INTERACTION OF A QUANTUM SYSTEM WITH A STRONG FIELD

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An approximate method of solving the equations describing the interaction of a quantum system with a strong field is developed. The system is described with the aid of probability amplitudes and the density matrix. The solution obtained is applied to the electromagnetic field and clarifies the specific nature of the saturation effect for a large difference in the spontaneous decay probabilities of the combining levels. The stability of the monochromatic regime in solid state quantum generators is discussed.

THE investigation of the interaction of quantum systems with a strong field reduces, as is well known, to the solution of a system of linear differential equations with variable coefficients. The mathematical difficulties arising here are due to the complicated dependence of the coefficients of the system on the time on the one hand, and to the necessity for accurate evaluation of the solution over extremely large intervals of time on the other hand.

It has been possible to obtain comparatively simple formulas for the solution or to find it numerically in several interesting special cases.[1-9] In a broader class of problems it turns out to be possible to obtain an expansion of the solution in powers of a small parameter, using the methods of perturbation theory[10,11] or certain asymptotic methods.[12] However, in many problems of interest, particularly to the theory of optical quantum generators, these methods impose overly stringent limitations on the magnitudes of the interaction matrix elements and their derivatives.

In the present article the solution of the problem of the interaction of a quantum system with a strong field is carried out based on a method proposed by A. M. Molchanov in his lectures at Moscow State University. Here the fundamental matrix for the system of differential equations is sought in the form of a product of matrices, similar to the way this is done in the method of "variation of constants." The initial linear system of equations is reduced to a nonlinear system, whose solution is found by the method of successive approximations.

In the method considered here, convergence of the successive approximations also occurs only under certain definite restrictions on the magnitude of the interaction. However, these limitations are not substantial for a number of problems involving a strong field. Thus, for example, one can investigate the dependence on the field of the "material constants" in Maxwell's equations[13] on the basis of the expressions obtained for the solution.

For simplicity of presentation, we here limit ourselves to an investigation of a model quantum system possessing two excited levels, using both the equations for the probability amplitudes of the states (Sec. 1) and the density matrix formalism (Sec. 2).

1. For a description of the behavior of the two-level system in the quasiclassical approximation, we shall use the following differential equations for the probability amplitudes a_m, a_n of states m, n:

$$\dot{a}_m + \gamma_m(t)\, a_m = -iV(t)\, a_n,$$
$$\dot{a}_n + \gamma_n(t)\, a_n = -iV^*(t)\, a_m. \tag{1.1}$$

In many problems the quantities $\gamma_m(t)$ and $\gamma_n(t)$ are markedly different from each other. For the case of the electromagnetic field, for example, γ_m and γ_n are equal to one-half the probabilities of spontaneous transitions from the levels m and n, and as a rule they differ by an order of magnitude or more. We shall assume that the following inequality is fulfilled for all values of $t \geq t_0$:

$$\gamma_n(t) \gg \gamma_m(t) > 0,$$

and the functions $\gamma_m(t)$, $\gamma_n(t)$, $V(t)$ are bounded.

Let $\mathbf{a}(t)$ be a two-dimensional vector with components $a_m(t)$, $a_n(t)$. The solution of the system (1.1) for arbitrary initial conditions can be easily found if the fundamental matrix $\hat{S}(t)$ of the system (1.1) is known:

$$\mathbf{a}(t) = \hat{S}(t)\mathbf{a}(t_0).$$

The matrix $\hat{S}(t)$ obeys the equation

$$d\hat{S}/dt = \hat{P}\hat{S}, \tag{1.2}$$

where $\hat{P}(t)$ is the matrix composed of the coefficients in the system of equations (1.1):

$$\hat{P}(t) = \begin{pmatrix} -\gamma_m(t) & -iV(t) \\ -iV^*(t) & -\gamma_n(t) \end{pmatrix}. \tag{1.3}$$

The initial condition for $\hat{S}(t)$ has the form $\hat{S}(t_0) = E$, where E is the unit matrix.

