(2 - 3\tilde{\lambda}) = \tfrac{11}{3},$$

i.e., it is sufficiently close to the true value.

Thus, for a Hamiltonian of the type (1) it becomes indeed possible, no matter how strong the initial nonlinearity, to approximate with sufficiently good accuracy the excited states by means of states of an effective harmonic oscillator. For the higher levels, the effective parabola already turns out to be different, in full agreement with classical nonlinear mechanics[2].

Let us consider now a concrete system described by a Hamiltonian of such a kind, namely a spherical even-even nucleus, the lower excited states of which have well known collective properties. As shown earlier[1,4], allowance for the adiabaticity of the collective excitations relative to the single-particle excitation enables us to determine the variables which describe the collective branch of the excitations, and represent the Hamiltonian of the system in the form

$$H = H^{(2)} + H^{(3)} + H^{(4)} = \omega_0\left(\frac{5}{2} + \sum_M \alpha_M^+\alpha_M\right)$$
$$+ \frac{\Gamma^{(3)}}{\omega_0^{3/2}}\,(\alpha^{(+)3})_{00} + \sum_{L=0,2,4} \frac{\Gamma_L^{(4)}}{\omega_0^2}\,((\alpha^{(+)2})_L\,(\alpha^{(+)2})_L)_{00}. \tag{11}$$

Here α_M and α_M^+ are the Bose operators for the annihilation and creation of a "phonon" with angular momentum J = 2 and projection $J_z = M$,

$$\alpha_M^{(\pm)} = \alpha_M \pm (-1)^M\,\alpha_{-M}^+,$$

$(\ldots)_{JM}$ is the symbol of vector addition,

ω_0—frequency of quadrupole oscillations, $\Gamma^{(3)}$ and $\Gamma_L^{(4)}$ (L = 0, 2, 4)—some constants. Their specific connection with the energies and the occupation numbers of the one-nucleon levels can be obtained from the microscopic theory which shows, in particular,[1] that the coefficient $\Gamma^{(3)}$ is small, since it is determined by sums of quantities which are odd relative to the Fermi boundary, taken over a large number of levels. It can also be shown that the corrections due to the higher anharmonic terms $\Gamma^{(n)}$ with $n \geq 5$ are also small, as are (in the case of sufficiently good adiabaticity) the terms which include the operators $\alpha^{(-)}$.

Inasmuch as the four quadrupole phonons have only one state with total angular momentum J = 0, the term $H^{(4)}$ in the Hamiltonian should actually contain only one independent constant. It is therefore convenient to transform $H^{(4)}$ somewhat. Using the known properties of the Racah coefficients, we can readily obtain

$$H^{(4)} = \frac{\omega_0}{3} \sum_{L=0,\,2,\,4} \sqrt{2L+1}\, \Lambda_L \left((\alpha^{(+)2})_L (\alpha^{(+)2})_L\right)_{00}, \tag{12}$$

$$\Lambda_L = \sum_{L'=0,2,4} \frac{\Gamma_{L'}^{(4)}}{\omega_0^3} \frac{1}{\sqrt{2L'+1}} \times [\delta_{LL'} + 2\,(2L'+1)\, W\,(2222; LL')]. \tag{13}$$

As can be readily verified, we have regardless of the value of the initial constant $\Gamma_L^{(4)}$

$$\Lambda_2 = \Lambda_4 = \tfrac{2}{7}\Lambda_0, \tag{14}$$

and the following useful identity holds true

$$\Lambda_L = \sum_{L'=0,\,2,\,4} (2L'+1)\, W\,(2222; LL')\, \Lambda_{L'}, \qquad L = 0,\, 2,\, 4. \tag{15}$$

By virtue of the property (15), all the operators $\alpha^{(+)}$ are contained in (12) symmetrically, thus noticeably simplifying the manipulations.

