RENORMALIZATION GROUP METHOD AND HIGH ENERGY SCATTERING

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Submitted to JETP editor April 24, 1964

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 845-850 (March, 1964)

The renormalization group method is used to investigate the behavior of high-energy scattering amplitudes. It is shown that for pion-pion scattering, renormalizability imposes a restriction on the energy and coupling-constant dependence of the high-energy scattering amplitude at a fixed angle. There is, however, no restriction on the angular dependence at fixed energy and coupling constant, or on the energy dependence at fixed momentum-transfer. The method, therefore, is not relevant to the problem of Regge poles. It is further pointed out that the renormalization group method is not applicable to the problem of pion-nucleon scattering. These results are in disagreement with those of a number of authors.

1. INTRODUCTION

In this paper we consider the question of whether the renormalizability of a field theory imposes any restriction on a scattering amplitude calculated in that field theory. We apply a method used by Gell-Mann and Low[1] in quantum electrodynamics which is equivalent to the renormalization group of Stueckelberg and Petermann[2] and of Bogolubov and Shirkov.[3] For pion-pion scattering, we show that renormalizability imposes a restriction on the energy and coupling-constant dependence of the high-energy scattering amplitude at a fixed angle. There is, however, no restriction on the angular dependence at fixed energy and coupling constant, or on the energy dependence at fixed momentum transfer. The method is, therefore, not relevant to the problem of Regge poles.

Our development of the method will make it clear that renormalizability imposes a restriction on the scattering amplitude only if the scattering amplitude requires a subtraction that is compensated for by renormalization. Thus, the method cannot be applied to pion-nucleon scattering, for which the scattering amplitude has no "skeleton divergence."

These conclusions are in contradiction with the results of some recent work.[4,5]

2. RENORMALIZATION

We consider the elastic scattering of two pions, ignoring isotopic spin for simplicity. The invariant scattering amplitude is dimensionless, and it depends on the square of the barycentric energy s and the corresponding quantity in the crossed channel t. Let the square of the pion mass be $\mu^2 \equiv {}^3\!/_4\, a$. The renormalized coupling constant g is then defined as the value of the scattering amplitude at s = t = a.

Let the scattering amplitude be calculated in a renormalizable field theory, which, for example, contains a φ^4 self-coupling plus coupling to nucleons. By summing all orders of perturbation theory, we can show that the scattering amplitude T′ satisfies an implicit equation of the form

$$T'\left(\frac{s}{a},\ \frac{t}{a},\ \Lambda,\ g_0\right) = g_0 + F\left[\frac{s}{a},\ \frac{t}{a},\ \Lambda;\ T'\right], \quad (1)$$

where g_0 is the bare coupling constant, and F is a functional of T′, with s/a, t/a, and Λ appearing as parameters.[1)] The number Λ is a cutoff parameter introduced to render perturbation theory finite.

It is assumed that F diverges logarithmically as $\Lambda \to \infty$, and that the limit

$$\lim_{\Lambda\to\infty}\left\{F\left[\frac{s}{a},\ \frac{t}{a},\ \Lambda;\ T'\right] - F\left[\frac{\lambda}{a},\ \frac{\sigma}{a},\ \Lambda;\ T'\right]\right\} \equiv D\left[\frac{s}{\lambda},\ \frac{t}{\sigma},\frac{\sigma}{\lambda},\ \frac{a}{\lambda};\ T'\right] \quad (2)$$

is finite. The parameters λ and σ are real numbers. It follows that in the limit $\Lambda \to \infty$, T′ satisfies the equation

$$T'\left(\frac{s}{a},\ \frac{t}{a},\ \Lambda,\ g_0\right) = g_{\lambda\sigma} + D\left[\frac{s}{\lambda},\ \frac{t}{\sigma},\ \frac{\sigma}{\lambda},\ \frac{a}{\lambda};\ T'\right], \quad (3)$$

[1)]Eq. (1) is valid in the absence of meson self-energy effects and interaction with other particles. We ignore these effects in the following.

where

$$g_{\lambda\sigma} \equiv T'(\lambda/a, \sigma/a, \Lambda, g_0). \tag{4}$$

The process of renormalization consists of taking the limit $\Lambda \to \infty$ with $g_{\lambda\sigma}$ held fixed.

