ON A SET OF ALGEBRAIC EQUATIONS EQUIVALENT TO THE LOW EQUATIONS

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A set of algebraic equations which is equivalent to the Low equations is derived. The set is derived by conformally transforming the physical plane into a circle and by expanding the scattering amplitude in a power series. The set of algebraic equations also expresses those relations between the coefficients of the series which guarantee that the unitarity conditions and the cross relations are satisfied.

1. INTRODUCTION

LOW'S integral equations [1] have until now been solved either by the method described in the paper by Salzman and Salzman [2] or by the so-called N/D method [3]. The essence of both methods consists of the regularization of Low's singular equations.

In applying the N/D method the regularization can be equivalent or nonequivalent depending on whether the denominator D does or does not have zeros. Since the location of zeros of D in the complex plane is tedious and inaccurate, the N/D method often leads to false or, at best, to unreliable solutions. The same also occurs in the case of the method of Salzman and Salzman [2]. In the present paper a set of algebraic equations is derived which, being equivalent to Low's integral equations, does not lead to false solutions. Equivalence is understood in the sense that there exists a one-to-one correspondence between the solutions of the set of algebraic equations and those of the integral equations, if solutions are sought on a class of functions having the asymptotic behavior $h(x) \lesssim 1/x$ at infinity.

A set of algebraic equations can also be derived for the determination of πN scattering in Mandelstam's formulation [4]. Here the scattering amplitude is a function of two complex variables, and although the algebraic treatment has great advantages compared to Mandelstam's integral equations [5], it is nevertheless still complicated. Therefore, a preliminary study of a model problem of the same type is required. Such a model problem is the set of algebraic equations derived in our paper.

In some cases the set of algebraic equations is very advantageous for theoretical investigations of the solutions of Low's equations. It also has advantages for numerical computation, since it does not contain any Cauchy integrals. The set of equations can be derived by starting either with the integral equations [6], or from the corresponding boundary value problem [1]. We shall adopt the latter method.

2. FORMULATION OF THE BOUNDARY VALUE PROBLEM

Low's integral equations are equivalent to the following boundary value problem [1,2].

It is required to find the functions $h'_\alpha(z')$ of the complex variable $z' = x' + iy'$, $\alpha = 1, 2, \ldots N$ ($N = 1$ [7], $N = 2$ [7,8], $N = 3$ [1,9,3], $N = 4$ [10]), satisfying conditions (I′), (II′), (III′).

(I′) $h'_\alpha(z')$ are analytic functions of the complex variable z' in the z' plane with cuts $(-\infty, -1)$ and $(+1, +\infty)$, with the exception of the point $z' = 0$, where they have simple poles with residues λ_α. The functions $h'_\alpha(x')$ are smooth and at infinity fall off not slower than $1/x$.

(II′) $h'_\alpha(x' + i\varepsilon)$ satisfy the unitarity relations:

$$\operatorname{Im} h'_\alpha(x' + i\varepsilon) = p'^n(x')\, v'^2(p')\, |h'_\alpha(x' + i\varepsilon)|^2, \qquad 1 \leqslant x < \infty, \tag{1}$$

where $n = 1$ [7,9], or $n = 3$ [1,3,8,10], $p' = (x'^2 - 1)^{1/2}$ is the momentum of the meson in the center of mass system, x' is the total energy of the meson in the same system, $v' = v'(p')$ is the "cut-off" function [1,11,8].

(III′) $h'_\alpha(x')$ satisfy the cross relations

$$h'_\alpha(-x' - i\varepsilon) = \sum_{\beta=1}^{N} A_{\alpha\beta} h'_\beta(x' + i\varepsilon), \qquad 1 \leqslant x' < \infty, \tag{2}$$

where $\sum\limits_{\beta=1}^{N} A_{\alpha\beta} A_{\beta\gamma} = \delta_{\alpha\gamma}$.

We consider the function

$$z' = f(z) = 2z/(1 + z^2). \quad (3)$$

It performs a conformal mapping of the z' plane on the z plane, which transforms the unit circle $|z| < 1$ into a plane with the cuts $(-\infty, -1)$ and $(+1, +\infty)$.

