THEORY OF A FERROMAGNETIC FERMI FLUID

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The low temperature properties of a ferromagnetic Fermi fluid are investigated by quantum field theory methods.

THE problem of the determination of the equilibrium properties of a macroscopic body reduces to the estimation of the energy spectrum of the elementary excitations of this body. As was shown by Bloch[1] on the basis of the Heisenberg model, a ferromagnetic system possesses excitations corresponding to the oscillations of the mean magnetic moment and having the dispersion law $\omega \sim \mathbf{k}^2$. Landau and Lifshitz[2] have shown that this dispersion law is not the specific mark of the Heisenberg model. Further theories have developed in the direction of improvement of the previous theory; however, their feature is that the ferromagnet was regarded in them as a system of rigidly fixed spins, which, of course, does not readily occur in metals. Only Abrikosov and Dzyaloshinskiĭ[3] have considered spin waves in ferromagnetic metals, basing their work on the Landau phenomenological theory of a Fermi fluid. They showed that even in a metal, where the electrons are not rigidly fixed at the lattice nodes, the spin waves have the same dispersion law.

As had already been shown by Landau,[4,5] excitations associated with the oscillations of the spin density of the system can exist in a nonferromagnetic Fermi fluid. These oscillations have the same nature as zero sound, which corresponds to the deformation oscillations of the surface of the Fermi system. The spin oscillations noted have a linear dispersion law, just as zero sound.

In the present research we have attempted to ascertain whether a connection exists between the noted zero-sound spin waves and the spin waves discovered earlier in ferromagnetic dielectrics. It was shown that transverse spin oscillations of the zero-sound type do not take place in a ferromagnetic Fermi fluid. This is explained by the fact that there are two different Fermi surfaces for a non-vanishing mean macroscopic magnetic moment, corresponding to the two values of the spin direction. Since the difference of the limiting Fermi momenta of these surfaces is non-zero, transitions of particles from one of them to the other is impossible without energy and momentum change. This is indeed the physical reason for the different nature of spin waves in a ferromagnetic Fermi fluid.

In the present paper it is shown, by means of methods of quantum field theory and without the drawing up of any concrete models, that the spin waves in a ferromagnetic Fermi fluid have a quadratic dispersion law. The interaction of particles with spin waves has been considered and an expression has been found for the vertex corresponding to the emission of particles with zero momentum by the spin wave. It has been shown to be possible to express this in terms of quantities characterizing the Fermi spectrum of the excitations of the system. Account is taken of the magnetic dipole interaction of the spins. This interaction leads to a change in the spectrum of the spin waves that is identical in accuracy with that obtained earlier in the work of Holstein and Primakoff[6] for the case of a ferromagnetic dielectric.

It is shown in the present work that, besides the spin waves, the Fermi excitations of the system also make a contribution to the temperature dependence of the magnetic moment at low temperatures. This contribution is proportional to T^2. It was shown that the contributions to the heat capacity and the magnetic moment from the spin waves and from the Fermi excitations are independent in the lowest order in the temperature; one can therefore compute them exactly. The condition for the appearance of ferromagnetism is found to be a certain relation for the two-particle vertex part. Relations are obtained for the effective masses of the excitations, as well as an expression for the longitudinal static magnetic susceptibility at $T = 0$.

1. PROPERTIES OF THE VERTEX PART FOR SMALL TRANSFERRED MOMENTA

It is known that the Green's function G and the vertex part Γ play fundamental roles in the theory of a Fermi fluid. The Green's function

$$G_{\alpha\beta}(x_1 - x_2) = -i\langle T(\psi_\alpha(x_1)\psi_\beta^+(x_2))\rangle$$

for the case of a ferromagnetic Fermi fluid is a matrix which, from general considerations, can be written in the form

$$G_{\alpha\beta} = G_+\left(\tfrac{1}{2}\delta_{\alpha\beta} + \mathbf{m}\mathbf{s}_{\alpha\beta}\right) + G_-\left(\tfrac{1}{2}\delta_{\alpha\beta} - \mathbf{m}\mathbf{s}_{\alpha\beta}\right). \quad (1)$$

Here $\mathbf{m} = \mathbf{H}/H$, $\mathbf{H}$ is the external magnetic field; $\hat{\mathbf{s}} = (\hat{s}^x, \hat{s}^y, \hat{s}^z)$ is the spin matrix of a particle with spin $1/2$. For convenience, we always assume that the magnetic field is directed along the z axis. In this case,

$$G_{\alpha\beta} = G_+P^+_{\alpha\beta} + G_-P^-_{\alpha\beta}, \qquad P^\pm_{\alpha\beta} = \tfrac{1}{2}\delta_{\alpha\beta} \pm s^z_{\alpha\beta}. \quad (2)$$

In the momentum representation, the Green's function is simply expressed in terms of the self-energy part $\Sigma_{\alpha\beta}(p)$:

$$G^{-1}_{\alpha\beta}(p) = (\varepsilon + \mu - \mathbf{p}^2/2m)\,\delta_{\alpha\beta} - 2\mu_0 H s^z_{\alpha\beta} - \Sigma_{\alpha\beta}(p),$$

where

$$\Sigma_{\alpha\beta}(p) = \Sigma_+(p)\,P^+_{\alpha\beta} + \Sigma^-(p)\,P^-_{\alpha\beta} \quad (3)$$

μ_0 is the Bohr magneton and μ is the chemical potential of the system.

In the Fourier representation, the vertex part Γ is connected with the two-particle Green's function

$$G^{\mathrm{II}}_{\alpha\beta\gamma\delta}(x_1, x_2; x_3, x_4) = \langle T(\psi_\alpha(x_1)\psi_\beta(x_2)\psi^+_\gamma(x_3)\psi^+_\delta(x_4))\rangle$$

in the following way:

$$\begin{aligned} G^{\mathrm{II}}_{\alpha\beta\gamma\delta}&(p_1, p_2; p_3, p_1 + p_2 - p_3) \\ &= G_{\alpha\gamma}(p_1)\,G_{\beta\delta}(p_2)\,\delta(p_1 - p_3)\,(2\pi)^4 \\ &- G_{\alpha\delta}(p_1)\,G_{\beta\gamma}(p_2)\,\delta(p_2 - p_3)\,(2\pi)^4 \\ &+ iG_{\alpha\gamma_1}(p_1)\,G_{\beta\gamma_2}(p_2)\,G_{\gamma_3\gamma}(p_3)\,G_{\gamma_4\delta}(p_1 + p_2 - p_3) \\ &\times \Gamma_{\gamma_1\gamma_2\gamma_3\gamma_4}(p_1, p_2; p_3, p_1 + p_2 - p_3). \end{aligned}$$

Let us consider the behavior, at small transferred momenta, of the transverse components of the vertex part, which, as will be seen below, play the fundamental role in the investigation of the properties of a ferromagnetic Fermi fluid. These components of Γ correspond to the scattering of quasiparticles with spin flip. In the absence of magnetic spin-spin interactions, there are two: $\Gamma_{12,21}$ and $\Gamma_{21,12}$.

As Landau has shown,[5] the special features of the vertex part

$$\Gamma_{\alpha\beta\gamma\delta}(p_1, p_2 + k; p_1 + k, p_2) \equiv \Gamma_{\alpha\beta\gamma\delta}(p_1, p_2; k)$$

over the momentum transfer k correspond to the Bose excitation of the system. The presence of such singularities in the transverse components of Γ is easily understood from the following physical considerations. The magnetic moment of the system in a constant magnetic field can be written in the form $\mathbf{M} = M\mathbf{H}/H$. The static magnetic susceptibility in this case will be

$$\chi_{ik} = \frac{\partial M_i}{\partial H_k} = \frac{M}{H}\left(\delta_{ik} - \frac{H_i H_k}{H^2}\right) + \frac{\partial M}{\partial H}\frac{H_i H_k}{H^2}.$$

It is easy to see that the transverse components of χ become infinite for H = 0. On the other hand, χ is connected by a simple linear relation with the vertex part (see, for example, [7]), from which follows the presence of singularities in the transverse components of Γ. Evidently the excitations corresponding to these singularities of Γ are equivalent to oscillations of the mean value of the magnetic moment, i.e., to spin waves. Landau has shown[5] that in a nonferromagnetic Fermi fluid the singularities of Γ correspond to zero-sound vibrations. Let us see the connection between the zero-sound vibrations and the singularities corresponding to spin waves.