Carrying out in the usual fashion a canonical transformation over the Hamiltonian (12) we obtain, in analogy with (8) and (9)

$$(1 - y^2)/y^3 = 8\Lambda_0; \qquad \omega = \omega_0/y. \tag{16}$$

The transformed Hamiltonian takes the form

$$H = U + \omega \sum_M \beta_M^+ \beta_M + \omega \widetilde{\Lambda}^{(3)} (\beta^{(+)3})_{00} + \frac{\omega}{3} \sum_{L=0,\,2,\,4} \widetilde{\Lambda}_L \sqrt{2L+1}\, N\{((\beta^{(+)2})_L (\beta^{(+)2})_L)_{00}\}, \tag{17}$$

and

$$\widetilde{\Lambda}_3 = \Gamma^{(3)}/\omega^{5/2} \leqslant \Gamma^{(3)}/\omega_0^{5/2},$$
$$\widetilde{\Lambda}_L = \Lambda_L y^3 \leqslant \Lambda_L/8\Lambda_0, \qquad L = 0,\, 2,\, 4. \tag{18}$$

The calculation of the energies of the two-phonon triplet levels is best carried out by using the equation of motion for the operators in the Heisenberg representation. The averages of the product of several operators which result from this are expressed, as in the Hartree-Fock method, in terms of the paired averages

$$\rho_{MM'} = \langle \alpha_M^+ \alpha_{M'} \rangle, \qquad \tau_{MM'} = \langle \alpha_M \alpha_{M'} \rangle. \tag{19}$$

We can establish a connection between these matrices

$$\tau^+\tau - \rho^2 = \rho, \tag{20}$$

analogous to the Bogolyubov condition[5] for Fermi systems. In a spherical even-even nucleus we can put

$$\rho_{MM'} = \delta_{MM'}\rho, \qquad \tau_{MM'} = (-1)^M \delta_{M,-M'}\tau, \tag{21}$$

where ρ and τ are real numbers. Using (20) and the commutation relations, we can easily express all the encountered averages in terms of the renormalization factor y:

$$\langle \alpha_M^{(+)} \alpha_{M'}^{(+)} \rangle = (-1)^M \delta_{M,-M'} y, \quad \langle \alpha_M^{(-)} \alpha_{M'}^{(-)} \rangle = (-1)^{M+1} \delta_{M,-M'} y^{-1},$$
$$\langle \alpha_M^{(+)} \alpha_{M'}^{(-)} \rangle = -\langle \alpha_M^{(-)} \alpha_{M'}^{(+)} \rangle = (-1)^M \delta_{M,-M'},$$
$$\rho = (1-y)^2/4y, \qquad \tau = (y^2-1)/4y. \tag{22}$$

After separating the averages we obtain a linear homogeneous system of equations for the matrix elements, the solvability condition of which is the secular equation for the level energies reckoned from the energy of the ground state. A zero root of the secular equation corresponds to those pairs of the "old" phonons with total angular momentum J = 0, which have already been taken into account in the ground state. Nonzero roots give the energy of the first level ω, coinciding with (16) apart from corrections of order $(\widetilde{\Lambda}^{(3)})^2$, which certainly can be evaluated by the usual perturbation theory (see [1,4]), and the energies of the two-phonon triplet level (J = 0, 2, 4):

$$\omega^2(J) = 4\omega^2 (1 + 4\widetilde{\Lambda}_J) + O\,[(\widetilde{\Lambda}^{(3)})^2]. \tag{23}$$

Formulas (16) and (23) go over into those previously obtained[4] in the limit of weak interaction ($\Lambda_0 \ll 1$).

With the aid of (18) and (14) we can estimate the upper limit of anharmonicity, determined by the four-phonon interaction: $\lambda_E \equiv \omega(2)/\omega \lesssim 2.15$. Inasmuch as for the most typical nuclei of the class under consideration the experimentally observed value is $\lambda_E \sim 2.2$, we can conclude that in these nuclei the values of renormalized constants $\widetilde{\Lambda}$ are close to the limiting values, i.e., the bare anharmonicity Λ and the frequency renormalization y^{-1} are large. The unperturbed frequency

ω_0 (the energy at the first level in the harmonic approximation) is usually determined[6] from the corresponding secular equation for the bound state of two quasiparticles. This equation contains the constant F of the particle-hole interaction, which is obtained[7] from the equality of ω_0 to the experimental energy ω of the first level. We have already seen that as a result of the strong anharmonicity these quantities may differ noticeably among themselves. Thus, it may turn out to be necessary to carry out a substantial refinement of the assumed value of F, and this will influence also the calculation of all the observed quantities.

In view of the strong influence of the anharmonicity it would hardly be worthwhile to attempt [7,8], without confining oneself qualitatively to a correct description, to obtain by a thorough choice of the single-particle level scheme a good qualitative agreement with all the experimental data, even in the harmonic approximation, from the energies of the first level and the probabilities of the electromagnetic transitions.

A situation calling for a special analysis is that of odd spherical nuclei where, in addition to the anharmonicity due to the interaction between the phonons, a strong coupling between the phonon and an odd particle is possible.

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