THE WAVE EQUATION AND MAGNETIC MOMENT OF SPIN 2 PARTICLES

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A relativistic wave equation in matrix form is derived for a particle of spin 2. By going over to the nonrelativistic equation a value for the intrinsic magnetic moment of the spin 2 particle is obtained which confirms the general hypothesis put forward earlier by Belinfante.

As calculations by Belinfante have shown, the intrinsic magnetic moment of a particle with spin $3/2$ is equal to $(e\hbar/3mc)S$ (e and m are the charge and mass of the particle, and S is the nonrelativistic spin operator). In this connection it was suggested that for any spin the maximum value of the projection of the intrinsic magnetic moment is $e\hbar/2mc$.[1] It is desirable to test this hypothesis for particles with larger values of the spin. In the present paper the magnetic moment of a particle with spin 2 is calculated. As the first step a wave equation in matrix form is derived for this particle.

1. THE GENERAL FORM OF THE WAVE EQUATION

The relativistic wave equation for a particle with arbitrary spin in an electromagnetic field is of the form

$$(\beta^\nu\pi_\nu + \varkappa S^{\mu\nu}F_{\mu\nu} - mc)\,\Psi = 0, \tag{1.1}$$

where $\pi_\nu = p_\nu - eA_\nu/c$, the metric is chosen in the form $g_{00} = -g_{kk} = 1$, and for completeness we have included in the equation an interaction of the type $\kappa S^{\mu\nu}F_{\mu\nu}$, where $S^{\mu\nu}$ is the spin matrix and $F_{\mu\nu}$ is the field tensor.

For the representation of the matrices we choose the most commonly used basis $|P_p, Q_q\rangle$ (cf., e.g., [2-5]), in which the state vectors are eigenvectors of the operators $\hat{S}^{12}$ and $\hat{S}^{03}$:

$$\hat{S}^{12}\,|Pp,\ Qq\rangle = (p+q)\,|Pp,\ Qq\rangle,$$

$$\hat{S}^{03}\,|Pp, Qq\rangle = -\,i\,(p-q)\,|Pp,\ Qq\rangle.$$

In this basis the forms of the matrices β^ν are [3,4]

$$\tfrac{1}{2}\,(\beta^0+\beta^3)^{\tau\tau'}_{pq;p'q'} = e^{\tau\tau'}\left(P'+\tfrac{1}{2}-\varepsilon_1 p\right)^{1/2}\left(Q'+\tfrac{1}{2}+\varepsilon_2 q\right)^{1/2}\delta_{p',p-1/2}\delta_{q',q+1/2};$$

$$\tfrac{1}{2}\,(\beta^0-\beta^3)^{\tau\tau'}_{pq,p'q'} = -\,\varepsilon_1\varepsilon_2 e^{\tau\tau'}\left(P'+\tfrac{1}{2}+\varepsilon_1 p\right)^{1/2}\times\left(Q'+\tfrac{1}{2}-\varepsilon_2 q\right)^{1/2}\delta_{p',p+1/2}\delta_{q',q-1/2};$$

$$\tfrac{1}{2}\,(\beta^1+i\beta^2)^{\tau\tau'}_{pq,p'q'} = \varepsilon_2\, e^{\tau\tau'}\left(P'+\tfrac{1}{2}-\varepsilon_1 p\right)^{1/2}\times\left(Q'+\tfrac{1}{2}-\varepsilon_2 q\right)^{1/2}\delta_{p',\,p-1/2}\delta_{q',q-1/2};$$

$$\tfrac{1}{2}\,(\beta^1-i\beta^2)^{\tau\tau'}_{pq,p'q'} = -\,\varepsilon_1 e^{\tau\tau'}\left(P'+\tfrac{1}{2}+\varepsilon_1 p\right)^{1/2}\times\left(Q'+\tfrac{1}{2}+\varepsilon_2 q\right)^{1/2}\delta_{p',p+1/2}\delta_{q',q+1/2}; \tag{1.2}$$

where

$$P' = P+\tfrac{1}{2}\varepsilon_1,\quad Q' = Q+\tfrac{1}{2}\varepsilon_2,\quad \varepsilon_1 = \pm 1,$$
$$\varepsilon_2 = \pm 1,\quad \tau = \tau_{PQ},\ \tau' = \tau_{P'Q'}.$$

