AN INTEGRAL EQUATION FOR THE $\omega_0 \to 3\pi$ DECAY

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Submitted to JETP editor February 28, 1963

J. Exptl. Theoret. Phys. (U.S.S.R.) **46**, 196-200 (January, 1964)

A deduction and investigation of an approximate integral equation for the absorption part of the $\omega_0 \to 3\pi$ decay amplitude are presented.

1. A theoretical investigation of three-particle decays of unstable particles is of great interest, since the amplitudes of such processes contain the simplest anomalous singularities that are possessed by the amplitudes of inelastic processes. Several papers by Gribov et al.[1-3] deal with the anomalous singularities of the simplest Feynman diagram for the inelastic processes and, as a particular case, for the τ decay. It is of interest to clarify the degree to which it is possible to use the method of integral equations, obtained from the analyticity and unitarity conditions in order to extract information on the terms whose presence in the expression for the amplitude of the process is connected with the existence of anomalous singularities. In the present article we describe briefly an investigation of the integral equation for the absorption part of the amplitude of the $\omega_0 \to 3\pi$ decay. An investigation of this process is of interest also because the spin of the ω_0 meson is equal to unity. We shall also use the fact that the isotopic spin of the ω_0 meson is equal to zero, and that the intrinsic parity is negative. The combination of these properties causes the transformation properties of the amplitude of the $\omega_0 \to 3\pi$ decay to coincide identically with the properties for the amplitude of photoproduction of a pion by a pion[4]. Therefore we can write for the amplitude of the $\omega_0 \to 3\pi$ decay the following expression[4]:

$$A_{\omega_0} = \frac{1}{2i}\varepsilon_{mnrs}\varepsilon_{\alpha\beta\gamma}\frac{q_1^m q_2^n q_3^r e^s}{4\sqrt{q_1^0 q_2^0 q_3^0 k^0}}F(s_1, s_2, s_3),$$

where q_i are the 4-momenta of the pions, k is the 4-momentum of the ω_0 meson and $F(s_1, s_2, s_3)$ is a function of the invariants s_1, s_2, and s_3.

The essential difference from the case of photoproduction is that the function $F(s_1, s_2, s_3)$ should have anomalous singularities. In order to obtain the integral equation we shall reason as follows. We first assume for the ω_0 meson a non-decay mass, and consider the fictitious process

$$\tilde{\omega}_0 + \pi \to 2\pi \tag{1}$$

(the ~ denotes that a non-decay mass of the ω_0 is considered for the time being). In particular, the mass of $\tilde{\omega}_0$ may prove to be equal to zero, and then we get photoproduction[5]. For the amplitude of the process (1) we have a Mandelstam representation in the form of a single representation[6] for F:

$$F(s_1, s_2, s_3) = B_0 + \frac{1}{\pi}\int_{4\mu^2}^{\infty}[\tilde{\varphi}(\sigma, s_1) + \tilde{\varphi}(\sigma, s_2) + \tilde{\varphi}(\sigma, s_3)]\,\alpha(\sigma)\,d\sigma. \tag{2}$$

For the partial P-wave we have the expression

$$\langle A_{\tilde{\omega}_0}\rangle_1 = \frac{kq}{4\sqrt{k^0 q^0}}B(q^2); \tag{3}$$

$$B(q^2) = B_0 + \frac{1}{\pi}\int_{4\mu^2}^{\infty}\tilde{\varphi}(\sigma, s_1)\,\alpha(\sigma)\,d\sigma + \frac{3}{4\pi}\int_{-1}^{+1}\sin^2\theta\, d(\cos\theta) \times \int_{4\mu^2}^{\infty}[\tilde{\varphi}(\sigma, s_2) + \tilde{\varphi}(\sigma, s_3)]\,\alpha(\sigma)\,d\sigma,$$

$$\tilde{\varphi}(\sigma, s_i) = (s_i - s_i^0)/(\sigma - s_i^0)(\sigma - s_i), \tag{4}$$

k and q are the absolute values of the three-dimensional momenta of the $\tilde{\omega}_0$ meson and of the two pions at the end of the reaction (1), respectively (in the c.m.s. of the two aforementioned pions).

