SOVIET PHYSICS JETP VOLUME 19, NUMBER 5 NOVEMBER, 1964

A GENERAL RELATION IN THE THEORY OF THE FERROMAGNETIC FERMI LIQUID

I. E. DZYALOSHINSKIĬ

Institute for Physical Problems, Academy of Sciences, U.S.S.R.

Submitted to JETP editor October 2, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 1722-1725 (May, 1964)

It is shown that the total spin of a system of Fermi particles possessing a spectrum of Fermi type (a Fermi liquid) is equal to the sum of the spins of its excitations, under the condition that only exchange forces are operative in the system.

IN the theory of the Fermi liquid, constructed by Landau[1], an important role is played by a set of general relations that are independent of the specific properties of the particles that constitute the Fermi liquid and of the type of interactions between them. One such relation is a condition of equality of the number of particles to the number of excitations, proved first by Landau and Pitaevskiĭ[2] in differential form (more precisely, it was shown that the derivative of the number of particles with respect to the chemical potential is equal to the derivative of the number of excitations with respect to the chemical potential). Later Luttinger and Ward[3] proved this relation in integral form; their proof essentially rested on the assumption that only paired forces between particles operate in the system. A proof free from assumptions of any kind was presented in the book of Abrikosov, Gor'kov, and the author (cf. [4], Sec. 19).

In the theory of the Fermi liquid there exists one other general relation, expressing the fact that the total spin of a system, between whose particles only exchange forces[1)] operate, is equal to the sum of the spins of its excitations. We emphasize that the assumption about exchange does not limit in any essential respect the generality of the proposition, for only in the absence of nonexchange interactions does a separation of the total moment of the system into spin and orbital parts have meaning. A proof of this relation for the physically unrealistic case of paired interactions between the particles, in general independent of the spin, was given in the article of Luttinger and Ward already cited. A proof valid for arbitrary exchange interactions is presented below[5].

The total spin S of the system is expressed in terms of the latter's single-particle Green function $G_{\alpha\beta}(\omega, p)$ by the formula

$$S = -\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} e^{i\omega\delta} \boldsymbol{\sigma}_{\alpha\beta} G_{\beta\alpha}(\omega, p), \quad \delta \to +0, \tag{1}$$

where σ represents the Pauli matrices. By use of Dyson's equation

$$G^{-1}_{\alpha\beta} = \delta_{\alpha\beta}(\omega - p^2/2m + \mu) + \gamma \boldsymbol{\sigma}_{\alpha\beta}\mathbf{H} - \Sigma_{\alpha\beta}, \tag{2}$$

where μ is the chemical potential, **H** is the external constant field, γ is the Bohr magneton, and Σ is the self-energy term, it is not difficult to transform (1) to the form

$$S = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4} e^{i\omega\delta} \boldsymbol{\sigma}_{\alpha\beta} G^{-1}_{\beta\gamma} \frac{\partial G_{\gamma\alpha}}{\partial\omega} + \frac{i}{2}\int \frac{d^4p}{(2\pi)^4} \boldsymbol{\sigma}_{\alpha\beta} \Sigma_{\beta\gamma} \frac{\partial G_{\gamma\alpha}}{\partial\omega}. \tag{3}$$

We now note that the self-energy term is the variational derivative, with respect to $G_{\alpha\beta}$, of some functional X {G} of the Green function (cf. [4], Sec. 19):

$$\delta X = \int \frac{d^4p}{(2\pi)^4} \Sigma_{\alpha\beta}\delta G_{\beta\alpha}. \tag{4}$$

The functional X represents the sum of all possible connected diagrams that have the form of closed loops and do not contain self-energy inserts; in place of each line, the correct Green function is to be substituted in the corresponding expression (Fig. 1). We will not be more precise about the coefficients in front of the diagrams; they are unimportant for what follows.

By virtue of the isotropy of a liquid, its total spin is always directed along the external field **H**. By choosing this as the axis of quantization, we reduce (3) and (4) to the form

$$S = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4} e^{i\omega\delta}\left(\frac{\partial}{\partial\omega}\ln G_+ - \frac{\partial}{\partial\omega}\ln G_-\right)$$

$$+ \frac{i}{2}\int \frac{d^4p}{(2\pi)^4}\left(\Sigma_+\frac{\partial G_+}{\partial\omega} - \Sigma_-\frac{\partial G_-}{\partial\omega}\right), \tag{5}$$

$$\delta X = \int \frac{d^4p}{(2\pi)^4} \Sigma_\pm \delta G_\pm. \tag{6}$$

[1)] By exchange forces we understand, as usual, any interactions that conserve the total spin. The exchange condition consists, therefore, only in this: that the total-spin operator shall commute with the Hamiltonian of the system.

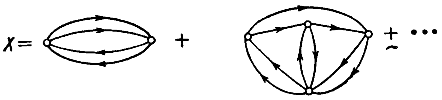

FIG. 1

Here $G_\pm$ and $\Sigma_\pm$ are the eigenvalues of the matrices $G_{\alpha\beta}$ and $\Sigma_{\alpha\beta}$. It is evident that G_+ corresponds to particles that have a positive spin projection, and G_- to particles with a negative projection.