Following the idea of A. M. Molchanov, we shall seek $\hat{S}(t)$ in the form of a product of two triangular matrices $\hat{S}(t) = \hat{X}(t)\hat{Y}(t)$, where

$$\hat{X}(t) = \begin{pmatrix} 1 & 0 \\ ix(t) & 1 \end{pmatrix}, \quad \hat{Y}(t) = \begin{pmatrix} \Lambda_1(t) & iy(t) \\ 0 & \Lambda_2(t) \end{pmatrix}.$$

In this connection

$$\hat{S}(t) = \begin{pmatrix} \Lambda_1(t) & iy(t) \\ ix(t)\Lambda_1(t) & \Lambda_2(t) - x(t)y(t) \end{pmatrix}. \tag{1.4}$$

For a periodic or quasiperiodic function $V(t)$, such a procedure enables us to separate to some extent the exponential change of the elements of the matrix $\hat{S}(t)$ from the periodic variation. We shall try to represent the first change, which is basically defined by the quantities γ_m and γ_n, by the functions Λ_1 and Λ_2, the second variation—by the functions x and y.

Let us assume that Λ_1 and Λ_2 are different from zero for all times $t \geq t_0$, and let $\lambda_i = \dot{\Lambda}_i/\Lambda_i$. Then from Eqs. (1.2)–(1.4) we obtain the following equations for the functions x and y:

$$\dot{x} + (\gamma_n - \gamma_m)x = -V^* - Vx^2, \tag{1.5}$$

$$\dot{y} + \lambda_1 y = -V\Lambda_2. \tag{1.6}$$

Here

$$\lambda_1 = \gamma_m - Vx, \qquad \lambda_2 = \gamma_n + Vx, \tag{1.7}$$

and the functions Λ_1, Λ_2 can be determined from the equations

$$\dot{\Lambda}_1 + \lambda_1\Lambda_1 = 0, \qquad \dot{\Lambda}_2 + \lambda_2\Lambda_2 = 0. \tag{1.8}$$

The initial conditions for x, y, Λ_1 and Λ_2 are determined by the form of $\hat{S}$, given by Eq. (1.4):

$$x(t_0) = y(t_0) = 0. \qquad \Lambda_1(t_0) = \Lambda_2(t_0) = 1.$$

Equations (1.5), (1.6), and (1.8) are exactly equivalent to the initial system of equations (1.1). All the difficulty in solving the problem is now concentrated in the nonlinear equation (1.5) for the single unknown function $x(t)$. The remaining equations are linear and are easily integrated for a known $x(t)$. The probability amplitudes $a_m(t)$, $a_n(t)$ are expressed in terms of the functions x, y, Λ_1, Λ_2 as follows:

$$a_m(t) = \Lambda_1(t)\, a_m(t_0) + iy(t)\, a_n(t_0), \tag{1.9}$$

$$a_n(t) = ix(t)\Lambda_1(t)\, a_m(t_0) + [\Lambda_2(t) - x(t)\, y(t)]\, a_n(t_0).$$

For $\gamma_n > \gamma_m$ the solution of Eq. (1.5) for $x(t)$ may be found by the method of successive approximations according to the formulas

$$x^{(0)} = 0, \quad x^{(n)} = -\int_{t_0}^{t} \{V^*(t') + V(t')\,[x^{(n-1)}(t')]^2\} \times \exp\left\{-\int_{t'}^{t} (\gamma_n - \gamma_m)\, dt''\right\} dt'. \tag{1.10}$$

For $|V(t)/[\gamma_n(t) - \gamma_m(t)]|$ sufficiently small the sequence $\{x^{(n)}\}_0^\infty$ converges to the solution $x(t)$. Actually, let us assume

$$\varepsilon \equiv \max_{t \geq t_0} \left| \frac{V(t)}{\gamma_n(t) - \gamma_m(t)} \right| < \frac{1}{2} \tag{1.11}$$

and for a certain $n = n_0$ let $|x^{(n)}| < 2\epsilon$. Then it follows from (1.10) that this same bound also holds for $n = n_0 + 1$, and since $x^{(0)} = 0$, so all the quantities $x^{(n)}$ are uniformly bounded: $|x^{(n)}| < 2\epsilon$, and the rate of convergence of the sequence $x^{(n)}$ is at least no less than that of the geometric progression with ratio $4\epsilon^2$.