The presence of singularities of the first type is seen from the equation which is satisfied by the vertex part:

$$\begin{aligned} \Gamma_{\alpha\beta,\gamma\delta}(p_1, p_2; k) &= \Gamma^{(1)}_{\alpha\beta,\gamma\delta}(p_1, p_2; k) \\ &- i\int \frac{d^4q}{(2\pi)^4}\,\Gamma^{(1)}_{\alpha\varkappa_1,\beta\varkappa_2}(p_1, q; k)\,G_{\varkappa_2\varkappa_3}(q)\,G_{\varkappa_4\varkappa_1}(q + k) \\ &\times \Gamma_{\varkappa_3\beta,\varkappa_4\delta}(q, p_2; k), \end{aligned} \quad (4)$$

where $\Gamma^{(1)}$ is the set of all possible diagrams for Γ which cannot be cut by the two lines G(q) and G(q + k). The integral in (4) consists of regions far from the point $q^0 = 0$, $|\mathbf{q}| = p_\pm$ ($p_\pm$ is the Fermi momentum for the different spin directions), and regions in the vicinity of the given point, where the Green's function has the form

$$G_\pm(\varepsilon, p) = a_\pm/[\varepsilon - v_\pm(|\mathbf{p}| - p_\pm) + i\delta\,\mathrm{sign}(|\mathbf{p}| - p_\pm)], \quad (5)$$

$v_\pm$ is the velocity on the Fermi surface for the different spin directions. This velocity can be written in the form $v_\pm = p_\pm/m_\pm$, where $m_\pm$ is the effective mass of the excitations on the corresponding Fermi surfaces. This latter region of integration also makes a contribution to the singularity of Γ; this occurs, naturally, in the case in which the spin directions coincide, so that the poles of the Green's function come together as $k \to 0$.

As in the case of a nonferromagnetic Fermi fluid, the product of the Green's functions can be described in the form of a sum of special and regular parts:

$$G_{\varkappa_2\varkappa_3}(q)\,G_{\varkappa_4\varkappa_1}(q+k)=\frac{2\pi i a_+^2}{v_+}\,\frac{\mathbf{k}\mathbf{v}_+}{\omega-\mathbf{k}\mathbf{v}_+}\,\delta(q^0)\,\delta(|\mathbf{q}|-p_+)$$
$$\times P^+_{\varkappa_1\varkappa_2}P^+_{\varkappa_3\varkappa_4}+\frac{2\pi i a_-^2}{v_-}\,\frac{\mathbf{k}\mathbf{v}_-}{\omega-\mathbf{k}\mathbf{v}_-}\,\delta(q^0)\,\delta(|\mathbf{q}|-p_-)$$
$$\times P^-_{\varkappa_1\varkappa_2}P^-_{\varkappa_3\varkappa_4}+\varphi_{\varkappa_2\varkappa_3\varkappa_4\varkappa_1}(q) \qquad (6)$$

(φ does not contain singularities; therefore it is taken at k = 0). We introduce the quantities

$$\Gamma^\omega(p_1,p_2)=\lim_{|\mathbf{k}|/\omega,\,\omega\to 0}\Gamma(p_1,p_2;k),$$
$$\Gamma^k(p_1,p_2)=\lim_{\omega/|\mathbf{k}|,\,|\mathbf{k}|\to 0}\Gamma(p_1,p_2;k),$$
$$[G_\pm^2(p)]^\omega=\lim_{|\mathbf{k}|/\omega,\,\omega\to 0}G_\pm(p)\,G_\pm(p+k),$$
$$[G_\pm^2(p)]^k=\lim_{\omega/|\mathbf{k}|,\,|\mathbf{k}|\to 0}G_\pm(p)\,G_\pm(p+k).$$

Just as was done for the nonferromagnetic Fermi fluid,[5] we can easily obtain for Γ an equation in which the integration takes place only over the Fermi surface:

$$\Gamma_{\alpha\beta,\gamma\delta}(p_1,p_2;k)=\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_1,p_2)$$
$$+\frac{a_+^2p_+^2}{(2\pi)^3v_+}\int d\Omega_{\mathbf{q}}\Gamma^\omega_{\alpha1,\gamma1}(p_1,q)\,\frac{\mathbf{v}_+\mathbf{k}}{\omega-\mathbf{v}_+\mathbf{k}}\,\Gamma_{1\beta,1\delta}(q,p_2;k)$$
$$+\frac{a_-^2p_-^2}{(2\pi)^3v_-}\int d\Omega_{\mathbf{q}}\Gamma_{\alpha2,\gamma2}(p_1,q)\,\frac{\mathbf{v}_-\mathbf{k}}{\omega-\mathbf{v}_-\mathbf{k}}\,\Gamma_{2\beta,2\delta}(q,p_2;k). \qquad (7)$$

Then the connection between the two limiting values of Γ follows directly:

$$\Gamma^k_{\alpha\beta,\gamma\delta}(p_1,p_2)=\Gamma^\omega_{\alpha\beta,\gamma\delta}(p_1,p_2)$$
$$-\frac{a_+^2p_+^2}{(2\pi)^3v_+}\int d\Omega_{\mathbf{q}}\Gamma^\omega_{\alpha1,\gamma1}(p_1,q)\,\Gamma^k_{1\beta,1\delta}(q,p_2)$$
$$-\frac{a_-^2p_-^2}{(2\pi)^3v_-}\int d\Omega_{\mathbf{q}}\Gamma^\omega_{\alpha2,\gamma2}(p_1,q)\,\Gamma^k_{2\beta,2\delta}(q,p_2). \qquad (8)$$

In (7) and (8), the integration takes place for $q^0 = 0$ and q on the Fermi surface; $d\Omega_q$ is the element of solid angle.

It is not difficult to see that (7) separates into equations, which are not connected with each other, for longitudinal and transverse components of Γ. The equations for the longitudinal part of Γ determine the singularities, which correspond to zero-sound oscillations of the Fermi fluid. The equations for the transverse components have the trivial form

$$\Gamma_{12,21}(p_1,p_2;k)=\Gamma^\omega_{12,21}(p_1,p_2),$$
$$\Gamma_{21,12}(p_1,p_2;k)=\Gamma^\omega_{21,12}(p_1,p_2).$$

This means that the zero-sound excitations do not have an effect on the excitations of the spin-wave type.

We proceed to the investigation of the singularities of the transverse components of Γ. These singularities can be regarded as brought about by the presence of bound particle-hole states with nearly equal values of the momentum and opposite spin directions. From the unitarity relation it follows that, near the pole corresponding to the bound state mentioned, one can write down the following equation for the vertex part $\Gamma_{12,21}$:

$$G_+(p_1)\,G_-(p_1+k)\,\Gamma_{12,21}(p_1p_2;k)\,G_-(p_2+k)\,G_+(p_2)$$
$$=-\Delta_+(p_1,k)\,D(k)\,\Delta_-(p_2,k);$$
$$D(k)=1/[\omega-\omega(\mathbf{k})+i\delta]. \qquad (9)$$

Here $\omega(\mathbf{k})$ is the frequency of the spin waves.

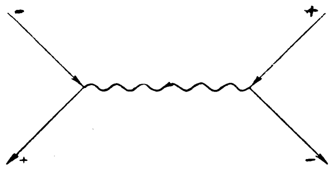

FIG. 1

We represent the right-hand side of (9) by graphs as shown in Fig. 1. The wavy line in the drawing corresponds to D(k). The plus and minus signs denote the signs of the spin direction relative to the external magnetic field.

Inasmuch as we are interested only in small k, we expand the quantities entering into (9) in terms of **k**. The relation (A.10) is obtained in the appendix and connects the component $\Gamma_{12,21}$ for **k** = 0 with the Green's function of the system. This formula can be used for finding the values at the pole of Γ in zero order of **k**. Actually, a comparison of (9) with (A.10) near the pole gives

$$\omega(0)=2\mu_0H, \qquad (10)$$
$$\Delta_\pm(p,k)=\sqrt{\mu_0/M_0}\,[G_+(p)-G_-(p)].$$

In the last equation, k corresponds to the polar value $\omega = 2\mu_0 H$, **k** = 0;

$$M_0=-i\mu_0\lim_{\tau\to+0}\int\frac{d^4p}{(2\pi)^4}e^{i\varepsilon\tau}[G_+(p)-G_-(p)]$$

is the mean value of the magnetic moment per unit volume of the system.