In (I′) we replace the variable z' by z. Then the function $h'_\alpha(z')$ goes over into the function $h_\alpha(z)$ which has the following property: (I) $h_\alpha(z)$ is analytic inside the unit circle $|z| < 1$ with the exception of the point $z = 0$, where it has a pole with the residue $\lambda_\alpha/2$, while the function $h_\alpha(z)$ is piecewise smooth at $|z| = 1$.

Conformal transformation [10] transforms the cuts in the z' plane into the unit circumference $|z| = 1$ in accordance with the formula

$$\cos\varphi = 1/x'. \quad (4)$$

In future we shall use the notation

$$F(\varphi) = p'^4(\cos^{-1}\varphi)\, v'^2(p') = p'^4(x')\, v'^2(p').$$

We shall assume that the function $F(\varphi)$ is piecewise smooth.

The representation of a function of a complex variable by means of two conjugate harmonic functions is invariant with respect to a conformal transformation. Consequently, if there exists some sort of a relationship between the real and the imaginary parts of a given function of a complex variable, it retains its form also after the conformal transformation. From this it follows that property (II′) will not be altered: (II) $h_\alpha(\varphi)$ satisfies the unitarity conditions

$$\operatorname{Im} h_\alpha(\varphi) = F(\varphi)\,|h_\alpha(\varphi)|^2, \qquad -\pi/2 < \varphi < \pi/2. \quad (5)$$

After the conformal transformation (III′) goes over into (III): (III) $h_\alpha(\varphi)$ satisfies the cross relations

$$h_\alpha(\varphi + \pi) = \sum_{\beta=1}^{N} A_{\alpha\beta} h_\beta(\varphi). \quad (6)$$

As a result of this the problem is reduced to finding the function $h_\alpha(\varphi)$. Our goal is the solution of the problem formulated by conditions (I), (II), and (III).

3. DERIVATION OF THE ALGEBRAIC SET OF EQUATIONS

It follows from (I) that $h_\alpha(z)$ can be represented in the form

$$h_\alpha(z) = \frac{\lambda_\alpha}{2z} + \sum_{n=0}^{\infty} a_n^{(\alpha)} z^n, \quad (7)$$

where all the coefficients are real, as follows from the condition

$$h_\alpha'^{*}(z') = h'_\alpha(z'^{*}).$$

On the unit circle $|z| = 1$ we obtain from (7) two Fourier series:

$$\operatorname{Re} h_\alpha(\varphi)\,|_{|z|=1} = a_0^{(\alpha)} + (\lambda_\alpha/2 + a_1^{(\alpha)})\cos\varphi + \sum_{n=2}^{\infty} a_n^{(\alpha)} \cos n\varphi, \quad (8)$$

$$\operatorname{Im} h_\alpha(\varphi)\,|_{|z|=1} = (-\lambda_\alpha/2 + a_1^{(\alpha)})\sin\varphi + \sum_{n=2}^{\infty} a_n^{(\alpha)} \sin n\varphi. \quad (9)$$

As concerns the function $F(\varphi)$, we must keep the following in mind: the symmetries inherent in $F(\varphi)$ are obtained from symmetries in the functions p'^4 and v^2. It is well known that the function $p' = (z'^2 - 1)^{1/2}$ has the property $p'(+x' + i\varepsilon) = p'(-x' - i\varepsilon)$. From this it follows that $p(\varphi) = p(\varphi + \pi)$. Moreover, $p'(x' + i\varepsilon) = -p'(x' - i\varepsilon)$, i.e., $p(\varphi) = -p(-\varphi)$. The function v^2 depends on p^2, and, therefore, it has identical values at all corresponding points of the four quadrants. Consequently, $F(\varphi)$ has the properties:

$$F(\varphi) = -F(-\varphi), \qquad F(\varphi) = F(\varphi + \pi).$$

The first of these equations shows that $F(\varphi)$ can be expanded in a series of sines, while the latter equation shows that terms with an odd value of the index n should not occur in the expansion:

$$F(\varphi) = \sum_{n=1}^{\infty} F_{2n} \sin 2n\varphi, \quad (10)$$

where F_{2n} are known numbers.