For a complete definition of the renormalization procedure we must consider more generally the scattering amplitude off the mass shell, which depends on the 4 arbitrary external masses in addition to s and t. The functional F is obtained by inserting the off-mass-shell scattering amplitude into each vertex of certain irreducible diagrams, properly defined to ensure the correct separation of divergent terms. The subtraction in (2) must generally be understood to be made at some fixed value of the external masses, for example, at μ for all external masses. The interaction between pions and nucleons does not alter our formal development, because the effect of closed nucleon loops can be absorbed in the definition of the functional F.

Let a function $T''(v, w, x, y, z)$ be defined by the equation

$$T''\left(\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, \frac{a}{\lambda}, g\right) = g + D\left[\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, \frac{a}{\lambda}; T''\right]. \tag{5}$$

It is clear that for all x, y, z,

$$T''(1, 1, x, y, \zeta) = \zeta. \tag{6}$$

Comparing (5) with (3), we see that as $\Lambda \to \infty$ with $g_{\lambda\sigma}$ fixed,

$$T'\left(\frac{s}{a}, \frac{t}{a}, \Lambda, g_0\right) \to T''\left(\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, \frac{a}{\lambda}, g_{\lambda\sigma}\right), \tag{7}$$

which is the renormalized scattering amplitude. By the definition (2), the functional form of D is independent of λ and σ. Hence the functional form of T'' is independent of λ and σ. By (7) the value of $T''(s/\lambda, t/\sigma, \lambda/\sigma, a/\lambda, g_{\lambda\sigma})$ is independent of λ and σ at fixed $g \equiv g_{aa}$.

As (4) shows, the number $g_{\lambda\sigma}$ is in general complex. For $\lambda = \sigma = a$ it is real, and is the conventional renormalized coupling constant g. The conventional renormalized scattering amplitude, a function of g, is defined by

$$T\left(\frac{s}{a}, \frac{t}{a}, g\right) \equiv T'\left(\frac{s}{a}, \frac{t}{a}, 1, 1, g\right). \tag{8}$$

In terms of g, λ, and σ, $g_{\lambda\sigma}$ is given by

$$g_{\lambda\sigma} = T(\lambda/a, \sigma/a, g). \tag{9}$$

Since $T''(s/\lambda, t/\sigma, \sigma/\lambda, a/\lambda, g_{\lambda\sigma})$ is independent of λ and σ, we may rewrite (8) in the form

$$T\left(\frac{s}{a}, \frac{t}{a}, g\right) = T''\left(\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, \frac{a}{\lambda}, g_{\lambda\sigma}\right). \tag{10}$$

The freedom in the choice of λ and σ is expressed by some authors [2,3] as an invariance property of the scattering amplitude with respect to a "renormalization group." It should be noted, however, that (10) is a trivial consequence of the definitions of T and T'', and has no physical content. Any physical results we may obtain from (10) can only be a consequence of further assumptions.

To make this important fact clear, let us substitute (9) into (10) and set $s/a = x$, $t/a = y$, $\lambda/a = u$, and $\sigma/a = v$. We then obtain

$$T(x, y, g) = T''\left(\frac{x}{u}, \frac{y}{v}, \frac{v}{u}, \frac{1}{u}, T(u, v, g)\right), \tag{11}$$

which may be restated in the form

$$T(\alpha x, \beta y, g) = f(\alpha, \beta, x, y, T(x, y, g)), \tag{12}$$

where f is unknown except for the condition $f(1, 1, x, y, z) = z$. The relation (12) imposes no restriction on the function $T(x, y, g)$ because it is satisfied by any function.

3. THE FUNCTIONAL EQUATION

A restriction on the behavior of the scattering amplitude results if we make the assumption that the right hand side of (10) approaches a finite limit as $a \to 0$:

$$M(x, y, \zeta, g) \equiv T''(x, y, \zeta, 0, g). \tag{13}$$

It follows from (6) that for all z and g

$$M(1, 1, \zeta, g) = g. \tag{14}$$

That M exists can be explicitly verified in low-order perturbation theory. The reason is that in the subtracted Feynman diagrams for T'', λ and σ appear as effective high-energy cutoffs, while s and t appear as effective low-energy cutoffs. It is plausible that this mechanism makes all Feynman diagrams approach finite limits as $a \to 0$.

On dimensional grounds we expect that for $s \gg a$, $t \gg a$, $\lambda \gg a$, and $\sigma \gg a$,

$$T''\left(\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, \frac{a}{\lambda}, g\right) = M\left(\frac{s}{\lambda}, \frac{t}{\sigma}, \frac{\sigma}{\lambda}, g\right) + O\left(\frac{a}{s}, \frac{a}{t}\right). \tag{15}$$

Thus, the replacement of T'' by M is equivalent to neglecting terms of order $1/s$ and $1/t$. In particular all logarithmic dependences on s and t will be retained in M.