If the function $x(t)$ is determined correct to within a quantity of order ϵ^{2n}, then we can integrate the linear equations (1.6) and (1.8) with the same degree of correctness and find

$$\Lambda_1(t) = \exp\left\{-\int_{t_0}^{t} [\gamma_m(t') - V(t')\, x(t')]\, dt'\right\},$$

$$\Lambda_2(t) = \exp\left\{-\int_{t_0}^{t} [\gamma_n(t') + V(t')\, x(t')]\, dt'\right\},$$

$$y(t) = -\int_{t_0}^{t} V(t')\Lambda_2(t') \times \exp\left\{-\int_{t'}^{t} [\gamma_m(t'') - V(t'')\, x(t'')]\, dt''\right\} dt'. \tag{1.12}$$

In order to clarify the physical meaning of the assumptions made, let us consider the case of an electromagnetic field, when γ_m and γ_n do not depend on the time, in more detail. We confine our attention to the leading terms, assuming $\epsilon^2 \ll 1$ and $\gamma_m \ll \gamma_n$. Then from (1.10) and (1.12) we find

$$x(t) = -\int_{t_0}^{t} V^*(t')\, e^{-\gamma_n(t-t')} dt',$$

$$y(t) = -\Lambda_1(t) \int_{t_0}^{t} V(t')\, e^{-\gamma_n(t'-t_0)} dt',$$

$$\Lambda_1(t) = \exp\left\{-\gamma_m(t-t_0) - \int_{t_0}^{t}\int_{t_0}^{t'} V(t')V^*(t'')\, e^{-\gamma_n(t'-t'')} dt'dt''\right\},$$

$$\Lambda_2 = e^{-\gamma_n(t-t_0)}. \tag{1.13}$$

First we consider the initial conditions $a_m(t_0) = 1$, $a_n(t_0) = 0$ (excitation of the system to the level m at the moment of time t_0):

$$a_m(t) = \Lambda_1(t)$$

$$= \exp\left\{-\gamma_m(t-t_0) - \int_{t_0}^{t}\int_{t_0}^{t'} V(t')V^*(t'')\, e^{-\gamma_n(t'-t'')} dt'\, dt''\right\},$$

$$a_n(t) = ix(t)\Lambda_1(t) = -ia_m(t)\int_{t_0}^{t} V^*(t')\, e^{-\gamma_n(t-t')} dt'. \tag{1.14}$$

The physical meaning of these expressions is completely clear. The quantity $|\Lambda_1(t)|^2$ is the probability of finding the system in the state m; it decreases with time as a result of both spontaneous and induced (the double integral in the exponent of the exponential) transitions. The probability $|a_n(t)|^2$ of finding the system in level n is proportional to $|a_m(t)|^2$ and to the function $|x(t)|^2$, which therefore describes the induced transitions from m to n. We obviously have the following estimate for the ratio $|a_n(t)/a_m(t)|^2$

$$\left|\frac{a_n(t)}{a_m(t)}\right|^2 = \left|\int_{t_0}^{t} V^*(t')\, e^{-\gamma_n(t-t')} dt'\right|^2 \sim \max\frac{|V|^2}{\gamma_n^2} = \varepsilon^2. \tag{1.15}$$

The approximation $\epsilon^2 \ll 1$ means therefore that $|a_n(t)|^2 \ll |a_m(t)|^2$ for all moments of time. This is related to the fact that after an induced transition $m \to n$ the system rapidly relaxes due to the condition $\gamma_n \gg \gamma_m$. One can interpret the subsequent approximations of order ϵ^2, ϵ^4, and so forth, as successive transitions $m \to n \to m \to n$ and so on.

In connection with these ideas, it is of interest to clarify the meaning of condition (1.11) for the convergence of the successive approximations. For this purpose let us consider the case of a monochromatic external field whose frequency we shall assume, for the sake of simplicity, equal to the transition frequency ω_{mn}. In this case the probability amplitudes are given by the expressions[7]

$$a_m(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t},$$

$$a_n(t) = -iV^*\left[\frac{A_1}{\alpha_1+\gamma_n} e^{\alpha_1 t} + \frac{A_2}{\alpha_2+\gamma_n} e^{\alpha_2 t}\right],$$

$$\alpha_{1,2} = -\tfrac{1}{2}(\gamma_m+\gamma_n) \pm \sqrt{\tfrac{1}{4}(\gamma_n-\gamma_m)^2 - |V|^2}, \tag{1.16}$$

where A_1 and A_2 are integration constants. If condition (1.11) is satisfied, then the radical is real and a_m, a_n decrease monotonically with time. However, if $|V|/(\gamma_n - \gamma_m) > 1/2$ then the $\alpha_{1,2}$ are complex quantities and a_m, a_n in addition to decreasing with time will undergo oscillations with a frequency that depends on $|V|$. Thus, condition (1.11) means that the method considered of solving the system of equations (1.1) only encompasses the "aperiodic behavior" of the transitions.

