OSCILLATIONS IN THE RADIATION FROM A LASER

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An equation is derived for the time dependence of the radiation intensity of a laser. A solution of the equation is obtained in two extreme cases, for small and for large amplitude oscillations. The waveform and period of strong intensity oscillations are determined as functions of the amplitude. The waveform is very different from sinusoidal. The oscillation damping law is deduced.

1. INTRODUCTION

OBSERVATIONS of oscillations in the intensity of laser radiation, having a frequency of the order of 10^5-10^6 cps, have been reported in the experimental papers on optical quantum generators (for example [1,2]). Several papers [2-5] have dealt with the theory of this effect. The papers of Hellwarth[3] and Samson and Savva [4] are based on a model of a laser having the form of an infinite slab of the active medium with plane parallel end walls. Such a model can hardly serve as a good approximation since it does not give certain important properties of the laser; in particular the directionality of the laser output does not follow from this model.

The papers of Sorokin [2] and Kaiser et al [5] treat a laser with end mirrors of finite dimensions. However only small oscillations in the output intensity are treated in the above papers, while the experimentally observed oscillations have large amplitude and are decidedly non-sinusoidal in shape. Actually the results obtained this way amount to an investigation of the stability of the stationary laser mode. In the above papers no account was taken of the dependence of the coefficient of amplification on frequency, and this omission leads in certain cases to qualitatively incorrect results.

The present paper treats the oscillations in the intensity of the directional radiation from a laser with reflectors of finite size and, in contrast to the previous papers [2,5], takes account of the dependence of the amplification coefficient on the frequency of the light and treats the related variation in the spectral composition of the output with time. It is shown below that taking account of this dependence leads to a new damping mechanism for the oscillations. The present paper considers large oscillations in the intensity having non-sinusoidal form. The waveform and period of the oscillations as a function of amplitude are found, as are other characteristics of the oscillating mode. Small oscillations in the radiation intensity are also considered in investigating the stability of the stationary mode.

In order to simplify the calculations it is assumed in the present paper, as in the previous papers [2,5], that all points in the active medium experience identical conditions and therefore that at each instant of time the radiation energy density is everywhere the same. In other words it is assumed that all of the radiant energy in the resonator oscillates as a whole. This assumption seems to be rather well founded since, during the duration of a period of the oscillation, the radiation usually has sufficient time to redistribute itself in the cavity.

2. EQUATION FOR THE RADIATION INTENSITY

Let the working transition occur between two impurity levels 1 and 2 (with 2 higher than level 1). The laser transition is characterized by an absorption band and a luminescence band; the shapes of these bands are conveniently described by the functions $P_1(\omega)$ and $P_2(\omega)$ respectively, normalized to unit area [6]. Let the absorption coefficient in the impurity band [1)] be denoted $\mu(\omega)$.

We designate by c_1 and c_2 the concentrations of impurity centers in states 1 and 2 (i.e., their number per unit volume), and by c_0 the total concentration of impurities. One usually has the relation

[1)]In order not to complicate the formulae, we assume that the sample is uniform and optically isotropic, and that the laser output is correspondingly unpolarized.

$$c_1 + c_2 = c_0, \tag{1}$$

and it will be assumed to hold in what follows.

The interaction of the light with the active medium in the resonator is conveniently described by an effective absorption coefficient [7]

$$\varkappa(\omega) = \mu(\omega)\left[\frac{c_0 - c_2}{c_0} - \frac{c_2}{c_0}\frac{P_2(\omega)}{P_1(\omega)}\right] + \varkappa_1,$$

$$\varkappa_1 = \frac{1-K}{l} + \varkappa_0 + \varkappa_D. \tag{2}$$

Here K is the reflection coefficient of the end mirrors, l is their separation, κ_0 is the absorption coefficient of the host material, and $\kappa_D l$ is the relative diffraction loss for one reflection. The first term in square brackets in Eq. (2) describes the active (i.e., accompanying the laser radiation) absorption of light by the impurity centers, the second term describes the amplification of light due to the stimulated transitions $2 \rightarrow 1$, and the last term in (2) describes the passive absorption of light by the host material and mirrors, as well as the diffraction losses. Let the pump power absorbed in the sample be denoted by N, and the radiant energy in the resonator by E (N and E are defined per unit volume). Let ω_p be the average light frequency used for pumping, and let ω_0 be the operating frequency of the laser. In order to write down equations relating c_2 and E, we note that c_2 increases owing to the pump and to the active absorption of part of the energy E, and decreases owing to spontaneous emission. Thus we have