It is important that if the mass of $\tilde{\omega}_0$ is stable, the double integral in the right half of (4) is real. Going over to the variables $\nu_i = q_i^2/\mu^2$ and making simple changes of variable, we obtain

$$B(\nu) = B_0 + \frac{1}{\pi}\int_0^{\infty}\tilde{\varphi}(\nu', \nu)\,\alpha(\nu')\,d\nu' + \frac{3}{\pi R(\nu)}\int_{\nu_2(\nu)}^{\nu_1(\nu)}\left[1 - \frac{4(\tilde{\nu} - \gamma + \nu/2)^2}{R^2(\nu)}\right]d\tilde{\nu} \times \int_0^{\infty}\tilde{\varphi}(\nu', \tilde{\nu})\,\alpha(\nu')\,d\nu';$$

$$\nu_1(\nu) = \gamma - \frac{\nu}{2} + \frac{1}{2}R(\nu), \quad \nu_2(\nu) = \gamma - \frac{\nu}{2} - \frac{1}{2}R(\nu),$$

$$\gamma = \frac{m^2 - 9\mu^2}{8\mu^2},$$

$$R(\nu) = \left\{[\nu^2 - (4\gamma + 3)\nu + 4\gamma(\gamma + 1)]\frac{\nu}{\nu + 1}\right\}^{1/2}, \tag{5}$$

m — mass of the $\tilde{\omega}_0$ meson and μ — mass of the pion.

In order to change over to the decay value of the mass M_{ω_0}, it is necessary to add to γ a positive imaginary increment and carry out analytic continuation in the mass m. After performing this operation and letting the imaginary addition go to zero, we obtain

$$B(\nu) = \bar{B} + \frac{1}{\pi}\int_0^\infty \varphi(\nu', \nu)\,\alpha(\nu')\,d\nu' + \frac{3\theta(\gamma-\nu)}{\pi R(\nu)}\int_{\nu_2}^{\nu_1}\left[1 - \frac{4(\tilde{\nu}-\gamma+\nu/2)^2}{R^2(\nu)}\right]d\tilde{\nu} \times \int_0^\infty \varphi(\nu', \tilde{\nu})\,\alpha(\nu')\,d\nu'$$
$$+ \frac{3\theta(\nu-\gamma)}{\pi R(\nu)}\left[\int_0^{\nu_1}\left[1 - \frac{4(\tilde{\nu}-\gamma+\nu/2)^2}{R^2(\nu)}\right]d\tilde{\nu}\int_0^\infty \varphi(\nu', \tilde{\nu})\,\alpha(\nu')\,d\nu' + \int_0^{\nu_2}\left[1 - \frac{4(\tilde{\nu}-\gamma+\nu/2)^2}{R^2(\nu)}\right]d\tilde{\nu}\int_0^\infty \varphi(\nu', \tilde{\nu})\,\alpha(\nu')\,d\nu'\right], \quad (6)$$

where $\bar{B}$ is a different constant

$$\varphi(\nu', \nu) = \frac{\nu}{\nu'(\nu'-\nu)}, \quad \theta(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & x < 0 \end{cases}.$$

Substituting (6) in the unitarity condition

$$\alpha(\nu) = \sqrt{\frac{\nu}{\nu+1}}\,\lambda_1^+(\nu)\,B(\nu), \quad (7)$$

where $\lambda_1(\nu)$ is the partial $\pi\pi$ scattering P-amplitude, we obtain an equation for the absorption part $\alpha(\nu)$.

2. The double integrals in (6) are already complex, since ν_1 and ν_2 are real and positive in the physical region of the variable ν for decay values of the mass m. We note that, unlike τ decay, the physical region of the $\omega_0 \to 3\pi$ decay is quite large, and that in particular the upper limit of the decay region is $\xi_0^2 \cong 4.36$. Therefore an examination of the amplitude of the process in the form of an expansion in the above-threshold momenta [1-3] does not make it possible in this case to evaluate the anomalously singular terms.