In order to obtain a representation of the next terms of the expansion of the quantity $\omega(\mathbf{k})$ in k and

$$\int\frac{d^4p}{(2\pi)^4}\Delta_\pm(p,k),$$

we determine the imaginary part of the function $D^{-1}(k)$ as k → 0. The graphs shown in Fig. 2 make a contribution to it. Using the unitarity relation, we can show that the graph of Fig. 2a

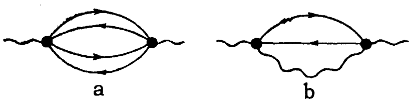

FIG. 2

makes a contribution $\backsim |\omega - 2\mu_0 H|^3$, and Fig. 2b, $\backsim |\omega - 2\mu_0 H|^{5/2}$. It is then clear that the expansion of the given quantities has an analytic form, at least up to terms of the order of $\mathbf{k}^4$ inclusively.

For the frequency of the spin waves, we have, with accuracy to $\mathbf{k}^2$,

$$\omega(\mathbf{k}) = 2\mu_0 H + \alpha k^2. \tag{11}$$

We see how the form of the pole of the vertex part changes with account of the relativistic spin-spin interaction. For this purpose, we need to sum the chain of graphs which includes this interaction. Then, by virtue of the fact that the interaction does not contain the spin, the quantity $D(k)$ will enter into the set of equations connecting it with the polar parts of the components $\Gamma_{11,22}$, $\Gamma_{22,11}$ for which we introduce the following notation:

$$G_+(p_1)\, G_-(p_1 + k)\, \Gamma_{11,22}(p_1, p_2; k)\, G_+(p_2 + k)\, G_-(p_2) = -\Delta_+(p_1, k)\, D^-(k)\, \Delta_+(p_2, k), \tag{12}$$

$$G_-(p_1)\, G_+(p_1 + k)\, \Gamma_{22,11}(p_1, p_2; k)\, G_-(p_2 + k)\, G_+(p_2) = -\Delta_-(p_1, k)\, D^+(k)\, \Delta_-(p_2, k). \tag{13}$$

We represent the pole vertices graphically as in Fig. 3.

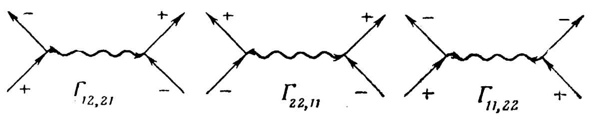

FIG. 3

This system of equations is represented graphically in Fig. 4, where the thin lines denote

$$D_0(k) = 1/[\omega - 2\mu_0 H - \alpha k^2 + i\delta].$$

$\Pi(k)$, $B_+(k)$, and $B_-(k)$ are the sets of diagrams containing the spin-spin interaction for the spin transitions $+- \to -+$, $-- \to ++$, and $++ \to --$, which cannot be cut by a single wavy line.

Solving the graphical equations, we get

$$D(k) = [\omega + A(k)]/[\omega^2 - A^2(k) + B_+(k)\, B_-(k)]. \tag{14}$$

$$D^\pm(k) = B_\pm(k)/[\omega^2 - A^2(k) + B_+(k)\, B_-(k)]. \tag{15}$$

Here we have introduced the notation

$$A(k) = 2\mu_0 H + \alpha k^2 + \Pi(k).$$

Let us determine $\Pi(k)$, $B_+(k)$, and $B_-(k)$. The magnetic spin-spin interaction can be regarded

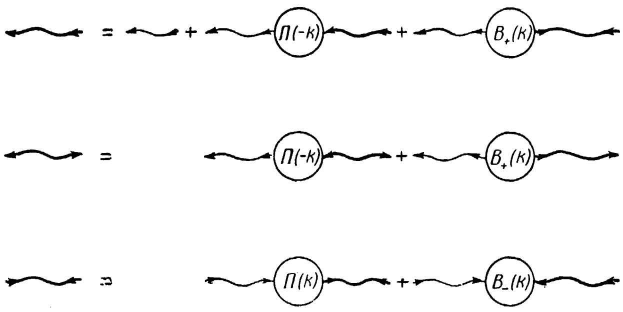

FIG. 4

as the interaction with the electromagnetic field, possessing the interaction Hamiltonian

$$\mathcal{H}_{int} = 2\mu_0 \int d\mathbf{r}\, \psi_\alpha^+(x)\, \mathbf{s}_{\alpha\beta} \psi_\beta(x)\, \mathbf{H}(x);$$

$\mathbf{H}(x)$ is the operator of the magnetic field. Inasmuch as this interaction is a small relativistic effect, we consider it in lowest order. It is not difficult to see that in this case the desired quantities are determined by the graphs pictured in Fig. 5.

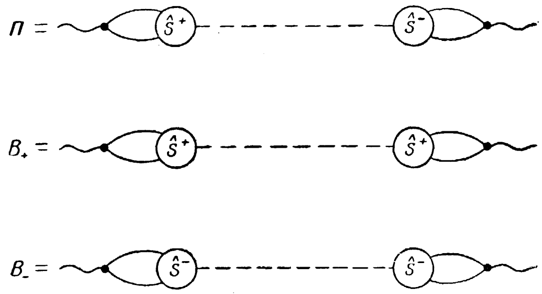

FIG. 5

The discontinuous line in the drawing corresponds to the Green's function of the magnetic field

$$F_{ik}(x_1 - x_2) = -i\langle T(H_i(x_1)\, H_k(x_2))\rangle.$$

It is clear that in zero order in the interaction it corresponds to the Green's function of the Maxwell equations without interaction:

$$F_{ik}(k) = 4\pi(k_i k_k/k^2 - \delta_{ik}).$$

However, when substituting F_{ik} in the expressions for $\Pi(k)$, $B_+(k)$, and $B_-(k)$ it is necessary to discard the term proportional to δ_{ik}, since it represents an interaction which commutes with the basic Hamiltonian; therefore, it reduces simply to renormalization of the initial conditions. Taking into account all of this, we can write

$$\Pi(k) = \frac{16\pi\mu_0^2 k_+ k_-}{k^2} \int \frac{d^4p_1 d^4p_2}{(2\pi)^8} \Delta_+(p_1, k)\, \Delta_-(p_2, k) = 4\pi\mu_0 M_0 \frac{k_+ k_-}{k^2}; \tag{16}$$

$$B_+(k) = \frac{16\pi\mu_0^2 k_-^2}{k^2}\left[\int \frac{d^4p}{(2\pi)^4}\Delta_-(p, k)\right]^2 = 4\pi\mu_0 M_0 \frac{k_-^2}{k^2}, \quad (17)$$

$$B_-(k) = \frac{16\pi\mu_0^2 k_+^2}{k^2}\left[\int \frac{d^4p}{(2\pi)^4}\Delta_+(p, k)\right]^2 = 4\pi\mu_0 M_0 \frac{k_+^2}{k^2};$$

$$k_\pm = k_x \pm ik_y. \quad (18)$$

In expanding the integrals in (16)—(18), we retain the lowest order in k. As will be seen below, this turns out to be entirely sufficient.

2. MAGNETIC MOMENT FOR T = 0

Dzyaloshinskiĭ[8] has shown that in the absence of relativistic spin-spin interaction, the magnetic moment per unit volume of a ferromagnetic Fermi fluid is equal to

$$M_0 = 2\mu_0 [p_+^3 - p_-^3]/3(2\pi)^2. \quad (19)$$

We calculate the correction to the expression (19) brought about by the spin-spin interaction. It is evident that we can write

$$\delta M = -2i\mu_0 \lim_{\tau\to+0} \mathrm{Sp} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau}\hat{s}^z \delta\hat{G}(p). \quad (20)$$

It is not difficult to verify directly that the change of the self-energy part due to relativistic interaction of the spins is

$$\delta\Sigma_{\alpha\beta}(p) = \int \frac{d^4p_1 d^4p_2 d^4k}{(2\pi)^{12}} Q_{\alpha\varkappa_1\varkappa_2,\beta\varkappa_3\varkappa_4}(p, p_1, p_2 + k; p_1 + k, p_2) \times \delta G^{II}_{\varkappa_3\varkappa_4,\varkappa_2\varkappa_1}(p_1, p_2; k); \quad (21)$$