It is usually assumed that nonlinear effects in radiation (including saturation effects) are expressly associated with the presence of oscillations in $|a_m|^2$ and $|a_n|^2$.[14] If several oscillations of $|a_m|^2$ and $|a_n|^2$ occur during a lifetime, this means that on the average the population of both levels is the same, and the system can neither absorb, nor radiate energy. This reasoning corresponds in reality only to the case $\gamma_m = \gamma_n$, when the "oscillatory behavior," as is evident from (1.16), begins at very small values of $|V|$, and the damping does not depend on the field. However, if $\gamma_n \gg \gamma_m$, then saturation is also possible for $\epsilon^2 \ll 1$, when, according to (1.15), the average populations of the levels differ by a factor of ϵ^2. Formally, this is due to the fact that the degree of saturation is determined[7] by the magnitude of the parameter $|V|^2/\gamma_m\gamma_n = \epsilon^2\gamma_n/\gamma_m$, which may be large for small values of ϵ^2 if $\gamma_n/\gamma_m \gg 1$. An intuitive interpretation of the nonlinear effects in this case is that the induced transitions increase the damping constant of the level m and reduce, by the same token, its average population to that value to which the oscillations in the case $\gamma_m = \gamma_n$ reduced the difference of populations. This circumstance is also reflected in expression (1.14) for a_m in the case of arbitrary dependence of the field on time. With respect to order of magnitude, the double integral in the exponent of the exponential in (1.14) is equal to the maximum value of $|V|^2(t - t_0)/\gamma_n$, i.e., its role will be appreciable if $\max|V|^2/\gamma_n \sim \gamma_m$, which agrees with the criterion given above for the existence of nonlinear effects.

It should be noted that rapid decay of the state m because of the induced transitions is not equivalent in general to a broadening of the level. In particular, it is shown in [7] that the induced transitions in a strong monochromatic field do not lead to broadening, but to a splitting of the spontaneous emission line.

Now let us investigate the case of excitation of the shorter-lived state n (i.e., $a_m(t_0) = 0$, $a_n(t_0) = 1$). Here

$$a_m(t) = iy(t), \qquad a_n(t) = \Lambda_2(t) - x(t)\,y(t). \quad (1.17)$$

The term $\Lambda_2(t)$ in $a_n(t)$ describes the usual relaxation after excitation of the system. The function $y(t)$ determines the population of the level m due to the induced transitions $n \to m$. Since the function $ix(t)$ gives the ratio of the probability amplitudes of the states n and m for the transition $m \to n$ [see (1.14)], and in the present case $a_m = iy$, so the term $-xy$ gives the contribution of the inverse transitions $m \to n$ to the probability of finding the system in the state n. We note that the function $y(t)$ decreases with time like $\Lambda_1(t)$, i.e., considerably slower than $\Lambda_2(t)$. Therefore, for sufficiently large times $t - t_0 \gtrsim \gamma_n^{-1}$, the term xy will be the leading term in the expression for a_n. Thus, for excitation of a state with a large damping constant, the first approximation takes into account the two-step transition $n \to m \to n$.

2. For many problems a description of the quantum system with the aid of the density matrix turns out to be advisable. Here the equation for the fundamental matrix $\hat{S}$ also has the form (1.2) and the matrix $\hat{P}$ for a system possessing two excited levels and three relaxation parameters[15] will have the form

$$\hat{P} = \begin{pmatrix} -\Gamma_m(t) & 0 & iV^*(t) & -iV(t) \\ 0 & -\Gamma_n(t) & -iV^*(t) & iV(t) \\ iV(t) & -iV(t) & -\Gamma(t) & 0 \\ -iV^*(t) & iV^*(t) & 0 & -\Gamma(t) \end{pmatrix}. \quad (2.1)$$

In the optical region the parameters Γ_m, Γ_n and Γ are markedly different from each other. Usually the following inequality holds:

$$\Gamma(t) \gg \Gamma_n(t) \gg \Gamma_m(t).$$

Problems for which either

$$\rho_{mm}(t_0) = 1,$$

$$\rho_{nn}(t_0) = \rho_{nm}(t_0) = \rho_{mn}(t_0) = 0 \text{ (problem I)},$$

or

$$\rho_{nn}(t_0) = 1,$$

$$\rho_{mm}(t_0) = \rho_{nm}(t_0) = \rho_{mn}(t_0) = 0 \text{ (problem II)}.$$

are of interest.