We shall now show that the second integral in (5) is equal to zero. For this purpose we suppose that we have placed the system in a magnetic field h(t) dependent on time (but not on the coordinates), directed along the external (time-independent) field **H**. The Hamiltonian for the interaction of the system with such a field has the form

$$H_{int} = -\gamma \mathbf{h}(t)\int \psi^+ \boldsymbol{\sigma} \psi \, d\mathbf{r}$$

and, because of the assumed exchange nature of the interaction, commutes with the complete Hamiltonian of the system. In consequence, the Green function $G^h(t_1, t_2)$ in the field, as can be verified without difficulty by direct calculation, is related to the function $G(t_1 - t_2)$ at $h = 0$ by the formula

$$G_\pm^h(t_1, t_2) = G_\pm(t_1 - t_2)\exp\left\{\pm i\gamma \int_{t_2}^{t_1} h(t)\, dt\right\}. \qquad (7)$$

On considering now the limiting case of a time-independent field h, we get instead of (7)

$$G_\pm^h(\omega, p) = G_\pm(\omega \pm \gamma h, p). \qquad (8)$$

The functional X in the presence of a field h is by definition equal to $X^h \equiv X\{G^h\}$. We now take account of the fact that the exchange interaction conserves the total spin of the particles under scattering. This means that at every vertex in the diagrams for X^h, the number of incoming lines with sign + (and correspondingly with sign −) is equal to the number of outgoing lines with the same spin direction (cf. Fig. 2). On noting that for a closed system the dependence of the vertices on frequency reduces to δ-functions, expressing the law of conservation of energy,

$$\delta(\Sigma\omega - \Sigma\omega') \equiv \delta(\Sigma_+(\omega_+ - \omega_+') + \Sigma_-(\omega_- - \omega_-')),$$

and on taking account of (8), we conclude that in the expression for each diagram for X^h the change of variables $\omega_\pm \pm \gamma h \to \omega_\pm$ can be made; the functional X^h is then seen to be independent of h.

Thus $\partial X^h/\partial h = 0$. On the other hand, the second integral in (5) coincides, except for a factor, with $\partial X^h/\partial h$ at $h = 0$:

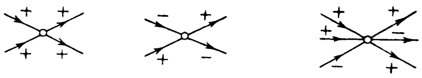

FIG. 2

$$\frac{\partial X^h}{\partial h} = \int \frac{d^4p}{(2\pi)^4}\left\{\Sigma_+(\omega + \gamma h)\frac{\partial G_+(\omega + \gamma h)}{\partial h} + \Sigma_-(\omega - \gamma h)\frac{\partial G_-(\omega - \gamma h)}{\partial h}\right\} = \gamma \int \frac{d^4p}{(2\pi)^4}\left\{\Sigma_+(\omega + \gamma h) \times \frac{\partial G_+(\omega + \gamma h)}{\partial \omega} - \Sigma_-(\omega - \gamma h)\frac{\partial G_-(\omega - \gamma h)}{\partial \omega}\right\},$$

Hence follows our proposition.

Thus, finally,

$$S = \frac{i}{2}\int \frac{d^4p}{(2\pi)^4} e^{i\omega\delta}\left(\frac{\partial}{\partial \omega}\ln G_+ - \frac{\partial}{\partial \omega}\ln G_-\right).$$

In [4] it was shown that the integral $\int d^4p e^{i\omega\delta}\,\partial \ln G/\partial\omega$ is equal, except for a factor, to the volume in momentum space of the region in which the Green function is positive for $\omega = 0$. Therefore

$$S = \frac{1}{2}\int_{G_+(0,\,p)>0} \frac{d\mathbf{p}}{(2\pi)^3} - \frac{1}{2}\int_{G_-(0,\,p)>0} \frac{d\mathbf{p}}{(2\pi)^3}. \qquad (9)$$

The Fermi branch of the spectrum of a ferromagnetic Fermi liquid coincides with the spectrum of a system of free Fermi quasiparticles in which the quasiparticles with spin projection $+\frac{1}{2}$ fill a sphere of radius p_+ and the quasiparticles with projection $-\frac{1}{2}$ a sphere of radius p_-. The Green function corresponding to this picture, at $\omega = 0$, has the form

$$G_\pm(0, p) = -\frac{a_\pm}{v_\pm(p - p_\pm)},$$

where $v_\pm$ is the velocity on the corresponding Fermi surface, and where $a_\pm$ is a positive constant. From this it follows that $G_\pm(0, p) > 0$ for $p < p_\pm$, and formula (9) gives

$$S = \tfrac{1}{2}(p_+^3/6\pi^2 - p_-^3/6\pi^2),$$

which expresses the desired relation.

---

[1] L. D. Landau, JETP **30**, 1058 (1956), Soviet Phys. JETP **3**, 920 (1957).

[2] L. P. Pitaevskiĭ, JETP **37**, 1794 (1959), Soviet Phys. JETP **10**, 1267 (1960).

[3] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960).

[4] Abrikosov, Gor'kov, and Dzyaloshinskiĭ, Metody kvantovoĭ teorii polya v statisticheskoĭ fizike (Methods of Quantum Field Theory in Statistical Physics), Fizmatgiz, 1962.

[5] I. E. Dzyaloshinskiĭ, Dissertation, Institute for Physical Problems, 1962.

Translated by W. F. Brown, Jr.
253