Q is the three-particle vertex part. We can show that only small values of the momentum k are important in the calculation of $\delta\Sigma_{\alpha\beta}(p)$. Therefore, for δG^{II} we use the expression close to the pole in k. By virtue of this, taking into account the relation

$$\delta\hat{G}(p) = \hat{G}(p)\,\delta\hat{\Sigma}(p)\,\hat{G}(p)$$

and integrating (21) over p_1, p_2, we get

$$\delta M = -2i\mu_0 \lim_{\tau\to+0} \mathrm{Sp} \int \frac{d^4p\,d^4k}{(2\pi)^8} e^{i\varepsilon\tau}\hat{G}(p)\,\hat{s}^z\hat{G}(p) \times [\hat{Q}(p, k)(D(k) - D_0(k)) + \hat{Q}_+(p, k)D^-(k) + \hat{Q}_-(p, k)D^+(k)]. \quad (22)$$

From the graph of Fig. 6, it is seen that the quantities

$$2\mu_0 \lim_{\tau\to+0} \mathrm{Sp} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau}\hat{G}(p)\hat{s}^z\hat{G}(p)\ \hat{Q}(p, k),$$

$$2\mu_0 \lim_{\tau\to+0} \mathrm{Sp} \int \frac{d^4p}{(2\pi)^4} e^{i\varepsilon\tau}\hat{G}(p)\,\hat{s}^z\,\hat{G}(p)Q_\pm(p, k)$$

are derivatives with respect to the magnetic field of A(k) and $B_\pm(k)$, respectively. Making use of

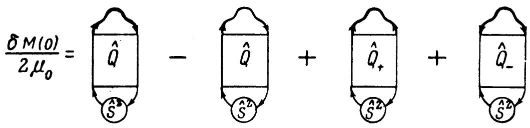

FIG. 6

the explicit expressions for A(k) and $B_\pm(k)$, we get

$$\delta M(0) = -2i\mu_0 \int \frac{d^4k}{(2\pi)^4}[D(k) - D_0(k)]. \quad (23)$$

Substituting

$$D(\omega, \mathbf{k}) = [\omega + A(\mathbf{k})]/(\omega - \omega(\mathbf{k}) + i\delta)(\omega + \omega(\mathbf{k}) - i\delta),$$

where

$$\omega(\mathbf{k}) = \sqrt{A^2(\mathbf{k}) - B_+(\mathbf{k})B_-(\mathbf{k})},$$

we find

$$\delta M(0) = -\mu_0 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{A(\mathbf{k}) - \omega(\mathbf{k})}{\omega(\mathbf{k})}. \quad (24)$$

The latter formula is identical with the result obtained by Holstein and Primakoff[6] for the case of a ferromagnetic dielectric.

Thus the total magnetic moment of the system is determined, with accuracy up to quantities of first order in the magnetic spin interaction, by the formula

$$M(0) = \frac{2\mu_0}{3(2\pi)^2}[p_+^3 - p_-^3] - \mu_0 \int \frac{d\mathbf{k}}{(2\pi)^3} \frac{A(\mathbf{k}) - \omega(\mathbf{k})}{\omega(\mathbf{k})}. \quad (25)$$

3. THE CONDITION FOR ONSET OF FERROMAGNETISM

It is well known that the self-energy part changes, for an infinitesimally small change in the Green's function, according to the equation

$$\delta\Sigma_{\alpha\beta}(p) = -i\int \frac{d^4p'}{(2\pi)^4} \Gamma^{(1)}_{\alpha\varkappa_1,\beta\varkappa_2}(p, p')\,\delta G_{\varkappa_2\varkappa_1}(p'). \quad (26)$$

We assume that the total magnetic moment of the system is changed by some infinitesimally small angle. In this case the Green's function and the self-energy part change in the following way in first order, as follows from (1)—(3):

$$\delta G_{\alpha\beta}(p) = [G_+(p) - G_-(p)](\delta m_+ s^-_{\alpha\beta} + \delta m_- s^+_{\alpha\beta})/2,$$
$$\delta\Sigma_{\alpha\beta}(p) = [\Sigma_+(p) - \Sigma_-(p)](\delta m_+ s^-_{\alpha\beta} + \delta m_- s^+_{\alpha\beta})/2,$$
$$m_\pm = m_x \pm im_y, \qquad \hat{s}^\pm = \hat{s}^x \pm i\hat{s}^y. \quad (27)$$

Substituting (27) in (26), and neglecting the magnetic spin-spin interactions, we find

$$\Sigma_+(p) - \Sigma_-(p) = -i\int \frac{d^4p'}{(2\pi)^4}\Gamma^{(1)}_{12,21}(p, p')[G_+(p') - G_-(p')]. \quad (28)$$

If the system considered is close to the ferromagnetic transition, then the quantities $\Sigma_+ - \Sigma_-$, $G_+ - G_-$ can be regarded as small; all factors with them can be set equal to zero.

Inasmuch as $\Gamma^{(1)}$ does not possess singularities, it can be expressed in terms of the value of the paramagnetic phase. Making use of operator notation, we write down the equation for the connection between $\Gamma^{(1)}$ and $\Gamma^{(2)}$, similar to (4):

$$\Gamma^{\omega} = \Gamma^{(1)} - i\Gamma^{\omega}[G^2]^{\omega}\Gamma^{(1)}. \qquad (29)$$

Substituting (29) in (28), we get

$$\{1 - i\Gamma^{\omega}[G^2]^{\omega}\}(\Sigma_+ - \Sigma_-) = -i\Gamma^{\omega}(G_+ - G_-).$$

Further, since

$$[G^2(p)]^{\omega} = [G^2(p)]^k + \frac{2\pi i a^2}{v}\delta(\varepsilon)\delta(|\mathbf{p}| - p_0),$$

$$[G^2(p)]^k(\Sigma_+(p) - \Sigma_-(p)) = G_+(p) - G_-(p),$$

we get, finally,

$$\Sigma_+(p) - \Sigma_-(p) = -\int\frac{d^4p'}{(2\pi)^4}\Gamma^{\omega}_{12,21}(p, p')\delta(\varepsilon')$$

$$\times\,\delta(|\mathbf{p}| - p_0)(\Sigma_+(p') - \Sigma_-(p')). \qquad (30)$$

Here p_0 is the limiting Fermi momentum of the paramagnetic phase. For the paramagnetic state, $\Sigma_+(p) - \Sigma_-(p) \equiv 0$; therefore, the equation (30) is satisfied identically, thereby placing no restrictions on the value of $\Gamma^{\omega}_{12,21}$.

For the ferromagnetic state, by virtue of the non-zero difference in the self-energy parts for different spin directions, (30) takes on the meaning of the relation for $\Gamma^{\omega}_{12,21}(p, p')$:

$$\frac{a^2p_0^2}{(2\pi)^3 v}\int d\Omega\,\Gamma^{\omega}_{12,21}(p, p') = -1,$$

$$\varepsilon = \varepsilon' = 0, \quad |\mathbf{p}| = |\mathbf{p}'| = p_0. \qquad (31)$$

This is also the desired condition for the onset of ferromagnetism.

4. THE MAGNETIC MOMENT AND THE HEAT CAPACITY OF A FERROMAGNETIC FERMI FLUID AT LOW TEMPERATURES

At non-zero temperatures, the mean value of the magnetic moment per unit volume of the system is given by the formula

$$M(T) = 2\mu_0 T\,\mathrm{Sp}\sum_{\varepsilon} e^{i\varepsilon\tau}\int\frac{d\mathbf{p}}{(2\pi)^3}\hat{s}^z\hat{\mathfrak{G}}(T; \varepsilon, \mathbf{p})$$

$$\text{for } \tau \to +0, \qquad (32)$$

where $G_{\alpha\beta}(T; \epsilon, \mathbf{p})$ is the temperature Green's function, defined as in the book of Abrikosov, Gor'kov and Dzyaloshinskiĭ.[9]

In the limiting case at T = 0, the summations over the imaginary frequencies must be replaced by integrations in all the diagrams, in accord with the formula

$$T\sum_{\varepsilon} \to \int\frac{d\varepsilon}{2\pi}.$$

For low temperatures, the mean value of the magnetic moment will obviously differ from the corresponding value at T = 0 by a small temperature contribution:

$$M(T) = M(0) + \delta M(T).$$

The method of calculating the temperature correction to mean physical quantities is described in [9]. Its substance is the following: to obtain the temperature correction to the self-energy part $\Sigma_{\alpha\beta}(0; \epsilon, \mathbf{p})$, it is necessary that all summations over frequencies in the corresponding set of temperature technique diagrams be replaced by integrations, with the exception of a single frequency, over which one must sum; the case in which the summation is carried out over two frequencies has a small statistical weight and leads to a temperature dependence of higher order. The frequency shown, over which summation takes place, can, for example, be the energy Green's function. Then, as shown in [9], the temperature corrections are determined by the poles of the Green's function at low energies, i.e., by the spectrum of Fourier excitations of the system. However, one can use the transferred energy of the two-particle Green's function as the frequency to which the given operation is applied. Then, as will be seen below, the corrections will be determined by the Bose excitation spectrum of the system.