For the solution of Eq. (1.2), we write as we did in Sec. 1, $\hat{S}$ in the form of a product of triangular matrices $\hat{X}$ and $\hat{Y}$:

$$\hat{S} = \hat{X}\hat{Y}, \qquad \hat{X} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 \\ ix_{31} & ix_{32} & 1 & 0 \\ -ix_{41} & -ix_{42} & x_{43} & 1 \end{pmatrix},$$

$$\hat{Y} = \begin{pmatrix} \Lambda_1 & y_{12} & y_{13} & y_{14} \\ 0 & \Lambda_2 & y_{23} & y_{24} \\ 0 & 0 & \Lambda_3 & y_{34} \\ 0 & 0 & 0 & \Lambda_4 \end{pmatrix}. \quad (2.2)$$

The factors i and $-i$ are introduced here for the sake of convenience in writing subsequent expressions. It is obvious that in order to solve problem I it is necessary to determine the elements of the first column of the matrix $\hat{S}$, i.e., the quantities x_{i1}, Λ_1, and in order to solve problem II, one must determine the elements of the second column—x_{i2}, y_{12} and Λ_2. In the same way as in Sec. 1, it follows from the form of the matrix $\hat{S}$ and the initial condition $\hat{S}(t_0) = E$ that $x_{ij}(t_0) = y_{ij}(t_0) = 0$, and $\Lambda_i(t_0) = 1$.

The equations for these quantities can be obtained from Eq. (1.2) and relations (2.1), (2.2). The functions x_{21}, x_{31}, and x_{41} are determined by the following system of equations:

$$\begin{aligned} \dot{x}_{31} + (\Gamma - \Gamma_m)\,x_{31} &= V - Vx_{21} + x_{31}(V^*x_{31} + Vx_{41}), \\ \dot{x}_{41} + (\Gamma - \Gamma_m)\,x_{41} &= V^* - V^*x_{21} + x_{41}(V^*x_{31} + Vx_{41}), \\ \dot{x}_{21} + (\Gamma_n - \Gamma_m)\,x_{21} &= (1 + x_{21})(V^*x_{31} + Vx_{41}). \end{aligned} \quad (2.3)$$

Λ_1 is the solution of the equation

$$\dot{\Lambda}_1 + \lambda_1\Lambda_1 = 0, \qquad \lambda_1 = \Gamma_m + V^*x_{31} + Vx_{41}.$$

The solution to the system (2.3) can be found by the method of successive approximations according to formulas similar to (1.10). In order to determine the n-th approximation, the quantities x_{21}, x_{31}, x_{41} on the right side of Eqs. (2.3) are assumed to be known to the $(n-1)$-th approximation. In the zeroth approximation they are assumed equal to zero. The functions $x_{i1}^{(n)}$ ($i = 2, 3, 4$; $n = 0, 1, 2, \ldots$) are uniformly bounded and as $n \to \infty$ they converge to the exact solution of the system (2.3), provided the parameter $\epsilon = \max|V/(\Gamma_n - \Gamma_m)|$ is sufficiently small. Actually, having assumed for example that $\epsilon < 1/5$ and that the following inequalities are satisfied for a certain value $n = m$:

$$|x_{21}^{(n)}| < \varepsilon, \qquad |x_{31}^{(n)}| < \tfrac{3}{2}\varepsilon, \qquad |x_{41}^{(n)}| < \tfrac{3}{2}\varepsilon, \quad (2.4)$$

it is easy to prove from integral expressions of the type (1.10) that these same bounds also hold for $n = m + 1$. Since $x_{i1}^{(0)} = 0$, it therefore follows that inequalities (2.4) hold for all $n = 1, 2, \ldots$, and the successive approximations $x_{i1}^{(n)}$ converge to the solution of the system (2.3) no more slowly than a geometric progression with ratio

$$5 \max |V|^2/[(\Gamma - \Gamma_m)(\Gamma_n - \Gamma_m)].$$

Relying on the convergence of the successive approximations, it is easy to verify that $x_{31}^* = x_{41}$ and x_{21} is a real function. We give the expression for the leading term in the solution (the first approximation), whose relative deviation from the exact solution is a quantity of order ϵ^2:

$$x_{41}(t) = x_{31}^*(t) = \int_{t_0}^{t} V^*(t') \exp\left\{-\int_{t'}^{t} (\Gamma - \Gamma_m)\, dt''\right\} dt',$$

$$x_{21}(t) = \int_{t_0}^{t} [V^*(t')\, x_{41}^*(t') + V(t')\, x_{41}(t')]$$

$$\times \exp\left\{-\int_{t'}^{t} (\Gamma_n - \Gamma_m)\, dt''\right\} dt'. \tag{2.5}$$