The function

$$D_\alpha(\varphi) = \operatorname{Im} h_\alpha(\varphi)\,|_{|z|=1} - F(\varphi)\,|h_\alpha(\varphi)|^2\,|_{|z|=1} \quad (11)$$

is periodic of period 2π, and can be represented by means of a Fourier series

$$D_\alpha(\varphi) = \sum_{n=1}^{\infty} D_n^{(\alpha)} \sin n\varphi. \quad (12)$$

It can be seen from (5), that $D_\alpha(\varphi) = 0$ for all values of φ in the range $-\pi/2 < \varphi < \pi/2$.

In order that the series (12) should vanish in the range $-\pi/2 < \varphi < \pi/2$ it is necessary and sufficient that the coefficients $D_1^{(\alpha)}, D_2^{(\alpha)}, D_3^{(\alpha)}, \ldots$ should satisfy the equations

$$\frac{\pi}{2} D_{2n}^{(\alpha)} - 4n(-1)^n \sum_{p=0}^{\infty} D_{2p+1}^{(\alpha)} \frac{(-1)^p}{(2n)^2 - (2p+1)^2} = 0$$

$$(n = 1, 2, 3, \ldots). \quad (13)$$

We substitute (8), (9) and (10) into (11) and compare the resultant expression with the expansion (12). Then for $D_k^{(\alpha)}$ we obtain

$$D_k^{(\alpha)} = \sum_m C_{km}^{(\alpha)} a_m^{(\alpha)} + \sum_{m,\,n} D_{kmn}^{(\alpha)} a_m^{(\alpha)} a_n^{(\alpha)}, \tag{14}$$

where the coefficients $C_{kn}^{(\alpha)}$ and $D_{kmn}^{(\alpha)}$ contain F_{2n} to a power not higher than the first.

Substitution of (14) into (13) leads to the set of algebraic equations:

$$\sum_{m=0}^{\infty} A_{km}^{(\alpha)} a_m^{(\alpha)} + \sum_{m,\,n=0}^{\infty} B_{kmn}^{(\alpha)} a_m^{(\alpha)} a_n^{(\alpha)} = 0$$

$$(\alpha = 1, 2, \ldots, N;\ k = 1, 2, \ldots, \infty), \tag{15}$$

for the unknowns $a_m^{(\alpha)}$ which is equivalent to the unitarity relation (II).

The set of algebraic equations corresponding to the cross relations is obtained either by the substitution of (8) into (6), or by the substitution of (9) into (6):

$$\sum_{\beta=1}^{N} [A_{\alpha\beta} - (-1)^n \delta_{\alpha\beta}]\, a_n^{(\beta)} = 0$$

$$(\alpha = 1, 2, \ldots, N;\ n = 0, 1, 2, \ldots, \infty). \tag{16}$$

This reduces the boundary value problem (I′), (II′), and (III′) to the sets of algebraic equations (15) and (16).

Expressing the coefficients A_{km} and B_{kmn} in (15) in terms of F_{2k} we obtain

$$F_{2k}\left(\frac{\lambda_\alpha}{2}\right)^2 + \frac{\lambda_\alpha}{2}\sum_{i=1}^{\infty}(F_{2i+4k} - F_{2i})\, a_{2i+2k-1}^{(\alpha)}$$

$$+ \frac{\lambda_\alpha}{2}\sum_{i=1}^{k}(F_{2k-2i} + F_{2k+2i})\, a_{2i-1}^{(\alpha)}$$

$$- a_{2k}^{(\alpha)} + (-1)^k \frac{8k}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2k)^2 - (2j+1)^2}$$