We shall be interested in the contributions to the magnetic moment and the heat capacity of the Fermi fluid due only to Fermi excitations and spin waves. It can be shown that the other branches of the Bose spectrum make a contribution to the temperature dependence of much higher order. We shall first calculate the contribution to the temperature dependence of the Fermi excitations.

According to [9], the correction to the temperature Green's function due to the Fermi excitations is determined by the expression

$$\delta\mathfrak{G}^F_{\alpha\beta} = \mathfrak{G}_{\alpha\gamma_1}(0; \varepsilon, \mathbf{p})\left[T\sum_{\varepsilon_1} - \int\frac{d\varepsilon_1}{2\pi}\right]$$

$$\times\int\frac{a\mathbf{p}_1}{(2\pi)^3}\Gamma_{\gamma_1\gamma_2,\gamma_3\gamma_4}(0; \varepsilon, \mathbf{p}; \varepsilon_1, \mathbf{p}_1)$$

$$\times\,\mathfrak{G}_{\gamma_4\gamma_2}(0; \varepsilon_1, \mathbf{p}_1)\mathfrak{G}_{\gamma_3\beta}(0; \varepsilon, \mathbf{p}). \qquad (33)$$

Graphically, this equation corresponds to Fig. 7. Here and in what follows, the canceling signs on

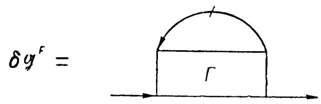

FIG. 7

lines indicate that in place of integration over the frequency of the line, as is done in the frequencies of the uncanceled lines, we should apply the operation

$$\left[T\sum_{\varepsilon}-\int d\varepsilon/2\pi\right]. \qquad (34)$$

It is not difficult to see that the correction to the magnetic moment of the system from the Fermi excitation is

$$\delta M_F(T)=\mathrm{Sp}\left[T\sum_{\varepsilon}-\int\frac{d\varepsilon}{2\pi}\right]\int\frac{d\mathbf{p}}{(2\pi)^3}\hat{s}^z\hat{\mathfrak{G}}(0;\,\varepsilon,\,\mathbf{p})$$
$$+\mathrm{Sp}\int\frac{d\varepsilon d\mathbf{p}}{(2\pi)^4}\hat{s}^z\delta\hat{\mathfrak{G}}^F(T;\,\varepsilon,\,\mathbf{p}). \qquad (35)$$

The latter formula corresponds to Fig. 8.

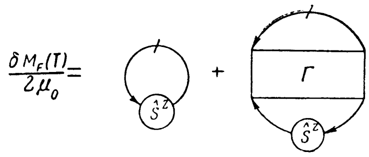

FIG. 8

In the Appendix, Eq. (A.3) is introduced for the derivative of the Green's function with respect to the magnetic field for T = 0. In a completely similar way, one can get the corresponding formula in the temperature technique:

$$\frac{\partial}{\partial H}\mathfrak{G}^{-1}_{\alpha\beta}(T;\,\varepsilon,\,p)$$
$$=-2\mu_0\left\{s^z_{\alpha\beta}+T\sum_{\varepsilon_1}\int\frac{dp}{(2\pi)^3}\Gamma_{\alpha\varkappa_1,\,\beta\varkappa_2}(T;\,\varepsilon,\,\mathbf{p};\,\varepsilon_1,\,\mathbf{p}_1)\right.$$
$$\left.\times\,[\hat{\mathfrak{G}}(T;\,\varepsilon_1,\,\mathbf{p}_1)\hat{s}^z\hat{\mathfrak{G}}(T;\,\varepsilon_1,\,\mathbf{p}_1)]_{\varkappa_2\varkappa_1}\right\}.$$

In the limit as $T\rightarrow 0$ and ϵ = const, we find

$$\frac{\partial}{\partial H}\mathfrak{G}^{-1}_{\alpha\beta}(0;\,\varepsilon,\,p)$$
$$=-2\mu_0\left\{s^z_{\alpha\beta}+\int\frac{d\varepsilon_1 d\mathbf{p}_1}{(2\pi)^4}\Gamma_{\alpha\varkappa_1,\,\beta\varkappa_2}(0;\,\varepsilon,\,\mathbf{p};\,\varepsilon_1,\,\mathbf{p}_1)\right.$$
$$\left.\times\,[\hat{\mathfrak{G}}(0;\,\varepsilon_1,\,\mathbf{p}_1)\hat{s}^z\hat{\mathfrak{G}}(0;\,\varepsilon_1,\,\mathbf{p}_1)]_{\varkappa_2\varkappa_1}\right\}. \qquad (36)$$

By combining (35) with (36) we get

$$\delta M_F(T)=\left[T\sum_{\varepsilon}-\int\frac{d\varepsilon}{2\pi^3}\right]$$
$$\times\int\frac{dp}{(2\pi)^3}\mathfrak{G}_{\alpha\beta}(0;\,\varepsilon,\,\mathbf{p})\frac{\partial}{\partial H}\mathfrak{G}^{-1}_{\beta\alpha}(0;\,\varepsilon,\,\mathbf{p}). \qquad (37)$$

In order to calculate the contribution of spin waves to the temperature dependence of the mean quantities, it is necessary to apply the operation (34) to the transferred energy of the two-particle Green's function corresponding to scattering of particles with reversal of spins, and since the desired corrections are determined by the poles, then in the given case, ω is the frequency of the pole functions. For simplicity, we shall not take into account the relativistic spin-spin interaction in these calculations. Then the contribution of the spin waves to the magnetic moment is determined graphically as shown in Fig. 9. This corresponds to the formula

$$\delta M_s(T)=-2\mu_0\,\mathrm{Sp}\left[T\sum_{\omega}-\int\frac{d\omega}{2\pi}\right]$$
$$\times\int\frac{d\varepsilon\,d\mathbf{p}}{(2\pi)^4}\frac{d\mathbf{k}}{(2\pi)^3}\hat{\mathfrak{G}}(0;\,\varepsilon,\,\mathbf{p})\hat{s}^z\hat{\mathfrak{G}}(0;\,\varepsilon,\,\mathbf{p})$$
$$\times\,\hat{Q}(0;\,\varepsilon,\,\mathbf{p};\,\omega,\,\mathbf{k})D_0(0;\,\omega,\,\mathbf{k}). \qquad (38)$$

Here $\hat{Q}(0;\,\epsilon,\,\mathbf{p};\,\omega,\,\mathbf{k})$ is the vertex corresponding to scattering of spin waves by the particle, similar to $\hat{Q}(\mathrm{p},\mathrm{k})$ which was used for T = 0 in (22). It is obvious that, similarly to (36), we can write

$$\frac{\partial D_0^{-1}(0;\,\omega,\,\mathbf{k})}{\partial H}=-\mathrm{Sp}\int\frac{d\varepsilon\,d\mathbf{p}}{(2\pi)^4}\hat{\mathfrak{G}}(0;\,\varepsilon,\,\mathbf{p})\hat{s}^z\hat{\mathfrak{G}}(0;\,\varepsilon,\,\mathbf{p})$$
$$\times\,\hat{Q}(0;\,\varepsilon,\,\mathbf{p};\,\omega,\,\mathbf{k}). \qquad (39)$$

Then substitution of (39) in (38) gives

$$\delta M_s(T)=\left[T\sum_{\omega}-\int\frac{d\omega}{2\pi}\right]\int\frac{d\mathbf{k}}{(2\pi)^3}D_0(0;\,\omega,\,\mathbf{k})\frac{\partial}{\partial H}D_0^{-1}(0;\,\omega,\,\mathbf{k}).$$