For the determination of the coefficients $e^{\tau\tau'}$ we choose the following supplementary conditions:

$$\Sigma\,(\beta^{\nu_1}\beta^{\nu_2} - fg^{\nu_1\nu_2})\,\beta^{\nu_3}\ldots\beta^{\nu_{b+1}} = 0, \tag{1.3}$$

$$\Lambda F\beta^\nu = \beta^{\nu+}\Lambda F, \tag{1.4}$$

$$\beta^0 J - J\beta^0 = 0,\quad \beta^k J + J\beta^k = 0. \tag{1.5}$$

Equation (1.3) is the necessary condition for Eq. (1.1) to describe a particle with a definite mass [6,7]; the summation in Eq. (1.3) is taken over all permutations of the indices; $b = 2\max(P+Q)$.[7] The scalar matrix f is introduced to allow for the fact that some irreducible representations are of auxiliary character (concerning auxiliary representations see [8]); f has on the diagonal 1's for the main representations and 0's for the auxiliary ones. The integral over space of the probability density $\Psi^+\Lambda F\beta^0\Psi$ must not depend on the time. From this condition, extended in virtue of the superposition principle to arbitrary matrix elements, there follows the condition (1.4). The introduction of the matrix Λ, which is of the form

$$(\Lambda)^{\tau\tau'}_{pq,p'q'} = \delta_{PQ'}\delta_{QP'}\delta_{pq'}\delta_{qp'}, \tag{1.6}$$

is due to the special nature of the orthogonality condition on the basis vectors. F is a scalar matrix,

$$(F)^{\tau\tau'}_{pq,p'q'} = f^{\tau}\delta_{PP'}\delta_{QQ'}\delta_{pp'}\delta_{qq'}, \tag{1.7}$$

and for a true scalar $f^{\tilde{\tau}} = f^{\tau}$ ($\tilde{\tau} \equiv \tau_{PQ}$). Finally, Eq. (1.5) expresses the condition for invariance of Eq. (1.1) under inversion; the matrix for the inversion operator is of the form

$$(J)^{\tau\tau'}_{pq,p'q'} = g^{\tau}\delta_{PQ'}\delta_{QP'}\delta_{pq'}\delta_{qp'}, \tag{1.8}$$

with $g^{\tau}\tilde{g}^{\tau} = 1$ (integer spin), $g^{\tau}\tilde{g}^{\tau} = \pm 1$ (half-integer spin).

It can be shown that all of the relations (1.3)—(1.5) follow from the relations written for the zeroth matrices only:

$$(\beta^0)^{b+1} = f\,(\beta^0)^{b-1}, \tag{1.9}$$

$$\Lambda F\beta^0 = \beta^{0+}\Lambda F, \tag{1.10}$$

$$\beta^0 J = J\beta^0. \tag{1.11}$$

Thus Eqs. (1.9)—(1.11) are sufficient conditions for the stated physical conditions to be satisfied.

In concluding this section we present formulas for the block structure of the matrices $S^k = \frac{1}{2}\epsilon_{klm}S^{lm}$ and S^{0k} when the columns and rows for each τ_{PQ} are arranged in the following sequence of the indices pq:

$$PQ,\ P\ Q-1, \ldots, P\ -Q,\ P-1\ Q,$$
$$P-1\ Q-1, \ldots, -P\ -Q.$$

We have

$$S^k = \Sigma_k^{Q,1,2P+1} + \Sigma_k^{P,2Q+1,1}, \qquad S^{0k} = i\,(\Sigma_k^{Q,1,2P+1} - \Sigma_k^{P,2Q+1,1}), \tag{1.12}$$

where $\Sigma_k^{Q,1,2P+1}$ is a matrix formed by placing along the diagonal the $2P+1$ nonrelativistic matrices for the spin Q, and $\Sigma_k^{P,2Q+1,1}$ is formed from the nonrelativistic matrices for the spin P by multiplying each element by a unit matrix of the order $2Q+1$. If the representation τ_{QP} is encountered directly after the representation τ_{PQ}, then for τ_{QP} the columns and rows are arranged in a different order: QP, Q − 1P, . . . , − QP, QP − 1, Q − 1, P − 1, . . . , − Q − P. In this case the block for τ_{QP} in S^k coincides with the block for τ_{PQ}, but in S^{0k} it has the opposite sign.