With the assumption (15) we have

$$T\left(\frac{s}{a},\ \frac{t}{a},\ g\right)=M\left(\frac{s}{\lambda},\ \frac{t}{\sigma},\ \frac{\sigma}{\lambda},\ g_{\lambda\sigma}\right) \tag{16}$$

for $s \gg a$, $t \gg a$, $\lambda \gg a$, and $\sigma \gg a$. Let us put $t = rs$ and $\sigma = r\lambda$, where r is related to the scattering angle by

$$r=-\frac{1}{2}\sin^2\frac{\theta}{2}. \tag{17}$$

Then (16) becomes

$$T\left(\frac{s}{a},\ \frac{rs}{a},\ g\right)=M\left(\frac{s}{\lambda},\ \frac{s}{\lambda},\ r,\ T\left(\frac{\lambda}{a},\frac{r\lambda}{a},\ g\right)\right), \tag{18}$$

where we have made use of (9). Let

$$K(x,\theta,g)\equiv T(x,rx,g),\qquad N(x,\theta,g)\equiv M(x,x,r,g). \tag{19}$$

Then K obeys the functional equation

$$K\left(\frac{s}{a},\theta,g\right)=N\left(\frac{s}{\lambda},\theta,K\left(\frac{\lambda}{a},\theta,g\right)\right). \tag{20}$$

For fixed s, (20) does not impose any restriction on the behavior of K as a function of θ and g. For fixed θ, however, (20) imposes a restriction on the behavior of K as a function of s and g. In fact, for fixed θ, (20) is identical in form to the functional equation derived by Gell-Mann and Low[1] for the photon propagator, with the same general solution:

$$K(x,\theta,g)=F_\theta(x,\Phi_\theta(g)), \tag{21}$$

where F_θ and Φ_θ are two arbitrary functions (generally complex) whose functional forms depend on θ.[2)]

By the same method used in Appendix B of [1], we can show that (21) is equivalent to the equation

$$\ln\frac{x}{x_0}=\int\limits_{Q(\theta,g)}^{K(x,\theta,g)}\frac{d\zeta}{\psi(\theta,\zeta)}, \tag{22}$$

where $Q(\theta,g)$ is an arbitrary function, and $\psi(\theta,g)$ is related to N of (19) by

$$\psi(\theta,\zeta)=[\partial N(x,\theta,\zeta)/\partial x]_{x=1}. \tag{23}$$

The number x_0 in (22) is determined by the choice of the path of integration, which necessarily extends into the complex z plane. To obtain further information on $K(x,\theta,g)$, we need to know $\psi(\theta,z)$, particularly for large complex values of z. Such knowledge cannot be obtained through the use of perturbation theory.

4. DISCUSSION

The functional equation (21) is solely a consequence of the assumption (15). The exact equation (10), being an identity, merely makes it convenient for us to utilize this assumption. Although we have made this assumption plausible, a rigorous proof appears difficult.

It is easy, on the other hand, to discover cases in which an assumption analogous to (15) is definitely incorrect. In such cases renormalizability imposes no restriction. For example, we can show in lowest-order perturbation theory that the right-hand side of (10) diverges if we put both $a = 0$ and $\sigma = 0$. This reflects the fact that renormalizability alone imposes no restriction on the behavior of the limit $s \to \infty$ with fixed t.

In cases where renormalizability imposes a restriction on the behavior of a function of physical interest, such as the high-energy photon propagator or the high-energy pion-pion scattering amplitude at a fixed angle, it may be hoped that the renormalization group method can be used to improve perturbation theory.[3] In cases where there is no restriction, such as pion-pion scattering at low energy, or at high energy with fixed momentum transfer, the method is entirely irrelevant. The authors of [4] used the renormalization group method in conjunction with perturbation theory in an attempt to show that the scattering amplitude has the Regge form $\beta(t)s^{\alpha(t)}$ for large s and small fixed t. We have seen that such a method is irrelevant to the latter problem. Their result is therefore a property of their approximation rather than a property of the scattering amplitude.