$$dc_2/dt = N/\hbar\omega_p - c_2/T + (E/\hbar\omega_0)\, v\, [\varkappa(\omega_0, c_2) - \varkappa_1] \tag{3}$$

(where v is the velocity of light in the medium, and T is the lifetime of the spontaneous transition $2 \rightarrow 1$). Here $\kappa(\omega, c_2)$ is a function of c_2 given by (2). We have put $\omega = \omega_0$ in the argument of κ, making use of the small spectral width of the laser radiation compared to the width of the (spontaneous) band of the impurity luminescence.

In order to write a second equation for c_2 and E we introduce the spectral energy density $\rho(\omega, t)$, which is related to E by the relation

$$E(t) = \int \rho(\omega, t)\, d\omega. \tag{4}$$

We have the following equation for the spectral density

$$\frac{\partial \rho(\omega, t)}{\partial t} = \frac{c_2 \hbar\omega}{T}\frac{\delta\Omega}{2\pi} P_2(\omega) - v\varkappa(\omega)\,\rho(\omega, t) \tag{5}$$

(Here $\delta\Omega$ is the solid angle within which the directional laser radiation is contained.) In Eq. (5) $\delta\Omega$ is divided by 2π instead of 4π since, within the element of solid angle centered on a given direction n, there is light emitted not only in the direction n but also in the direction symmetric to n in the plane perpendicular to the optical axis of the laser.

(In the case where the angular spread of the radiation is comparable with the diffraction angle, the quantity $\delta\Omega/2\pi$ should be considered as an independent parameter, giving the relative fraction of the spontaneous emission which remains in the resonator and is added to the energy of the directional emission. In this case $\delta\Omega$ is of the same order of magnitude as the solid angle of the laser emission.)

The first term on the right hand side of (5) represents the fraction of the spontaneously emitted radiant energy falling within the small solid angle $\delta\Omega$ and in unit frequency interval[2]; the second term describes the losses of radiant energy within the cavity due to absorption effects.

We will assume that the pump is turned on at time t = 0, after which its intensity is independent of time. Taking this into account we can solve Eq. (5) for ρ

$$\rho(\omega, t) = \frac{\delta\Omega}{2\pi T}\hbar\omega P_2(\omega)\int_0^t c_2(t')\exp\left[-v\int_{t'}^t \varkappa(\omega, t'')\, dt''\right]dt'. \tag{6}$$

Integrating expression (6) over frequency and inserting it in equation (3), we obtain an equation with just one unknown, c_2. In order to integrate (6) explicitly we will consider two cases separately.

A. The impurity absorption band connected with the working transition does not overlap the corresponding luminescence band, so that in the neighborhood of the peak of the luminescence ω_0, expression (2) takes the form

$$\varkappa(\omega) = -\frac{c_2}{c_0}\frac{\bar{\mu}P_2(\omega)}{P_2(\omega_0)} + \varkappa_1, \quad \bar{\mu} = \frac{\mu(\omega)}{P_1(\omega)} P_2(\omega_0). \tag{7}$$

The parameter $\bar{\mu}$ does not depend on frequency since $P_1(\omega)$ reflects the shape of the absorption band. In order of magnitude $\bar{\mu}$ coincides with the maximum value of $\mu(\omega)$.

B. The absorption and luminescence bands corresponding to the working transition coincide[3], i.e., $P_1(\omega) \equiv P_2(\omega)$. In this case

[2]We recall that the function $P_2(\omega)$ gives the shape of the spontaneous luminescence spectrum.