2. THE WAVE EQUATION FOR THE PARTICLE WITH SPIN 2

The equation for the particle with spin 2 corresponds to the following choice of representations:

$$\tau_{1/2\,3/2} + \tau_{3/2\,1/2} + \tau_{11} + \tau_{1/2\,1/2} + \tau_{00}, \tag{2.1}$$

and the last two representations are auxiliary ones. According to the requirements of the condition of linkage, $\tau_{1/2,1/2}$ has been included in the expression (2.1) in addition to the irreducible representations used in the paper of Fierz and Pauli [8] in the derivation of the equation in the form of a system of tensor equations of the second order. The matrices β^{ν} are of the form

$$\beta^0 = \begin{pmatrix} 0 & 0 & a_1A_0 & 0 & 0 \\ 0 & 0 & b_1B_0 & 0 & 0 \\ a_2A_0^+ & b_2B_0^+ & 0 & -c_1C_0^+ & 0 \\ 0 & 0 & c_2C_0 & 0 & -d_1D_0 \\ 0 & 0 & 0 & d_2D_0^+ & 0 \end{pmatrix},$$

$$\beta^k = \begin{pmatrix} 0 & 0 & -a_1A_k & 0 & 0 \\ 0 & 0 & b_1B_k & 0 & 0 \\ a_2A_k^+ & -b_2B_k^+ & 0 & c_1C_k^+ & 0 \\ 0 & 0 & c_2C_k & 0 & d_1D_k \\ 0 & 0 & 0 & d_2D_k^+ & 0 \end{pmatrix}. \tag{2.2}$$

Here a_1, a_2, b_1, b_2 and so on are notations for the $e^{\tau\tau'}$. The forms of the separate blocks are defined by the formulas (1.2). The zeros in Eq. (2.2) denote various rectangular zero matrices. The matrices f, Λ, F, J can be written in analogous compact form, for example

$$\Lambda = \begin{pmatrix} 0 & I_8 & 0 & 0 & 0 \\ I_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & I_4 - D_0D_0^+ & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.3}$$

where I_n is the unit matrix of order n.

In calculating the coefficients by means of the formulas (1.9)—(1.11) it is not necessary to write out the explicit values of the blocks; it suffices to use the identities

$$A_0K = B_0,\ K^+ = K,\ K^2 = I_9,\ A_0^+A_0 = B_0^+B_0 = 6I_9 - C_0^+C_0,$$
$$C_0C_0^+ = 4I_4 + D_0D_0^+,\ A_0C_0^+D_0 = B_0C_0^+D_0 = 0,$$
$$D_0^+D_0 = 2,\ C_0K = (D_0D_0^+ - I_4)\,C_0. \tag{2.4}$$

From the condition (1.9), which in this case takes the form $(\beta^0)^5 = f\,(\beta^0)^3$, we get a system of algebraic equations, from which it follows that

$$a_1a_2 + b_1b_2 = 1/6,\quad c_1c_2 = 1/12,\quad d_1d_2 = -1/4. \tag{2.5}$$

The condition (1.10) gives the connections between the coefficients g^{τ}, and also the relation

$$a_1 a_2 = b_1 b_2. \tag{2.6}$$

It is not hard to remove the remaining indefiniteness by making use of the arbitrariness in the normalization of the basic spin vectors. This can be most simply done by writing out the matrix of the coefficients of the equation

$$\begin{pmatrix} 1 & 0 & a_1 & 0 & 0 \\ 0 & 1 & b_1 & 0 & 0 \\ a_2 & b_2 & 1 & c_1 & 0 \\ 0 & 0 & c_2 & 1 & d_1 \\ 0 & 0 & 0 & d_2 & 1 \end{pmatrix}, \tag{2.7}$$

in which the rows and columns can be multiplied by arbitrary nonzero numbers, with the stipulation that when the columns are so multiplied there is a corresponding change of the normalization of the components of the wave function. By means of such multiplications and the use of the relations (2.5) and (2.6) we easily obtain a matrix with the coefficients