Then, in lowest order in temperature, the magnetic field is determined by the formula

$$M(T)=M(0)+\left[T\sum_{\varepsilon}-\int\frac{d\varepsilon}{2\pi}\right]$$
$$\times\int\frac{d\mathbf{p}}{(2\pi)^3}\mathfrak{G}_{\alpha\beta}(0;\,\varepsilon,\,\mathbf{p})\frac{\partial}{\partial H}\mathfrak{G}^{-1}_{\beta\alpha}(0;\,\varepsilon,\,\mathbf{p})+\left[T\sum_{\omega}-\int\frac{d\omega}{2\pi}\right]$$
$$\times\int\frac{d\mathbf{k}}{(2\pi)^3}D_0(0;\,\omega,\,\mathbf{k})\frac{\partial}{\partial H}D_0^{-1}(0;\,\omega,\,\mathbf{k}). \qquad (40)$$

The expressions in (40) can be expanded by the method used in calculation of the heat capacity of a Fermi fluid (see [9]). As a result, we get

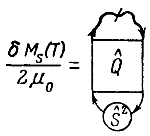

FIG. 9

$$M(T)=M(0)-2\mu_0\int\frac{d\mathbf{k}}{(2\pi)^3}\left[\exp\frac{\omega(\mathbf{k})}{T}-1\right]^{-1}$$
$$+\frac{T^2}{12}\frac{\partial}{\partial H}(p_+m_++p_-m_-). \qquad (41)$$

We recall that $p_\pm$ is the limiting Fermi momentum and $m_\pm$ is the effective mass of the Fermi excitations for different spin directions.

In order to determine the heat capacity per unit volume of the system, we use the well-known thermodynamic formula $(\partial M/\partial T)_H = (\partial S/\partial H)_T$, where S is the entropy per unit volume. We then find from (41)

$$C=T\left(\frac{\partial S}{\partial T}\right)_H=\frac{T}{6}(p_+m_++p_-m_-)$$
$$+\frac{\partial}{\partial T}\int\frac{d\mathbf{k}}{(2\pi)^3}\,\omega(\mathbf{k})\left[\exp\frac{\omega(\mathbf{k})}{T}-1\right]^{-1}. \qquad (42)$$

We consider two limiting cases for the values of the magnetic moment and the heat capacity. If the temperature is sufficiently large, $T \gg 2\mu_0 H$, then

$$M(T)=M(0)-\frac{\zeta(^3/_2)}{4\pi^{3/2}}\mu_0\left(\frac{T}{\alpha}\right)^{3/2}+\frac{T^2}{12}\frac{\partial}{\partial H}(p_+m_++p_-m_-),$$
$$C=\frac{T}{6}(p_+m_++p_-m_-)+\frac{15}{32}\frac{\zeta(^5/_2)}{\pi^{3/2}}\left(\frac{T}{\alpha}\right)^{3/2}; \qquad (43)$$

$\zeta(x)$ is the Riemann ζ function.

In the temperature range $T \ll 2\mu_0 H$, we have

$$M(T)=M(0)+\frac{T^2}{12}\frac{\partial}{\partial H}(p_+m_++p_-m_-)$$
$$-\frac{\mu_0}{4\pi^{3/2}}\left(\frac{T}{\alpha}\right)^{3/2}e^{-2\mu_0H/T},$$
$$C=\frac{T}{6}(p_+m_++p_-m_-)+\frac{1}{8\pi^{3/2}}\left(\frac{2\mu_0H}{\alpha}\right)^{3/2}\left(\frac{2\mu_0H}{T}\right)^{1/2}e^{-2\mu_0H/T}. \qquad (44)$$

For the case $T \gg 2\mu_0 H$, we investigate the problem as to whether the contributions to M and C from the Fermi excitations and from the spin waves intersect. The presence of the spin waves does not introduce into the Fermi spectrum of the excitations any singularities which would be essential in the calculation of the first temperature corrections. These singularities would appear in the imaginary part of the Green's function. However, the lowest order contribution of the spin waves to the damping of the Fermi excitations is determined by the graph of Fig. 10, the value of which $\backsim |\epsilon-\mu|^{7/2}$. Therefore, the subsequent temperature term in the heat capacity that is associated with the Fermi excitations $\backsim T^{7/2}\ln T$; the term in the magnetic moment $\backsim T^{9/2}\ln T$. The subsequent temperature terms in M and C from the spin waves are proportional to $T^{5/2}$, which follows from the expansion of $\omega(\mathbf{k})$ discussed in Sec. 1.

Thus the Fermi excitations and the spin waves in the lowest order in the temperature make independent contributions to the heat capacity and the magnetic moment of the ferromagnetic Fermi liquid.

In conclusion, the author expresses his deep gratitude to I. E. Dzyaloshinskiĭ for suggesting the problem, for advice and constant interest in the work.

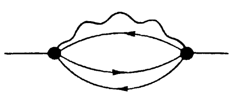

FIG. 10

APPENDIX

We derive several formulas which connect the quantities characterizing the Fermi excitation spectrum with the vertex part of the interaction of the particles. These formulas are necessary for investigation of the singularities of the transverse components of the vertex part, and also for obtaining the temperature corrections to the thermodynamic quantities at low temperatures. With their help, we get relations for the effective masses of the quasiparticles on the Fermi surface and an expression for the longitudinal static susceptibility of the ferromagnetic Fermi fluid.

To obtain the desired formulas we shall assume that our system is located in an additional, infinitesimally small, external field, which has the corresponding interaction Hamiltonian

$$\mathcal{H}_{int}=\int d\mathbf{r}\psi_\alpha^+(x)\,\delta U_{\alpha\beta}(x)\,\psi_\beta(x). \qquad (A.1)$$

Change in the Green's function of the system in first order in the value of the external field is determined graphically in Fig. 11, which corresponds, in the momentum representation, to the equation

$$-\delta G_{\alpha\beta}^{-1}=\delta U_{\alpha\beta}(k)-i\int\frac{d^4q}{(2\pi)^4}\Gamma_{\alpha\varkappa_1,\beta\varkappa_2}(p,q;k)$$
$$\times[\hat G(q)\,\delta\hat U(k)\,\hat G(q+k)]_{\varkappa_2\varkappa_1}. \qquad (A.2)$$

As above, we shall assume that the system is in a magnetic field H which is taken into account exactly in all the quantities. In subsequent calcu-

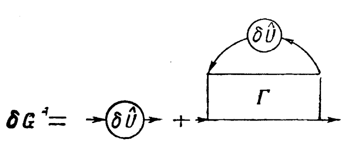

FIG. 11

lations, we shall neglect the relativistic spin-spin interactions everywhere.

To obtain the first consequence of the relation (A.2), we insert the additional constant, inhomogeneous magnetic field $h(x) = he^{i\mathbf{k}\cdot\mathbf{r}}$, which, as also H, is directed along the z axis. In this case, $\delta U_{\alpha\beta}(x) = 2\mu_0 s^z_{\alpha\beta} he^{i\mathbf{k}\cdot\mathbf{r}}$. As $\mathbf{k} \to 0$, we get from (A.2)

$$\frac{\partial}{\partial H} G^{-1}_{\alpha\beta}(p) = -2\mu_0 \left\{ s^z_{\alpha\beta} - i \int \frac{d^4q}{(2\pi)^4} \Gamma^k_{\alpha\varkappa_1, \beta\varkappa_2}(p, q) [\hat{G}(q) \hat{s}^z \hat{G}(q)]^k_{\varkappa_2\varkappa_1} \right\}. \tag{A.3}$$

It is evident that

$$[\hat{G}(p) \hat{s}^z \hat{G}(p)]^{k,\omega}_{\alpha\beta} = \frac{1}{2} [G^2_+(p)]^{k,\omega} P^+_{\alpha\beta} - \frac{1}{2} [G^2_-(p)]^{k,\omega} P^-_{\alpha\beta}.$$

In order to obtain a second relation, we place the system in an additional, inhomogeneous variable magnetic field $h(t) = he^{-i\omega t}$ directed along the z axis. As $\omega \to 0$, we find for the correction to the Green's function [from (A.2)],

$$\delta G^{-1}_{\alpha\beta} = -2\mu_0 h \left\{ s^z_{\alpha\beta} - i \int \frac{d^4q}{(2\pi)^4} \Gamma^\omega_{\alpha\varkappa_1, \beta\varkappa_2}(p, q) \times [\hat{G}(q) \hat{s}^z \hat{G}(q)]^\omega_{\varkappa_2\varkappa_1} \right\}. \tag{A.4}$$