[3]The absorption band due to the working transition must not be confused with the pumping band which, naturally, must not overlap with the luminescence band.

$$\varkappa(\omega) = \mu(\omega)(1 - 2c_2/c_0) + \varkappa_1. \tag{8}$$

Case A usually applies to two and four level lasers, whereas case B applies to three level lasers. We will make the calculation for case A only; for case B we will simply give the final results.

Let us turn to consideration of case A. Near the maximum of the luminescence ω_0, the function $P_2(\omega)$ may be represented in the form

$$P_2(\omega) = P_2(\omega_0)\exp[-(\omega-\omega_0)^2/A^2]$$
$$= P_2(\omega_0)[1-(\omega-\omega_0)^2/A^2] \tag{9}$$

(usually the Gaussian shape of the luminescence spectrum persists for some distance from the maximum, in which case A equals the half-width of the luminescence band multiplied by 0.6). We insert (9) in (7):

$$\varkappa(\omega) = \varkappa(\omega_0) + \bar{\mu}c_2(\omega-\omega_0)^2/c_0A^2,$$
$$\varkappa(\omega_0) = \varkappa_1 - c_2\bar{\mu}/c_0. \tag{10}$$

Using the small spectral width of the laser radiation compared to the width of the luminescence spectrum, we insert the expansion (10) for $\kappa(\omega)$ in (6). Integrating (6) with respect to the frequency and substituting in (3) we obtain an equation for c_2. It is convenient to write this equation in terms of dimensionless variables

$$u = t\sqrt{v\varkappa_1\zeta/T}, \qquad y = v\int_0^t \varkappa(\omega_0, t')\,dt',$$

where ζ is a parameter characterizing the amount by which the pump power exceeds threshold, $\zeta = (N - N_{thr})/N_{thr}$. In terms of the variables u and y, the equation for the radiation intensity has the form

$$y''(u) + 1 + \frac{\beta}{\zeta}y'(u)$$
$$-\varepsilon(1-\beta y')e^{-y}\int_0^u \frac{[1-\beta y'(u')]\,e^{y(u')}\,du'}{\{(u-u') + \beta[y(u') - y(u)]\}^{1/2}} = 0. \tag{11}$$

Here we have introduced the notation

$$\beta = \sqrt{\zeta/v\varkappa_1T}, \quad \varepsilon = AP_2(\omega_0)\,\delta\Omega\,(v\varkappa_1T)^{1/4}/2\sqrt{\pi}\zeta^{5/4},$$
$$\zeta = (N - N_{thr})/N_{thr} \tag{12}$$

($N_{thr} = h\omega pc_0\kappa_1/\bar{\mu}T$ is the threshold pump power).

The magnitude of ϵ is small because of the smallness of the solid angle $\delta\Omega$. It will be shown below that oscillations in the output occur only for small values of the parameter β. Hence ϵ and β have the role of the small parameters of the theory; it will be clear from what follows that $|\ln\epsilon|$ primarily determines the amplitude of the oscillation, and β and ϵ determine its damping.

3. THE STEADY STATE AND ITS STABILITY

Equation (11) may describe the steady state of the laser. To see this we put y' = const in (11) and let u go to infinity. Carrying out the integration over u', we have

$$1 + y'\beta/\zeta = \varepsilon(1 - y'\beta)^{3/2}\sqrt{\pi/y'}. \tag{13}$$

From this it is clear that y' is a quantity of second order in ϵ. From this we find, accurate to cubic terms,

$$y' = \pi\varepsilon^2. \tag{14}$$

Using the expansion (10) and putting $y = \pi\epsilon^2 u$ in (6) we find the spectral distribution of energy in the steady state

$$\rho(\omega) = \text{const}\,[1 + (\omega-\omega_0)^2/\pi\beta\varepsilon^2A^2]^{-1}. \tag{15}$$

In order to investigate the stability of the steady state we look for a solution of (11) of the form $y = \pi\epsilon^2u + \eta(u)$ ($\eta \to 0$). Linearizing (11) in η we have

$$\eta'' + \frac{\beta}{\zeta}\eta' + \varepsilon\beta\int_{-\infty}^{u}\frac{e^{\pi\varepsilon^2(u-u')}}{\sqrt{u-u'}}\Big\{\eta'(u) + \eta'(u')$$
$$+\Big[\frac{1}{\beta} - \frac{0{,}5}{u-u'}\Big][\eta(u) - \eta(u')]\Big\}du' = 0. \tag{16}$$