For a known quantity x_{41}, the function Λ_1 is determined by the expression

$$\Lambda_1(t) = \exp\left\{-\int_{t_0}^{t} [\Gamma_m(t') + V^*(t')\, x_{41}^*(t') + V(t')\, x_{41}(t')]\, dt'\right\}, \tag{2.6}$$

and the solution of problem I is found according to the formulas

$$\rho_{mm} = \Lambda_1, \qquad \rho_{nn} = x_{21}\Lambda_1, \qquad \rho_{mn} = \rho_{nm}^* = i x_{41}^* \Lambda_1. \tag{2.7}$$

The functions y_{12}, x_{32}, x_{42} and Λ_2, needed for calculating the solution to problem II, are determined by the equations

$$\dot{x}_{32} + (\Gamma - \Gamma_n)\, x_{32} = -V + (x_{31} - x_{32} - x_{32}x_{21})(V^* x_{32} + V x_{42}),$$

$$\dot{x}_{42} + (\Gamma - \Gamma_n)\, x_{42} = -V^* + (x_{41} - x_{42} - x_{42}x_{21})(V^* x_{32} + V x_{42}),$$

$$\dot{\Lambda}_2 + \lambda_2\Lambda_2 = 0, \qquad \lambda_2 = \Gamma_n - (1 + x_{21})(V^* x_{32} + V x_{42}),$$

$$\dot{y}_{12} + \lambda_1 y_{12} = -(V^* x_{32} + V x_{42})\, \Lambda_2. \tag{2.8}$$

If the x_{i1} ($i = 2, 3, 4$) are known, the function x_{32} or x_{42}, which is equal to x_{32}^*, can be found from (2.8) by the method of successive approximations, the condition for the convergence in this case being smallness of the ratio $|V/(\Gamma - \Gamma_n)|$. After this the equations for y_{12} and Λ_2 are easily integrated since they are linear equations with known quantities on the right sides. We note that in this case, when it is possible to confine our attention to the first approximation for the solution and one can neglect Γ_n, Γ_m in comparison with Γ, it turns out that $x_{42} = -x_{41}$, and

$$y_{12}(t) = \Lambda_1 \int_{t_0}^{t} [V^* x_{41}^* + V x_{41}] \exp\left\{-\int_{t'}^{t} [\Gamma_n - \Gamma_m]\, dt''\right\} dt',$$

$$\Lambda_2 = \exp\left\{-\int_{t_0}^{t} \Gamma_n dt'\right\}. \tag{2.9}$$

The solution of problem II for known functions y_{12}, x_{42}, and Λ_2 can be obtained from the formulas

$$\rho_{mm} = y_{12}, \qquad \rho_{nn} = \Lambda_2 + x_{21} y_{12},$$

$$\rho_{mn} = \rho_{nm}^* = i\,(\Lambda_2 x_{42}^* + x_{41}^* y_{12}).$$

The interpretation of the solutions obtained is similar to that given in Sec. 1 for the case of the description of the system in terms of probability amplitudes.

Now let us consider, with the aid of the solutions obtained, the question of absorption and emission of photons of a weak electromagnetic field in the presence of a "strong" field. We shall assume Γ_m, Γ_n, and Γ to be real constants which characterize the relaxation processes taking place in the system.[15] The matrix element of the interaction $V(t)$ has the form

$$V(t) = V_1 e^{-i\Omega_1 t} + V_2 e^{-i\Omega_2 t},$$

$$\Omega_1 = \omega_1 - \omega_{mn}, \qquad \Omega_2 = \omega_2 - \omega_{mn}, \qquad V_1 = p_{mn} E_1/2\hbar,$$

$$V_2 = p_{mn} E_2/2\hbar. \tag{2.10}$$

Here ω_1, ω_2 and E_1, E_2 are the frequencies and amplitudes, respectively, of the strong and weak fields, p_{mn} is the matrix element of the dipole transition $m \rightarrow n$. We shall assume that the conditions for applicability of the first approximation, which is given by formulas (2.5), (2.6) and (2.9), are satisfied.