$$\times \left[a_{2j+1}^{*(\alpha)} - \frac{\lambda_\alpha}{2}\sum_{i=1}^{\infty}(F_{2i+4j+z} - F_{2i})\, a_{2i+2j}^{(\alpha)}\right.$$

$$\left.- \frac{\lambda_\alpha}{2}\sum_{i=1}^{j+1}(F_{2j-2i+2} + F_{2j+2i})\, a_{2i-2}^{(\alpha)}\right] + \sum_{i=1}^{\infty}(F_{2i+4k} - F_{2i})\, S_{2i+2k}^{(\alpha)}$$

$$+ \sum_{i=1}^{k}(F_{2k-2i} + F_{2k+2i})\, S_{2i}^{(\alpha)} + F_{2k} S_0^{(\alpha)}$$

$$- (-1)^k \frac{8k}{\pi}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2k)^2 - (2j+1)^2}\left[\sum_{i=1}^{\infty}(F_{2i+4j+2} - F_{2i})\, S_{2i+2j+1}^{(\alpha)}\right.$$

$$\left.+ \sum_{i=1}^{j+1}(F_{2j-2i+2} + F_{2j+2i})\, S_{2i-1}^{(\alpha)}\right] = 0;$$

$$S_m^{(\alpha)} = \sum_{n=0}^{\infty} a_n^{(\alpha)} a_{n+m}^{(\alpha)}, \qquad m = 0, 1, 2, \ldots;$$

$$a_{2j+1}^{*(\alpha)} = \begin{cases} a_1^{(\alpha)} - \lambda_\alpha/2, & \text{if } j = 0, \\ a_{2j+1}^{(\alpha)}, & \text{if } j = 1, 2, 3, \ldots \end{cases} \tag{17}$$

In particular, the equations of cross symmetry (16) for the matrix from the paper by Salzman and Salzman [2] have the form:

$$a_n^{(2)} = 2a_n^{(3)} - a_n^{(1)} \quad \text{for } n = 0, 2, 4, 6, \ldots,$$
$$a_n^{(1)} = -2a_n^{(3)} \quad \text{for } n = 1, 3, 5, \ldots,$$
$$a_n^{(2)} = -\tfrac{1}{2}\, a_n^{(3)} \quad \text{for } n = 1, 3, 5, \ldots$$

4. THE METHOD OF SOLVING THE SET OF EQUATIONS

The set (15) and (16) is not convenient for obtaining the adiabatic solutions by the usual method, since the determinant $\mathrm{Det}\,(A_{km}^{(\alpha)}) = 0$, where $A_{km}^{(\alpha)}$ is the coefficient in (15). Adiabatic solutions can be obtained with the aid of the sets of algebraic equations given in [12].

The set (15) and (16) is suitable for finding resonance solutions.

We assume that the solution depends on S parameters, i.e., that under certain assumptions each curve $\mathrm{Im}\, h_\alpha(\varphi)$ passes through S points with the abscissas $\varphi_1, \varphi_2, \ldots, \varphi_S$ and with ordinates given respectively by $H_1^{(\alpha)}$, $H_2^{(\alpha)}, \ldots H_S^{(\alpha)}$. Consequently we shall have

$$\mathrm{Im}\, h_\alpha(\varphi_s) = (-\lambda_\alpha/2 + a_1^{(\alpha)})\sin\varphi_s + \sum_{n=2}^{\infty} a_n^{(\alpha)} \sin n\varphi_s = H_s^{(\alpha)}$$

$$(s = 1, 2, \ldots, S;\ \alpha = 1, 2, \ldots, N). \tag{18}$$

The simultaneous solution of the system (15), (16) and (18) can be conveniently found by Newton's method [13]. We substitute $a_n^{(\alpha)} = b_n^{(\alpha)} + x_n^{(\alpha)}$, where $b_n^{(\alpha)}$ are approximate values of $a_n^{(\alpha)}$, while $x_n^{(\alpha)}$ are small corrections.