We assume that at the instant u = 0 the concentration c_2 varies smoothly under the influence of the momentary perturbation, i.e., the quantity η' takes on a non-zero value η_0'. Correspondingly, we solve (16) for the initial conditions $\eta(0) = 0$, $\eta'(0) = \eta_0'$. Using the Laplace transform method we find

$$\eta(u) = \frac{\eta_0'}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{pu}\Big[p^2 + p\beta\Big(1+\frac{1}{\zeta}\Big) - \frac{\varepsilon\sqrt{\pi}}{\sqrt{\pi\varepsilon^2+p}} + 1\Big]^{-1}dp \tag{17}$$

(terms of order $p\epsilon\beta$ have been omitted in the denominator of the integrand).

Let p_0 be the integrand pole with the largest real part. Displacing the contour of integration to the left we see that for $u \to \infty$ the solution (17) has the asymptotic form const e^{p_0u}. Setting the denominator of the integrand equal to zero we obtain an equation for p_0 which may be solved with the help of the small parameter ϵ. Finally ($u \to \infty$)

$$\eta = \text{const}\exp\Big\{\pm iu\sqrt{1 - \tfrac{1}{4}\beta^2(1+1/\zeta)^2}$$
$$- u\Big[\tfrac{1}{2}\beta(1+1/\zeta) + \varepsilon\sqrt{\pi/8}\Big]\Big\}. \tag{18}$$

Thus the oscillations damp out in time and the stationary state is stable. The rate of damping of the vibrations is the sum of two terms, proportional to β and ϵ. The term proportional to β increases without limit as the passive coefficient of absorption κ_1, describing the dissipation of radiant energy, goes to zero. Energy dissipation is therefore a necessary requirement for the oscillations. The physical meaning of this, initially, rather strange result, will be explained in the next section. The term proportional to ϵ is connected with the fact that spontaneous emission is constantly adding new energy to the radiant energy in the cavity and with the fact that this spontaneous emission has a different spectral composition. (We recall that the fraction of the spontaneously emitted energy remaining in the cavity is proportional to the solid angle $\delta\Omega$, i.e., to the small parameter ϵ.)

When $\epsilon = 0$ Eq. (18) coincides with the result of Kaiser et al[5]. In their paper the rate of damping of the oscillations does not contain a term proportional to ϵ since the spectral dependence of the amplification coefficient was ignored, with the result that the time variation of the energy does not depend on its spectral composition.

4. THE OSCILLATING MODE

As has already been pointed out, the experimentally observed oscillations in the laser output have large amplitude and non-sinusoidal shape. The reason for the occurrence of such oscillations may be easily understood in terms of the following qualitative considerations.

First we note that according to (2) the quantity $\kappa - \kappa_1$ represents the active part of the effective absorption coefficient, related to the absorption and stimulated emission of light by the impurity luminescence centers. For $\kappa - \kappa_1 > 0$, it is clear that the absorption of light by unexcited luminescence centers predominates over stimulated emission from the excited centers, and hence that c_2 increases under the influence of the radiant energy in the resonator (for the time being we are not considering the effect of the pump). On the other hand for $\kappa - \kappa_1 < 0$, stimulated emission predominates, and the effect of the radiant energy is to decrease c_2.