The probability of stimulated emission (or absorption) of photons $\hbar\omega_1$, $\hbar\omega_2$ of the strong or weak fields is given by the following two expressions:

$$W_1 = 2\mathrm{Re}\left\{i \int_{t_0}^{\infty} V_1(t)\, \rho_{nm}(t)\, dt\right\},$$

$$W_2 = 2\mathrm{Re}\left\{i \int_{t_0}^{\infty} V_2(t)\, \rho_{nm}(t)\, dt\right\}. \tag{2.11}$$

For definiteness, let us consider problem I. Then, using (2.7) we find

$$W_{1\mathrm{I}} = 2\mathrm{Re}\left\{\int_{t_0}^{\infty} V_1(t)\, x_{41}(t)\, \Lambda_1(t)\, dt\right\},$$

$$W_{2\mathrm{I}} = 2\mathrm{Re}\left\{\int_{t_0}^{\infty} V_2(t)\, x_{41}(t)\, \Lambda_1(t)\, dt\right\}. \tag{2.12}$$

By a "weak" field we mean a field which does not give any contribution to saturation. This means that it is necessary to expand the exponent in formula (2.6) for Λ_1 in powers of V_2 and for the calculation of W_{2I} according to (2.12), it is necessary to limit ourselves to the first nonvanishing term in V_2. Averaging W_{1I}, W_{2I} over the phase difference between V_1 and V_2 and assuming $\Gamma \gg \Gamma_n \gg \Gamma_m$, it is easy to obtain the following expressions with the aid of (2.10), (2.5) and (2.6):

$$W_{1I} = \frac{2\Gamma}{\Gamma_m} \frac{|V_1|^2}{\Omega_1^2 + \Gamma^2 + 2\Gamma |V_1|^2/\Gamma_m},$$

$$W_{2I} = \frac{2|V_2|^2}{\Gamma} \frac{\Gamma^2}{\Gamma^2 + \Omega_2^2} \frac{1}{\alpha} \left\{1 - \frac{2|V_1|^2}{\Gamma^2 + \Omega_1^2} \frac{\Gamma\alpha - \Omega_1(\Omega_2 - \Omega_1)}{\alpha^2 + (\Omega_2 - \Omega_1)^2}\right\},$$

$$\alpha = \Gamma_m + \frac{2\Gamma |V_1|^2}{\Gamma^2 + \Omega_1^2}. \tag{2.13}$$

The dependence of W_{2I} on the frequency Ω_2 of the weak field has the form of the usual dispersion curve of width Γ (the factor $\Gamma^2/(\Gamma^2 + \Omega_2^2)$ in front of the curly brackets), and near $\Omega_2 = \Omega_1$ there is a narrow "dip", determined by the second term inside the curly brackets. The width of this "dip" is $\alpha = \Gamma_m + 2\Gamma |V_1|^2/(\Gamma^2 + \Omega_1^2)$, so that under our conditions it is considerably less than the total line width Γ. The relative depth of the dip is

$$\frac{\Gamma^2}{\Gamma^2 + \Omega_1^2} \frac{2|V_1|^2}{\Gamma_m\Gamma} \left[1 + \frac{\Gamma^2}{\Gamma^2 + \Omega_1^2} \frac{2|V_1|^2}{\Gamma_m\Gamma}\right]^{-1},$$

which for $2|V_1|^2/\Gamma_m\Gamma \sim 1$ amounts to approximately one-half the intensity of the line. The complicated frequency dependence of W_{2I} is a good illustration that the shortening of the lifetime of the upper level caused by induced transitions is not equivalent to its broadening.

Expression (2.13) enables us to solve the interesting problem of the stability of the monochromatic regime of generation in a solid state quantum generator. Let a monochromatic field of frequency ω_1 be established in the generator. Due to spontaneous emission, electromagnetic waves with other frequencies $\omega_2 \neq \omega_1$ will exist in the medium of the generator, where one can regard the amplitudes of these waves as small in the sense that they do not cause saturation. If these weak fields decay with time, then the monochromatic regime of generation is stable, but if the weak fields increase with time—it is unstable.

Such a formulation of the problem was adopted in [16]. For a model of a quantum generator in the form of a layer of "active" material bounded by mirrors with high coefficients of reflectivity r_1, r_2, it was shown that nontrivial and exponentially time-dependent solutions of the wave equation exist if the following conditions are satisfied:

$$\omega_1 c^{-1} \sqrt{\varepsilon_0} l\, [\overline{\Delta\varepsilon''}(\omega_1) - c_1/2] = 1 - (r_1 + r_2)/2,$$
$$\omega_2 c^{-1} \sqrt{\varepsilon_0} l \overline{\Delta\varepsilon''}(\omega_2) = 1 - (r_1 + r_2)/2 + 2\gamma_2 l/c. \tag{2.14}$$