On the other hand, one can compute the value of the correction to the Green's function directly. Since

$$\mathcal{H}_{int} = 2\mu_0 he^{-i\omega t} \int d\mathbf{r} \psi^+_\alpha(x) s^z_{\alpha\beta} \psi_\beta(x) \equiv he^{-i\omega t} \hat{M}_z$$

and inasmuch as the magnetic moment is conserved, i.e., $\hat{M}$ commutes with the basic Hamiltonian, the change in the quantum operator is determined by the formula

$$\hat{\tilde{\psi}}(x) = \exp\left[i\hat{M}_z \int_0^t h(t')\, dt' \right] \hat{\psi}(x) \exp\left[-i\hat{M}_z \int_0^t h(t')\, dt' \right]. \tag{A.5}$$

But as $\omega \to 0$, we have

$$\int_0^t h(t')\, dt' \to ht$$

and we get from (A.5)

$$\tilde{\psi}_\alpha(\mathbf{r}, t) = \{\exp[-2i\mu_0 \hat{s}^z ht]\}_{\alpha\beta} \psi_\beta(\mathbf{r}, t). \tag{A.5$'$}$$

For the Green's function in the Fourier representation in spatial coordinates, we get

$$\tilde{G}_{\alpha\beta}(\mathbf{p}, t_1, t_2) = \{\exp(-2i\mu_0 \hat{s}^z ht_1)\}_{\alpha\gamma} G_{\gamma\delta}(\mathbf{p}, t_1 - t_2) \times \{\exp(2i\mu_0 \hat{s}^z ht_2)\}_{\delta\beta}$$

$$= G_{\alpha\gamma}(\mathbf{p}, t_1 - t_2) \{\exp[-2i\mu_0 \hat{s}^z h(t_1 - t_2)]\}_{\gamma\beta}.$$

Finally, in the Fourier time representation, we find

$$\tilde{G}_{\alpha\beta}(\varepsilon, \mathbf{p}) = G_+(\varepsilon - \mu_0 h, \mathbf{p}) P^+_{\alpha\beta} + G_-(\varepsilon + \mu_0 h, \mathbf{p}) P^-_{\alpha\beta}.$$

Combining this equation with (A.4), we get

$$\frac{\partial}{\partial\varepsilon} [\hat{s}^z \hat{G}^{-1}(p)]_{\alpha\beta} = s^z_{\alpha\beta} - i \int \frac{d^4q}{(2\pi)^4} \Gamma^\omega_{\alpha\varkappa_1, \beta\varkappa_2}(p, q) \times [\hat{G}(q) \hat{s}^z \hat{G}(q)]^\omega_{\varkappa_2\varkappa_1}. \tag{A.6}$$

To obtain the following relations, we assume that the system is placed in an additional, alternating homogeneous transverse magnetic field $h_x(t) \pm ih_y(t) = h^\pm e^{\pm i\omega t}$. Here

$$\mathcal{H}_{int} = \mathcal{H}_1 + \mathcal{H}^+_1, \qquad \mathcal{H}_1 = \mu_0 \int d\mathbf{r} \psi^+_\alpha(x) s^-_{\alpha\beta} \psi_\beta(x) h^+ e^{i\omega t},$$

and $\mathcal{H}^+_1$ is the value of the Hermitian conjugate of $\mathcal{H}_1$. Then, in first order in the infinitely small field, the particle operators are represented in the following form:

$$\tilde{\psi}_\alpha(x) = \psi_\alpha(x) + i \int_{-\infty}^t dt' [V(t'), \psi_\alpha(x)], \tag{A.7}$$

where

$$V(t) = e^{i\mathcal{H}_0 t} \mathcal{H}_{int} e^{-i\mathcal{H}_0 t}.$$

Denoting

$$V_1(t) = e^{i\mathcal{H}_0 t} \mathcal{H}_1 e^{-i\mathcal{H}_0 t},$$

we write down the equation for $V_1(t)$ in the form

$$\partial V_1(t) / \partial t = ie^{i\mathcal{H}_0 t} [\mathcal{H}_0, \mathcal{H}_1] e^{-i\mathcal{H}_0 t} + i\omega V_1(t).$$

The only term in $\mathcal{H}_0$ which makes a contribution to the commutator $[\mathcal{H}_0, \mathcal{H}_1]$ is

$$2\mu_0 \int d\mathbf{r} H \psi^+_\alpha(x) s^z_{\alpha\beta} \psi_\beta(x).$$

By taking into account the commutation rule of Fermi operators, we get

$$\partial V_1(t) / \partial t = i(\omega - 2\mu_0 H) V_1(t).$$

Then

$$V_1(t) = \mu_0 h^+ e^{i(\omega - 2\mu_0 H)t} \int d\mathbf{r} \psi^+_\alpha(x) s^-_{\alpha\beta} \psi_\beta(x).$$

Substituting the last equation in (A.7) and taking it into account that $V(t) = V_1(t) + V^+_1(t)$, we get

$$\tilde{\psi}_\alpha(x) = S_{\alpha\beta}(t) \psi_\beta(x), \tag{A.8}$$

where

$$S_{\alpha\beta}(t) = \delta_{\alpha\beta} + \frac{\mu_0 s^-_{\alpha\beta} h^+}{\omega - 2\mu_0 H - i\delta} e^{i(\omega - 2\mu_0 H)t} - \frac{\mu_0 s^+_{\alpha\beta} h^-}{\omega - 2\mu_0 H + i\delta} e^{-i(\omega - 2\mu_0 H)t}. \tag{A.9}$$

Finally, we get from (A.8) and (A.9),

$$\delta G_{\alpha\beta}(p, p') = \frac{\mu_0 s^-_{\alpha\beta} h^+}{\omega - 2\mu_0 H - i\delta}[G_+(p+k') - G_-(p)]$$
$$\times \delta(p' - p - k') + \frac{\mu_0 s^+_{\alpha\beta} h^-}{\omega - 2\mu_0 H + i\delta}$$
$$\times [G_+(p) - G_-(p-k')]\,\delta(p - p' - k'); \qquad (A.10)$$

here $k' = (\omega - 2\mu_0 H, 0)$.

This correction can be obtained from the general rule with the help of (A.2). As a result of comparison of the two results, we get the following equations:

$$G_+(p)\,G_-(p-k)$$
$$\times\left[1 - i\int \frac{d^4q}{(2\pi)^4}\Gamma_{12,21}(p, q, -k)\,G_+(q)\,G_-(q-k)\right]$$
$$= \frac{1}{\omega - 2\mu_0 H + i\delta}[G_+(p) - G_-(p-k')], \qquad (A.10')$$

$$G_-(p)\,G_+(p+k)$$
$$\times\left[1 - i\int \frac{d^4q}{(2\pi)^4}\Gamma_{21,12}(p, q; k)\,G_-(q)\,G_+(q+k)\right]$$
$$= \frac{1}{\omega - 2\mu_0 H - i\delta}[G_+(p+k') - G_-(p)]; \qquad (A.11)$$

Here $k = (\omega, 0)$.

Just as was done for the nonferromagnetic Fermi fluid,[10] one can get the following relations:

$$\frac{\partial}{\partial\varepsilon}G^{-1}_{\alpha\beta}(p) = \delta_{\alpha\beta} - i\int\frac{d^4q}{(2\pi)^4}\Gamma^{\omega}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[G^2(q)]^{\omega}_{\varkappa_2\varkappa_1}, \qquad (A.12)$$

$$\frac{\partial}{\partial\mathbf{p}}G^{-1}_{\alpha\beta}(p) = -\frac{\mathbf{p}}{m}\delta_{\alpha\beta}$$
$$+ i\int\frac{d^4q}{(2\pi)^4}\frac{\mathbf{q}}{m}\Gamma^{k}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[G^2(q)]^{k}_{\varkappa_2\varkappa_1}, \qquad (A.13)$$

$$\mathbf{p}\frac{\partial}{\partial\varepsilon}G^{-1}_{\alpha\beta}(p) = \mathbf{p}\delta_{\alpha\beta}$$
$$- i\int\frac{d^4q}{(2\pi)^4}\mathbf{q}\Gamma^{\omega}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[G^2(q)]^{\omega}_{\varkappa_2\varkappa_1}, \qquad (A.14)$$

$$\frac{\partial}{\partial\mu}G^{-1}_{\alpha\beta}(p) = \delta_{\alpha\beta}$$
$$- i\int\frac{d^4q}{(2\pi)^4}\Gamma^{k}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[G^2(q)]^{k}_{\varkappa_2,\varkappa_1}; \qquad (A.15)$$

here

$$[G^2(p)]^{k,\omega}_{\alpha\beta} = [G^2_+(p)]^{k,\omega}P^+_{\alpha\beta} + [G^2_-(p)]^{k,\omega}P^-_{\alpha\beta}.$$