We assume that the pump was turned on at time $t = 0$. The concentration of excited centers begins to increase, and at some instant $t = t_1$ reaches a value c_2^* for which $\kappa(\omega_0)$ becomes zero (Fig. 1). Beginning at time $t = t_1$, induced (directional) radiation will begin to build up in the

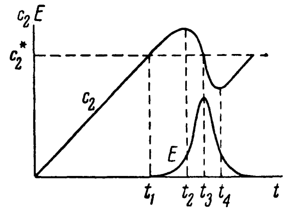

FIG. 1.

cavity. Since initially the stimulated transitions are caused only by the spontaneous emission, of which only a small fraction remains in the cavity, the intensity of the directional emission is initially extremely small—although it increases exponentially with time. When the energy of the directional emission reaches a sufficiently large value ($t = t_2$), the probability for stimulated emission from the impurity centers exceeds the probability for their excitation by the pump, after which c_2 begins to decrease. The energy however continues to grow until the moment $t = t_3$, when c_2 becomes less than c_2^*, and κ becomes negative. For $t > t_3$ the energy begins to decrease, and c_2 continues decreasing, since the quantity $\kappa - \kappa_1$ remains negative for some time. When, as a consequence of the damping of the energy, the probability for stimulated emission becomes less than the probability for exciting the centers by the pump ($t > t_4$), c_2 again begins to increase; however the energy continues to decrease exponentially and very quickly falls essentially to zero. After this the next period begins. In view of the fact that the energy of the directional emission initially increases very rapidly and then dies out very rapidly, the radiation spike occupies only a small fraction of the period.

It is clear from the foregoing that for the existence of oscillations it is necessary to satisfy simultaneously the inequalities $\dot{E} < 0$ and $\dot{c}_2 < 0$ in some interval (t_3, t_4). This is possible only for the condition $0 < \kappa(\omega_0) < \kappa_1$. Physically this requirement means that although the interaction of the radiation with the impurity centers leads to its amplification, it is nonetheless damped because of the predominance of passive absorption. Since κ depends linearly on c_2, the greater the range of variation of κ in the interval $t_3 < t < t_4$ the deeper the corresponding dip in the curve for c_2. Hence the larger κ_1, the stronger in general will be the oscillations in the output. For $\kappa_1 = 0$ oscillations would be impossible since the derivatives $\dot{E}$ and $\dot{c}_2$ could not simultaneously be negative.

In this section we will find the solution of (11) describing output oscillations of large amplitude. In what follows we assume that $\tilde{y} \gg 1$, where $\tilde{y}$

is the amplitude of the oscillating quantity y. In solving (11) we also make use of the small parameters ϵ and β.

We consider the instant of switching on the pump, u = 0, to be the perturbation which causes the oscillations in the laser output. (After being turned on, the power of the pump is constant.) We seek a solution of (11) in the form of a periodic function whose period and amplitude are slowly varying functions of the number of the period. Using this property of the solution we give equation (11) the simple form:

$$y''(u) + 1 - e^{-\gamma - y(u)} = 0, \tag{19}$$

where γ is some y-dependent functional whose value varies slowly from period to period.

We will show later that γ determines the amplitude of the vibration: $\tilde{y} \approx \gamma + \ln\gamma$. At the instant the pump is turned on $\gamma \approx |\ln\epsilon|$; as time goes on γ decreases. From the smallness of ϵ it follows that within a short interval after turning on the pump $\gamma \gg 1$, $\tilde{y} \gg 1$.

In solving (11) we first fix the value of γ and find y during a single period; then putting this solution for y (found for each period for the corresponding value of the parameter γ) in (11) we obtain an equation for γ.

We divide the u axis into periods as shown in Fig. 2. We call the mid-point of the n-th period $\bar{u}_n$, the point corresponding to the minimum of y, and we call the left hand boundary of the period, corresponding to the maximum, u_n. We define $y_n = y - y(u_n)$; clearly $\max\{y_n\} = 0$ within the n-th period.

In the n-th period Eq. (11) can be put in the form

$$y_n''(u) + 1 - e^{-\gamma_n - y_n} + \xi(u) = 0. \tag{20}$$

Here

$$e^{-\gamma_n} = \varepsilon \int_0^{\bar{u}_n} e^{y_n(u')} (\bar{u}_n - u')^{-1/2} du'. \tag{21}$$

The small terms in (20) responsible for the damping of the oscillations are designated ξ; they have the form

$$\xi(u) = \beta y_n' \left(\frac{1}{\zeta} + e^{-\gamma - y_n}\right)$$

$$-\varepsilon \int_0^{\bar{u}_n} \left[\frac{1}{\sqrt{u-u'}} - \frac{1}{\sqrt{\bar{u}_n - u'}}\right] e^{y_n(u') - y_n(u)} du' \tag{22}$$

[only the terms of lowest order in ϵ and β have been kept in (22)]. In (20) we neglect terms of order ϵ and β which do not lead to damping.