If the linearized sets (15), (16) and (18) are broken off at the K-th term they can be respectively written in the following form

$$\sum_{m=0}^{K} A_{km}^{(\alpha)} x_m^{(\alpha)} + \sum_{m,\,n=0}^{K} B_{km}^{(\alpha)} (b_m^{(\alpha)} x_n^{(\alpha)} + b_n^{(\alpha)} x_m^{(\alpha)})$$

$$= -\sum_{m=0}^{K} A_{km}^{(\alpha)} b_m^{(\alpha)} - \sum_{m,\,n=0}^{K} B_{kmn}^{(\alpha)} b_m^{(\alpha)} b_n^{(\alpha)}$$

$$(\alpha = 1, 2, \ldots, N,\ k = 1, 2, \ldots, K_1\ (K_1 < K)), \tag{19}$$

$$\sum_{\beta=1}^{N} [A_{\alpha\beta} - (-1)^n \delta_{\alpha\beta}]\, x_n^{(\beta)} = -\sum_{\beta=1}^{N} [A_{\alpha\beta} - (-1)^n \delta_{\alpha\beta}]\, b_n^{(\beta)}$$

$$(\alpha = 1, 2, \ldots, N,\ \ n = 1, 2, \ldots, K), \tag{20}$$

$$\sum_{n=1}^{K} x_n^{(\alpha)} \sin n\varphi_s = H_s^{(\alpha)} - (-\lambda_\alpha/2 + b_1^{(\alpha)}) \sin\varphi_s - \sum_{n=2}^{K} b_n^{(\alpha)} \sin n\varphi_s$$
$$(\alpha = 1, 2, \ldots, N, \quad s = 1, 2, \ldots, S). \tag{21}$$

We note that in the set (20) there exist $L < NK$ independent equations. Therefore, the parameter K_1 which determines the number of equations in (19) is so chosen that the total number $N(K+1)$ of unknowns is equal to the total number $NK_1 + L + NS$ equations in the set (19)—(21).

Given the initial approximate values $b_n^{(\alpha)}$, on solving the system (19)—(21) we obtain the corrections $x_n^{(\alpha)}$. We then determine the first approximations $b_n^{(\alpha)} + x_n^{(\alpha)}$ etc. The initial approximate values $b_n^{(\alpha)}$, $n = 1, 2, \ldots, S$, $\alpha = 1, 2, \ldots, N$ are determined as the solution of the set (21) with $x_n^{(\alpha)} = 0$, $\alpha = 1, 2, \ldots, N$, $n = 1, 2, \ldots, K$ and $b_{S+1}^{(\alpha)} = \ldots = b_K^{(\alpha)} = 0$, $\alpha = 1, 2, \ldots, N$. Because of this, it is possible to replace the set (21) by a simpler one:

$$\sum_{n=1}^{K} x_n^{(\alpha)} \sin n\varphi_s = 0, \qquad \alpha = 1, 2, \ldots, N,$$
$$s = 1, 2, \ldots, S, \tag{22}$$

since the right-hand sides of (21) will always be equal to zero.

As an example we consider the case with $N = 1$, $\lambda_\alpha = 0$ and $p'^4(x')v'^2(p') = 1$, i.e., the equation

$$\operatorname{Im} h'(z') = |h'(z')|^2. \tag{23}$$

We also add the cross relation

$$h'(-z') = h'(z') \tag{24}$$

and demand that $h'(z')$ should be analytic in the plane with the cuts $(-\infty, -1)$, $(+1, +\infty)$.

Equations (23) and (24) and the requirement of analyticity are satisfied by the function

$$h'(z') = -\sqrt{z'^2 - 1} \Big/ \left[\frac{1+\mu}{2(1-\mu)}(z'^2 - 2) + i\sqrt{z'^2 - 1}\right],$$

which in going over to the z plane takes on the form

$$h(z) = \frac{1-\mu}{2} \frac{1 - z^4}{1 + \mu z^4} i, \tag{25}$$

from which it follows that

$$\operatorname*{Im}_{|z|=1} h(z) = \operatorname{Im} h(\varphi) = \frac{\sin^2 2\varphi}{[(1+\mu)/(1-\mu)]^2 \cos^2 2\varphi + \sin^2 2\varphi}$$
$$= \sum_{n=0}^{\infty} a_{4n} \cos 4n\varphi, \tag{26}$$