Here l is the thickness of the layer; $-\Delta\epsilon''$ is the imaginary part of the dielectric permittivity due to population inversion; c_1 is the amplitude of the first harmonic in the Fourier expansion of $\Delta\epsilon''$; γ_2 is the imaginary part of the frequency of the weak field. The term $2\gamma_1 l/c$ is absent from the first equation of (2.14), since it was assumed that the strong field is stationary, $\gamma_1 = 0$. Inside the layer the strong field has the form of a standing wave

$$V_1 = V_{10} \sin(\omega_1 c^{-1} \sqrt{\varepsilon_0} x), \tag{2.15}$$

and due to the saturation effect the dielectric permittivity of the medium turns out to be a periodic function of the coordinate x. The bar over $\Delta\epsilon''$ in (2.14) denotes average over the period of the standing wave. The presence of an inhomogeneous medium leads to the reflection of the waves propagating in the layer by each inhomogeneity. If the period of inhomogeneity is equal to one-half the wavelength, as happens for the strong field, then the reflected waves are shifted in phase by 2π, and it turns out that they reduce the amplifying effect of the medium. This effect also leads to the term $c_1/2$ in condition (2.14) for the strong field. The wavelength of the weak field differs from twice the period of the inhomogeneity, and in this case it turns out that the reflected waves completely damp one another.[16]

The stability or instability of the monochromatic regime of generation obviously depends on the sign of γ_2. If $\gamma_2 > 0$ then the weak field will grow with time, which finally leads to the disruption of the generation of the field with frequency ω_1. However, if $\gamma_2 < 0$ then the weak field turns out to be damped, and the strong field is stable. The sign of γ_2 is determined by the magnitude of the ratio

$$\chi = \overline{\Delta\varepsilon''(\omega_2)}/(\overline{\Delta\varepsilon''(\omega_1)} - c_1/2).$$

If $\chi > 1$, then $\gamma_2 > 0$; however, if $\chi < 1$, then $\gamma_2 < 0$.

Let us consider the simplest case $|V_{10}|^2 \ll \Gamma_m\Gamma$, which is adequate for our purposes. Under this condition one can neglect the dependence of α on $|V_1|^2$. Assuming for simplicity that $\Omega_1 = \omega_1 - \omega_{mn} = 0$, with the aid of (2.3) and (2.15) we find:

$$\chi = \left\{1 + \frac{|V_{10}|^2}{2\Gamma_m\Gamma}\left[1 - \frac{2\Gamma_m^2}{\Omega_2^2 + \Gamma_m^2}\right]\right\} \frac{\Gamma^2}{\Omega_2^2 + \Gamma^2}. \tag{2.16}$$

The maximum value $\chi = \chi_m$ is reached when

$$\Omega_2^2 + \Gamma_m^2 = \sqrt{|V_{10}|^2 \Gamma_m \Gamma}. \qquad (2.17)$$

Substituting (2.17) into (2.16), it is easy to show that $\chi_m \leq 1$, provided

$$|V_{10}|^2/\Gamma_m\Gamma \leqslant 2\,(3 + \sqrt{8})\,(\Gamma_m/\Gamma)^2. \qquad (2.18)$$

If Γ_m/Γ is sufficiently small, then condition (2.18) is fulfilled for $|V_{10}|^2 \ll \Gamma_m\Gamma$, and the extremal value of Ω_2 (in order of magnitude) is equal to Γ_m, i.e., much smaller than the linewidth Γ.

The quantity $|V_{10}|^2$ is related to Q, the number of acts of excitation per unit volume and unit time. If $|V_{10}|^2/\Gamma_m\Gamma \ll 1$, then from the first equation of (2.14) one can find[17]

$$|V_{10}|^2/\Gamma\Gamma_m = \tfrac{2}{3}\,[Q/Q_0 - 1], \qquad (2.19)$$

where Q_0 is the threshold value of Q at which generation begins. Thus, the condition for stability of generation takes the form

$$Q/Q_0 - 1 < 3\,(3 + \sqrt{8})\,(\Gamma_m/\Gamma)^2. \qquad (2.20)$$

Upon reversal of the inequality sign in (2.20) the weak fields turn out to increase with time, which leads to a change in the conditions of generation. Since Γ_m and Γ usually differ by several orders of magnitude, so (2.20) is practically never satisfied. We note that the criterion (2.20) actually coincides with the similar criterion obtained in [16] for the case $2\Gamma = \Gamma_n + \Gamma_m \gg \Gamma_m$.

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