We wish to point out further that the renormalization group method cannot be extended to the case of pion-nucleon scattering, contrary to the belief expressed in [4,5]. The reason is that in pion-nucleon scattering the irreducible Feynman diagrams are convergent, and require no subtraction. The degree of freedom associated with the choice of the subtraction parameters λ and σ is consequently absent. In fact the renormalized coupling constant is uniquely determined by the residue of the nucleon pole in the scattering am-

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2)When meson self-energy effects are included, we find in place of (21)

$$K(x,\theta,g)=\alpha_\theta(g)F_\theta(x\Phi_\theta(g)),$$

where α is a second arbitrary function of g. If we further include the pion-nucleon interaction, we find

$$K(x,\theta,g,f)=\alpha_\theta(g,f)F_\theta(x,\Phi_\theta(g,f),\beta_\theta(g,f)),$$

where α, Φ, and β are arbitrary functions of f and g, and f is the pion-nucleon coupling constant. Generalization to any number of interactions is immediate. For each new coupling constant we obtain new arbitrary functions on which F may depend.

plitude. A formal application of the renormalization group method would be in this case equivalent to making an artificial subtraction, and then defining a function T'' analogous to (5). However, as $a \rightarrow 0$, one can verify in lowest-order perturbation theory that T'' has an infra-red divergence, which renders the functions M of (13) and ψ of (23) meaningless.

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Errata

Vol. 15, no. 6, 1063-1064 (N. G. Basov and A. N. Oraevskiĭ)

Formulas (8), (9), (10), (16), and (17) should read

$$P_\omega(t_0) = \frac{1}{\tau}\int_{t_0}^{t_0+\tau} P(t', t_0)e^{i\omega t'}dt'. \tag{8}$$

$$Z_\omega E_\omega = \alpha_\omega E_\omega + 4\pi\omega^2\beta_\omega \int_0^\infty \rho_{21}{}^{(0)}e^{i(\omega-\omega_{21})t_0}dt_0, \tag{9}$$

$$\alpha_\omega = 8\pi\omega^2 \frac{i}{\hbar}\frac{|\mu_{12}|^2 N}{\tau}[\rho_{22}{}^{(0)} - \rho_{11}{}^{(0)}]\frac{e^{i(\omega-\omega_{21})\tau} - i(\omega-\omega_{21})\tau - 1}{(\omega-\omega_{21})^2},$$

$$\beta_\omega = 2\mu_{12}\frac{N}{\tau}[e^{i(\omega-\omega_{21})\tau} - 1]/i(\omega-\omega_{21}). \tag{10}$$

$$\rho_{12} = \tilde{\rho}_{12}{}^0 e^{i(\omega_1-\omega_{21})(t_1+\tau_1)}e^{i\omega_{21}t}, \tag{16}$$

$$R_\omega = \langle\tilde{\rho}_{21}{}^0 e^{i(\omega_1-\omega_{21})T}\beta_\omega\rangle, \tag{17}$$

These errors do not affect the main conclusions of the article. The authors thank G. L. Suchkin for pointing out these misprints.

Vol. 19 no. 3 p. 581 (K. Huang and F. E. Low)

In Eq. (21) and the second equation in footnote 2, the comma between x and Φ_θ should be omitted. Thus Eq. (21) should read

$$K(x, \theta, g) = F_\theta(A), \text{ where } A \equiv x\Phi_\theta(g).$$

The second equation in footnote 2 should read

$$K(x, \theta, g, f) = \alpha_\theta(g, f)\, F_\theta(A, B),$$

where

$$A \equiv x\Phi_\theta(g, f), \quad B \equiv \beta_\theta(g, f).$$

Vol. 19, no. 6, p. 1313 (A. M. Prokhorov and V. V. Fedorov)

Right hand column, second formula from top, replace v_z^* in denominator by v_z^{*4}.

Vol. 20, no. 1, p. 122 (Poluektov, Presnyakov, and Sobel'man)

An error was made in the approximate calculation of the integral (AI.1) in Appendix I. The points z_n in the vicinity of which the derivative of the argument of the exponential vanishes must be sought prior to approximating the radical in the integrand of (AI.1). As a result, γ in (AI.2) is replaced by 2γ, the parameter $\pi\omega/\gamma v$, in (18), (AI.3), (AI.4), (AI.5), and (AI.6) is replaced by $\pi\omega/2\gamma v$, and ω in (20) and (AII.7) is replaced by $\omega/2$. Elimination of this error improves the agreement between the experimental and theoretical curves in the region of the maximum. The authors thank E. E. Nikitin for noting this error.