$$\operatorname*{Re}_{|z|=1} h(z) = \operatorname{Re} h(\varphi) = \frac{1+\mu}{1-\mu} \frac{\sin 2\varphi \cos \varphi}{[(1+\mu)/(1-\mu)]^2 \cos^2 2\varphi + \sin^2 2\varphi}$$
$$= \sum_{n=1}^{\infty} a_{4n} \sin 4n\varphi; \tag{27}$$

$$a_0 = \frac{1-\mu}{2}, \qquad a_{4n} = (-1)^n \frac{1-\mu^2}{2} \mu^{n-1} \quad (n = 1, 2, 3, \ldots). \tag{28}$$

The problem which we are considering as an example satisfies conditions (I′), (II′), (III′), corresponding to (I), (II), (III), with the one difference that in (I) n is an even number. Therefore, Im h is expanded in a Fourier cosine series, while Re h is expanded into a Fourier sine series. This difference, without altering the structure of the set of algebraic equations, leads to an example which is not trivial and, at the same time, is sufficiently simple for calculations and convenient for checking the effectiveness of different methods of nonlinear analysis.

An example of the application of algebraic equations to the solution of problem (I′), (II′), (III′) without any simplifications is given in [12].

The function defined by formulas (26) and (27) describes a resonance solution. The position of the resonance is fixed and corresponds to $\varphi = \pi/4$. The parameter μ determines the width of the resonance, and the fact that it depends on one parameter enables us to study the effect of the width of the resonance on the numerical methods of solution.

We shall solve the boundary value problem (23) and (24) with the aid of a set of algebraic equations analogous to (15) and (16). The equations of cross symmetry are not satisfied in this case. As regards the set of equations (15), in place of it we can utilize the equivalent system

$$a_0 = \sum_{n=0}^{\infty} a_n^2, \quad \tfrac{1}{2} a_k = \sum_{n=0}^{\infty} a_n a_{n+k}, \quad k = 1, 2, 3, \ldots, \tag{29}$$

which is obtained by the substitution of (26) and (27) into (23).

It can be verified that the exact solutions of the system (29) are given by formulas (28).

We now go on to the description of the solutions of (29) by means of Newton's method.

From the symmetry of the curve Im h(φ) and from the existence of a resonance it follows that it necessarily passes through the points with the abscissas $\varphi_1 = 0$ and $\varphi_3 = \pi/4$, and the corresponding ordinates $H_1 = 0$ and $H_3 = 1$. In (26) we can select μ in such a manner that for a given $\varphi = \varphi_2$ the function Im h(φ) has the given value H_2, which can vary arbitrarily over a certain

μ	n	a_{4n}	b_{4n}	x_{4n}	$\tilde{b}_{4n}$	$\left\|\frac{a_{4n}-\tilde{b}_{4n}}{a_{4n}}\right\|$
1/2	0	+0.25000	+0.300	—0.0320	+0.2680	0.072
	1	—0.37500	—0.500	+0.0928	—0.4072	0.086
	2	+0.18750	+0.200	0	+0.2000	0.067
	3	—0.09375	0	—0.0800	—0.0800	0.147
	4	+0.04687	0	+0.0320	+0.0320	0.318
	5	—0.02344	0	—0.0128	—0.0128	0.454
1/3	0	+0.33333	+0.350	—0.0135	+0.3365	0.010
	1	—0.44444	—0.500	+0.0491	—0.4509	0.015
	2	+0.14815	+0.150	0	+0.1500	0.013
	3	—0.04938	0	—0.0450	—0.0450	0.089
	4	+0.01646	0	+0.0135	+0.0135	0.180
	5	—0.00549	0	—0.0040	—0.0040	0.271
1/4	0	+0.37500	+0.382	—0.0065	+0.3755	0.002
	1	—0.46875	—0.500	+0.0294	—0.4706	0.004
	2	+0.11719	+0.118	0	+0.1180	0.007
	3	—0.02930	0	—0.0276	—0.0276	0.058
	4	+0.00732	0	+0.0065	+0.0065	0.111
	5	—0.00183	0	—0.0018	—0.0018	0.017

range. We choose $\varphi_2 = \pi/8$. Then $H_2 = (1-\mu)^2/2(1+\mu^2)$.