We now obtain relations connecting the effective masses of the quasiparticles with the values on the Fermi surface. Close to the Fermi surface, we get from (5), as $\epsilon \to 0$,

$$\frac{\partial}{\partial\varepsilon}G^{-1}_{\pm} = \frac{1}{a_\pm}, \qquad \frac{\partial}{\partial\mathbf{p}}G^{-1}_{\pm} = -\frac{\mathbf{v}_\pm}{a_\pm} = -\frac{\mathbf{p}}{m_\pm a_\pm}. \qquad (A.16)$$

In order to get the desired relations for the effective masses, we substitute (8) in Eq. (A.13). In this case, we find

$$-\frac{\mathbf{p}}{m_+a_+}P^+_{\alpha\beta} - \frac{\mathbf{p}}{m_-a_-}P^-_{\alpha\beta} + \frac{\mathbf{p}}{m}\delta_{\alpha\beta}$$
$$= i\int\frac{d^4q}{(2\pi)^4}\frac{\mathbf{q}}{m}\Gamma^{\omega}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[G^2(q)]^{k}_{\varkappa_2\varkappa_1}$$
$$- \frac{p_+^2a_+^2}{(2\pi)^3 v_+}\int d\Omega_{\mathbf{q}}\Gamma^{\omega}_{\alpha 1,\beta 1}(p, q)\left(\frac{\mathbf{q}}{m} - \frac{\mathbf{q}}{m_+}\right)$$
$$- \frac{p_-^2a_-^2}{(2\pi)^3 v_-}\int d\Omega_{\mathbf{q}}\Gamma^{\omega}_{\alpha 2,\beta 2}(p, q)\left(\frac{\mathbf{q}}{m} - \frac{\mathbf{q}}{m_-}\right).$$

Making use of the relation

$$[G^2(p)]^{k}_{\alpha\beta} = [G^2(p)]^{\omega}_{\alpha\beta} - \frac{2\pi i a_+^2}{v_+}P^+_{\alpha\beta}\delta(\varepsilon)\,\delta(|\mathbf{p}| - p_+)$$
$$- \frac{2\pi i a_-^2}{v_-}P^-_{\alpha\beta}\,\delta(\varepsilon)\,\delta(|\mathbf{p}| - p_-)$$

and the expressions (A.14), we get, finally,

$$\frac{1}{m} = \frac{1}{m_+} + \frac{p_+^2a_+^2}{(2\pi)^3}\int d\Omega\cos\theta\,\Gamma^{\omega}_{11,11}(\theta)$$
$$+ \frac{p_-^2a_+a_-}{(2\pi)^3 p_+}\int d\Omega\cos\theta\,\Gamma^{\omega}_{12,12}(\theta),$$
$$\frac{1}{m} = \frac{1}{m_-} + \frac{p_-^2a_-^2}{(2\pi)^3}\int d\Omega\cos\theta\,\Gamma^{\omega}_{22,22}(\theta)$$
$$+ \frac{p_+^2a_+a_-}{(2\pi)^3 p_-}\int d\Omega\cos\theta\,\Gamma^{\omega}_{21,21}(\theta). \qquad (A.17)$$

The vertex parts entering into (A.17) are taken at zero energies. The momenta lie on the corresponding Fermi surfaces; θ is the angle between the momenta. Finally, let us find the longitudinal magnetic susceptibility of a ferromagnetic Fermi fluid. From (19) we find

$$\chi = \frac{\partial M_0}{\partial H} = \frac{\mu_0}{2\pi^2}\left[p_+^2\frac{\partial p_+}{\partial H} - p_-^2\frac{\partial p_-}{\partial H}\right]. \qquad (A.18)$$

For $\partial p_\pm/\partial H$, we make use of (A.3), taking it into account that, on the Fermi surface,

$$\frac{\partial}{\partial H}G^{-1}_{\pm} = -2\mu_0 A_\pm \equiv \frac{v_\pm}{a_\pm}\frac{\partial p_\pm}{\partial H},$$

$$A_+P^+_{\alpha\beta} + A_-P^-_{\alpha\beta} = s^z_{\alpha\beta} - i\int\frac{d^4q}{(2\pi)^4}\Gamma^{k}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)$$
$$\times[\hat{G}(q)\,\hat{s}^z\hat{G}(q)]^{k}_{\varkappa_2\varkappa_1}. \qquad (A.19)$$

Substituting (8) in (A.19), we have

$$s^z_{\alpha\beta} - A_+P^+_{\alpha\beta} - A_-P^-_{\alpha\beta}$$
$$= i\int\frac{d^4q}{(2\pi)^4}\Gamma^{\omega}_{\alpha\varkappa_1,\beta\varkappa_2}(p, q)\,[\hat{G}(q)\,\hat{s}^z\hat{G}(q)]^{k}_{\varkappa_2\varkappa_1}$$
$$- \frac{p_+^2a_+^2}{(2\pi)^2 v_+}\int d\Omega_{\mathbf{q}}\Gamma^{\omega}_{\alpha 1,\beta 1}(p, q)\left(\frac{1}{2} - A_+\right)$$
$$- \frac{p_-^2a_-^2}{(2\pi)^3 v_-}\int d\Omega_{\mathbf{q}}\Gamma^{\omega}_{\alpha 2,\beta 2}(p, q)\left(-\frac{1}{2} - A_-\right).$$

Making use of the equation

$$[\hat{G}(p)\hat{s}^z\hat{G}(p)]^k_{\alpha\beta} = [\hat{G}(p)\hat{s}^z\hat{G}(p)]^\omega_{\alpha\beta}$$
$$-\frac{\pi i a_+^2}{v_+}P^+_{\alpha\beta}\delta(\varepsilon)\,\delta(|\mathbf{p}|-p_+)+\frac{\pi i a_-^2}{v_-}P^-_{\alpha\beta}\delta(\varepsilon)\,\delta(|\mathbf{p}|-p_-)$$

and (A.6), we get a set of two equations:

$$A_+ + \frac{p_+^2 a_+^2}{2\pi^2 v_+}\overline{\Gamma^\omega_{11,11}}A_+ + \frac{p_-^2 a_-^2}{2\pi^2 v_-}\overline{\Gamma^\omega_{12,12}}A_- = \frac{1}{2a_+},$$

$$A_- + \frac{p_+^2 a_+^2}{2\pi^2 v_+}\overline{\Gamma^\omega_{21,21}}A_+ + \frac{p_-^2 a_-^2}{2\pi^2 v_-}\overline{\Gamma^\omega_{22,22}}A_- = -\frac{1}{2a_-}. \quad (A.20)$$

Here we have used the notation

$$\overline{\Gamma^\omega} = \int \frac{d\Omega}{4\pi}\Gamma^\omega(p_1, p_2);$$

$\epsilon_1 = \epsilon_2 = 0$, and $\mathbf{p}_1$, $\mathbf{p}_2$ are taken on the Fermi surface.

Determining A_+, A_- from (A.20), and substituting in (A.18), we get the final expression for the longitudinal static susceptibility of a ferromagnetic Fermi fluid:

$$\chi = \frac{\mu_0^2}{2\pi^2}\Big\{p_+m_+ + p_-m_- + \frac{p_+m_+p_-m_-}{2\pi^2}$$
$$\times[a_+^2\overline{\Gamma^\omega_{11,11}} + 2a_+a_-\overline{\Gamma^\omega_{12,12}} + a_-^2\overline{\Gamma^\omega_{22,22}}]\Big\}$$
$$\times\Big\{\Big(1+\frac{p_+m_+}{2\pi^2}a_+^2\overline{\Gamma^\omega_{11,11}}\Big)\Big(1+\frac{p_-m_-}{2\pi^2}a_-^2\overline{\Gamma^\omega_{22,22}}\Big)$$
$$-\frac{p_+m_+p_-m_-}{4\pi^4}(a_+a_-\overline{\Gamma^\omega_{12,12}})^2\Big\}^{-1}. \quad (A.21)$$

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