Let us proceed in (7) to the variable $x = \nu/(\nu+1)$. We note that the amplitude of the partial $\pi\pi$ scattering P-wave is proportional to ν [7]:

$$\lambda_1(\nu) = \nu A_1(\nu). \quad (8)$$

We then obtain in place of (7)

$$\tilde{\alpha}(x) = \frac{x^{3/2}}{1-x}\tilde{A}_1^+(x)\bar{B} + \frac{x^{3/2}\tilde{A}_1^+(x)}{\pi(1-x)}F(\tilde{\alpha}) + \frac{3\theta(\beta-x)}{\pi(1+\gamma)}\frac{\tilde{A}_1^+(x)K_1(\tilde{\alpha})}{(x-\zeta^2)^{3/2}(x-\eta^2)^{3/2}} + \frac{3\theta(x-\beta)}{\pi(1+\gamma)}\frac{\tilde{A}_1^+(x)K_2(\tilde{\alpha})}{(x-\zeta^2)^{3/2}(x-\eta^2)^{3/2}};$$

$$F(\tilde{\alpha}) = \int_0^1 \varphi(x', x)\,\tilde{\alpha}(x')\,dx',$$

$$K_1(\tilde{\alpha}) = \int_{N(x)}^{N_1(x)} dy\left\{x(x-\zeta^2)(x-\eta^2) - \left[y(1-x)(1-\beta) - \beta + \frac{x}{2}(1+\beta)\right]^2\right\}\int_0^1 \varphi\left(x', \frac{y}{1+y}\right)\tilde{\alpha}(x')\,dx',$$

$$K_2(\tilde{\alpha}) = \int_0^{N_1(x)} dy\left\{x(x-\zeta^2)(x-\eta^2) - \left[y(1-x)(1-\beta) - \beta + \frac{x}{2}(1+\beta)\right]^2\right\} \times \int_0^1 \varphi\left(x', \frac{y}{1+y}\right)\tilde{\alpha}(x')\,dx'$$
$$+ \int_0^{N_2(x)} dy\left\{x(x-\zeta^2)(x-\eta^2) - \left[y(1-x)(1-\beta) - \beta + \frac{x}{2}(1+\beta)\right]^2\right\} \times \int_0^1 \varphi\left(x', \frac{y}{1+y}\right)\tilde{\alpha}(x')\,dx',$$

$$N_1(x) = \frac{\beta - x/2 - \beta x/2 + \sqrt{x(x-\zeta^2)(x-\eta^2)}}{(1-x)(1-\beta) - \beta + x/2 + \beta x/2 - \sqrt{x(x-\zeta^2)(x-\eta^2)}},$$

$$N_2(x) = \frac{\beta - x/2 - \beta x/2 - \sqrt{x(x-\zeta^2)(x-\eta^2)}}{(1-x)(1-\beta) - \beta + x/2 + \beta x/2 + \sqrt{x(x-\zeta^2)(x-\eta^2)}},$$

$$\zeta^2 = \frac{\xi_0^2}{1+\xi_0^2} \cong 0{,}81, \quad \beta \cong 0{,}74, \quad \eta^2 \cong 0{,}90,$$

$$\tilde{\alpha}(x) \equiv \alpha(\nu), \quad \tilde{A}_1(x) \equiv A_1(\nu). \quad (9)$$

In the physical region of the decay $0 \le x \le \zeta^2$. Thus, in place of an expansion in the above-threshold momenta, we can carry out expansion in the variable $\sqrt{x}$, where x is the ratio of the square of the momentum to the square of the total energy.