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = 1/2\sqrt{3}, \quad d_1 = -d_2 = -{}^1/_2 \tag{2.8}$$

and thus define the equation uniquely. From the remaining condition (1.10) it follows that with the choice (2.8) for the coefficients the matrix F is proportional to the matrix

$$\begin{pmatrix} I_{29} & 0 \\ 0 & -1 \end{pmatrix}.$$

3. THE MAGNETIC MOMENT OF THE PARTICLE WITH SPIN 2

The calculation of the magnetic moment is a matter of passing from the relativistic equation to the limit of the nonrelativistic equation; for this purpose we carry out a unitary transformation of the equation by means of the matrix

$$\begin{pmatrix} I_{16} & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I_5 \end{pmatrix},$$

$$U = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{6} \\ 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 & 0 & -\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{2} & 0 & -\sqrt{2} & 0 & \sqrt{2} & 0 & 0 \end{pmatrix}. \tag{3.1}$$

We put Ψ in the form of a column of five functions ξ, η, φ, χ, λ, corresponding to the sequence of irreducible representations (2.1). The corresponding blocks of $S^{\mu\nu}$ will be denoted by $S^{\mu\nu}_{(1)}, S^{\mu\nu}_{(2)}, \ldots, S^{\mu\nu}_{(5)}$; the forms of these blocks are defined by Eq. (1.12) and the remarks made after that equation. Besides this we introduce the notation

$$U\varphi = \begin{pmatrix} \psi \\ \delta \\ \mu \end{pmatrix}, \tag{3.2}$$

where ψ has five components, δ three, and μ one. The unitary transformation (3.1) is chosen so that in nonrelativistic approximation the components of δ and μ are small. Furthermore it transforms the relativistic spin matrices into the nonrelativistic matrices:

$$US^k_{(3)}U^+ = \begin{pmatrix} S_k & 0 & 0 \\ 0 & \Sigma_k & 0 \\ 0 & 0 & 0 \end{pmatrix}; \tag{3.3}$$

S_k are the nonrelativistic spin matrices for spin 2, and Σ_k are those for spin 1.

We represent the products of the matrices A_ν, B_ν, C_ν by U^+ in the form

$$\begin{aligned} A_0U^+ &= (P_0\ Q_0\ 0), & A_kU^+ &= (P_k\ Q_k\ R_k), \\ B_0U^+ &= (P_0\ -Q_0\ 0), & B_kU^+ &= (P_k\ -Q_k\ R_k), \\ C_0U^+ &= (0\ W_0\ \sqrt{3}D_0), & C_kU^+ &= (V_k\ W_k\ 3^{-1/2}D_k), \end{aligned} \tag{3.4}$$

where, in accordance with the separation (3.2), P_ν and V_k have five columns, Q_ν and W_ν three, and R_k and D_ν a single column. Without presenting the explicit forms of the matrices, we write out some relations between them which are needed for the calculation:

$$P_0^+P_0 = 6I_5, \quad Q_0^+Q_0 = 2I_3, \quad W_0^+W_0 = 4I_3,$$

$$W_0W_0^+V_i = 4V_i, \; P_0^+S^k_{(1)}P_0 = S_k,$$

$$P_i^+ Q_0 - P_0^+ Q_i = V_i^+ W_0, P_j^+ P_k + V_j^+ V_k = 6\delta_{jk} + 3i\varepsilon_{jkl} S_l. \quad (3.5)$$