We set K = 20. Since $\varphi_1 = 0$, $\varphi_2 = \pi/8$ and $\varphi_3 = \pi/4$, then (22) goes over into the system[1]

$$x_0 + x_4 + x_8 + x_{12} + x_{16} + x_{20} = 0, \qquad x_0 - x_8 + x_{16} = 0,$$
$$x_0 - x_4 + x_8 - x_{12} + x_{16} - x_{20} = 0. \tag{30}$$

In (29) we retain only the first three equations and the first six unknowns. After linearization it goes over into the set of equations:

$$(b_0 - \tfrac{1}{2})\, x_0 + b_4x_4 + b_8x_8 + b_{12}x_{12} + b_{16}x_{16} + b_{20}x_{20} = A_1,$$
$$b_4x_0 + (b_0 + b_8 - \tfrac{1}{2})\, x_4 + (b_4 + b_{12})\, x_8 + (b_8 + b_{16})\, x_{12}$$
$$+ (b_{12} + b_{20})\, x_{16} + b_{16}x_{20} = A_2,$$
$$b_8x_0 + b_{12}x_4 + (b_0 + b_{16} - \tfrac{1}{2})\, x_8 + (b_4 + b_{20})\, x_{12}$$
$$+ b_8x_{16} + b_{12}x_{20} = A_3, \tag{31}$$

where

$$A_1 = \tfrac{1}{2}\Big(b_0 + \sum_{n=0}^{5} b_{4n}^2\Big), \qquad A_2 = \tfrac{1}{2} b_4 - \sum_{n=0}^{4} b_{4n}b_{4n+4},$$
$$A_3 = \tfrac{1}{2} b_8 - \sum_{n=0}^{3} b_{4n}b_{4n+8}.$$

The simultaneous solution of the sets (30) and (31) yields the desired corrections x_0, x_4, x_8, x_{12}, x_{16}, x_{20}. In order to begin the iteration process it is necessary to obtain also the zero order approximations b_0, b_4, b_8, b_{12}, b_{16}, b_{20}. As has been stated earlier, b_0, b_4, b_8 are obtained as the solutions of the set of equations:

$$b_0 + b_4 + b_8 = 0, \qquad b_0 - b_4 + b_8 = 1,$$
$$b_0 - b_8 = (1-\mu)^2/2\,(1+\mu^2),$$

while b_{12}, b_{16}, b_{20} are set equal to zero.

The exact values a_{4n} of the Fourier coefficients, the initial approximate values b_{4n}, the corrections x_{4n} and the first order approximations $\tilde{b}_{4n}$ for $\mu = 1/2$, $\mu = 1/3$ and $\mu = 1/4$ are given in the table.

5. CONCLUSION

The problem of determining the scattering amplitude is, instead of solving Low's integral equations, reduced to finding such solutions a_n (n = 0, 1, 2, . . .) of the algebraic set of equations (15), (16), which, as $n \to \infty$, fall off not slower than const-n^{-2}. This corresponds to the scattering amplitudes $h(z)$ which as $z \to \infty$ fall off not slower than const z^{-1}.

The numerical example investigated in this paper shows that the algebraic sets of equations have significant advantages compared with the integral equations.

[1]Formula (22) is written for the case when Im h_α is expanded in a cosine series. In the example under consideration Imh_α is expanded in a cosine series utilizing formulas (22) and assumes the form

$$\sum_{n=0}^{K} x_n^{(\alpha)} \sin n\varphi_s = 0, \qquad \alpha = 1, 2, \ldots, N, \qquad s = 1, 2, \ldots, S.$$

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