3. Let us consider the possibility of applying an iteration procedure to Eq. (9). In this case we take account of the fact that the resonant interaction of the two pions, which leads to the formation of the ρ meson, lies in the energy region outside the physical region of the $\omega_0 \to 3\pi$ decay. Indeed, its upper limit is $\xi_0^2 \cong 4.36$, whereas the quantity ν, corresponding to the resonance in the $\pi\pi$ system has a value $\nu_{res} \cong 6.18$. We therefore assume that the pion-pion scattering length is sufficiently small. We represent A_1, as usual, in the form of the ratio

$$A_1(\nu) = \frac{N_1(\nu)}{D_1(\nu)},$$

where $N_1(\nu)$ has a left-hand cut and $D_1(\nu)$ a right-hand cut. Then for $\nu \geq 0$[7]

$$\nu \sqrt{\frac{\nu}{\nu+1}} \operatorname{ctg} \delta_1(\nu) = \frac{\operatorname{Re} D_1(\nu)}{N_1(\nu)}, \qquad (10)^*$$

with $\operatorname{Im} N_1(\nu) = 0$. We proceed in (10) to the variable x. We then obtain

$$\frac{x^{3/2}}{1-x} \operatorname{ctg} \tilde{\delta}_1(x) = \frac{\operatorname{Re} \tilde{D}_1(x)}{\tilde{N}_1(x)}. \qquad (10a)$$

Assuming the integral terms in (9) to be small compared with the first term in the right half, we make two iterations. As a result we find that

$$\tilde{\alpha}_{(2)}(x) = \bar{B}\left\{a_1 x^{3/2} + a_1\left(1 + \frac{2}{\pi} - r_1 a_1\right) x^{5/2} + \frac{i\,27\sqrt{3}\,a_1^2\zeta^6}{16(1+\gamma)} \frac{z^3(1 - {}^{11}/_9 z^2 + {}^{28}/_{27} z^4)}{\sqrt{1-z^2}}\right\}, \qquad (11)$$

where a_1 is the scattering length of the $\pi\pi$ scattering partial P-amplitude, and r_1 is the first term of the expansion

$$-\frac{1}{a_1} + \frac{\operatorname{Re} \tilde{D}_1(x)}{\tilde{N}_1(x)} = \sum_{k=1}^{\infty} r_k x^k, \qquad (12)$$

which is valid in the region $0 \leq x < 1$, $z = \sqrt{x/\zeta^2}$, $\gamma \cong 2.84$. We thus see that in the case of the $\omega_0 \to 3\pi$ decay the anomalously singular terms are obtained in the second iteration $\sim \zeta_\omega^6 \cong 0.53$, whereas in the case of τ decay they are $\cong 0.39$.

For the invariant amplitude we obtain the expression

$$F_{(2)}(\nu_1, \nu_2, \nu_3) \equiv \tilde{F}_{(2)}(x_1, x_2, x_3) = \bar{B}\left\{1 + \sum_{k=1}^{3}\left[i a_1 x_k^{3/2} + i a_1\left(1 + \frac{2}{\pi} - r_1 a_1\right) x_k^{5/2} - \frac{27\sqrt{3}}{8(1+\gamma)} \frac{a_1^2 \zeta^6 z_k^3 (1 - {}^{11}/_9 z_k^2 + {}^{28}/_{27} z_k^4)}{\sqrt{1-z_k^2}} \times \operatorname{arc\,cos} z_k + c_1 x_k + c_2 x_k^2 + c_3 x_k^3\right]\right\}, \qquad (13)$$

*ctg = cot.

where c_1, c_2, and c_3 are certain constants, which in this analysis cannot be determined exactly from the iteration procedure. If we use the relation $\nu_1 + \nu_2 + \nu_3 = 2\gamma$, then all the terms linear in x_k can be expressed in terms of their quadratic and cubic combinations.

It is seen from (13) that the amplitude of the $\omega_0 \to 3\pi$ decay can be represented in the form of an expansion in powers of x and ζ^2.

It must be noted that in the case of τ decay of K mesons the procedure employed here gives in the second and third iterations exactly the same results as were obtained by Gribov et al.[1-3].

The author considers it his pleasant duty to thank Professor K. A. Ter-Martirosyan for very useful discussions on the problems touched upon here.

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Translated by J. G. Adashko
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