For the case of a particle in a constant magnetic field **H** the unitary transformation (3.1) causes the equation (1.1) with the matrices (2.2) and the coefficients (2.8) to break up into the following system of equations:

$$2\sqrt{3}\,(mcI_8 - 2\varkappa S_{(1)}\mathbf{H})\,\xi = (P_0\pi^0 + P_i\pi^i)\,\psi + (Q_0\pi^0 + Q_i\pi^i)\,\delta + R_i\pi^i\mu, \quad (3.6)$$

$$2\sqrt{3}\,(mcI_8 - 2\varkappa S_{(1)}\mathbf{H})\,\eta = (P_0\pi^0 - P_i\pi^i)\,\psi + (-Q_0\pi^0 + Q_i\pi^i)\,\delta - R_i\pi^i\mu, \quad (3.7)$$

$$P_0^+\pi^0(\xi + \eta) - P_i^+\pi^i(\xi - \eta) + 2\sqrt{3}\,(2\varkappa\,\mathbf{S}\mathbf{H} - mc)\,\psi - V_i^+\pi^i\chi = 0, \quad (3.8)$$

$$Q_0^+\pi^0(\xi - \eta) - Q_i^+\pi^i(\xi + \eta) + 2\sqrt{3}\,(2\varkappa\,\Sigma\mathbf{H} - mc)\,\delta - (W_0^+\pi^0 + W_i^+\pi^i)\,\chi = 0, \quad (3.9)$$

$$\tfrac{1}{2}3^{-1/2}R_i\pi^i(\xi - \eta) + mc\mu + \tfrac{1}{6}(3D_0^+\pi^0 + D_i^+\pi^i)\,\chi = 0, \quad (3.10)$$

$$\tfrac{1}{2}3^{-1/2}[-V_i\pi^i\psi + (W_0\pi^0 - W_i\pi^i)\,\delta] + \tfrac{1}{6}(3D_0\pi^0 - D_i\pi^i)\,\mu + (2\varkappa\,S_{(4)}\mathbf{H} - mcI_4)\,\chi + \tfrac{1}{2}(D_0\pi^0 + D_i\pi^i)\,\lambda = 0, \quad (3.11)$$

$$(D_0^+\pi^0 - D_i^+\pi^i)\,\chi - 2mc\lambda = 0. \quad (3.12)$$

To obtain the nonrelativistic approximation we must substitute in the main equation (3.8) expressions for ξ, η, and χ in terms of ψ as found from the other equations. The calculation is made to such a degree of accuracy that the terms coming into Eq. (3.8) are in nonrelativistic approximation of order not higher than $(\pi^i/mc)^2\psi$, and the interaction terms are linear in the field. After the functions ξ and η, expressed to the required accuracy by using Eqs. (3.6) and (3.7), have been substituted in Eqs. (3.8)—(3.10), there remains the task of determining the functions δ and χ. Here Eqs. (3.9)—(3.12) are decidedly simplified, since it suffices to find δ and χ in first approximation. The result of a calculation in which we use the identities (3.5) and the usual passage to the limit

$$\psi \to \psi \exp(-imc^2\hbar^{-1}t), \quad \pi_0^2 - m^2c^4 \to 2mE$$

(E is the nonrelativistic energy operator) is to bring Eq. (3.8) into the form

$$[E - (\boldsymbol{\pi}^2/2m - \boldsymbol{\mu}\mathbf{H})]\,\psi = 0, \quad (3.13)$$

$$\boldsymbol{\mu} = (e\hbar/4mc + 2\varkappa c)\,\mathbf{S}. \quad (3.14)$$

The value $(e\hbar/4mc)\mathbf{S}$ found for the intrinsic magnetic moment confirms Belinfante's hypothesis.

The expression (3.14) and the results obtained earlier for lower spins, in which cases we can for generality introduce the interactions $\kappa S^{\mu\nu}F_{\mu\nu}$, can be combined into the general formula

$$\boldsymbol{\mu} = (e\hbar/2Smc + 2\varkappa c)\,\mathbf{S}, \quad S = 1/2, 1, 3/2, 2. \quad (3.15)$$

The result obtained for spin 2 adds to the probability of the hypothesis that the formula (3.15) is also valid for higher values of the spin.

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