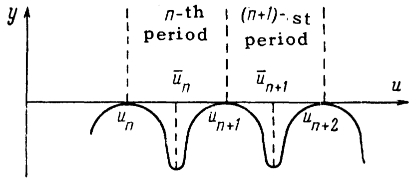

FIG. 2.

In expressions (21) and (22) it is permissible to change the upper limit of integration $u \to \bar{u}_n$. In fact the integral term in (11), proportional to ϵe^{-y}, contributes only in the neighborhood of the point $\bar{u}_n$, giving the minimum of y (we are using the fact that by definition the amplitude y of the oscillation is large). The integrand of the same expression makes a significant contribution only in the neighborhood of the point u_n, where y is close to its maximum. Elsewhere, with the exclusion of the neighborhood of $\bar{u}_n$, the integral term in (21) and (22) is not significant, and in the neighborhood of $\bar{u}_n$ replacing u by $\bar{u}_n$ in the upper limit of the integral does not affect its value.

Initially we neglect the term ξ in (20); the equation then goes over into (19), which has the form of the usual equation of mechanical vibration in a potential field. In general the solution of such an equation describes a vibration of arbitrary amplitude; in our case, however, the amplitude is fixed by the condition that $\max\{y_n\} = 0$ in the n-th period. Using the fact that $\gamma \gg 1$, we find finally

$$\frac{1}{\sqrt{2}} \int_{\gamma + y_n}^{\gamma} \frac{dt}{(\gamma - t - e^{-t})^{1/2}} = \begin{cases} u - u_n, & u_n < u < \bar{u}_n \\ u_{n+1} - u, & \bar{u}_n < u < u_{n+1} \end{cases}. \tag{23}$$

The function y_n given by (23) can be tabulated for different values of the parameter γ. The family of curves y(u) corresponding to different γ is shown in Fig. 3. The envelope of this family is the parabola $y = -u^2/2$, which is the solution of (19) when the exponential term $e^{-\gamma - y}$ is dropped. This exponential term, which is proportional to the energy of stimulated emission stored in the cavity, begins to contribute at a time which comes later, the smaller the fraction $\delta\Omega/2\pi$ of the spontaneous emission energy contained in the cavity. It will be shown below that γ increases with decreasing $\delta\Omega$. Therefore, the larger γ, the later the solution bends away from the envelope (see Fig. 3).

The dependence of the period of oscillation b on γ is shown in Fig. 4. For large values of γ the asymptote of the function $b(\gamma)$ has the form $b = \sqrt{8\tilde{y}} + o(\tilde{y}^{-1/2})$, where $\tilde{y} = \gamma + \ln\tilde{y} \cong \gamma + \ln\gamma$ is the amplitude of the quantity y, i.e. $\tilde{y} = y_{n\,\max} - y_{n\,\min} = |y_{n\,\min}|$.

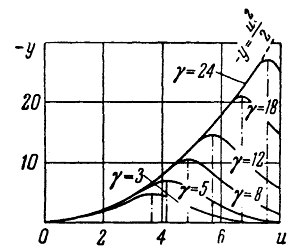

FIG. 3. Solutions of Eq. (11) corresponding to different values of the parameter γ. The abscissa is dimensionless time counted from the beginning of the period; the ordinate is the function y taken with opposite sign and counted from its maximum value. The dash-dot lines show the symmetry axes of the curves. The envelope of the family of curves is a parabola, whose extension is shown dashed.

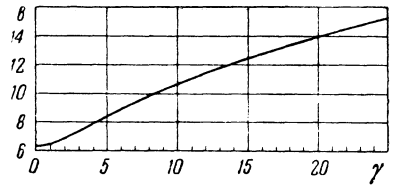

FIG. 4. Graphs of the dependence of the period, in units of $\sqrt{T/\nu\kappa_1\zeta}$ on the parameter γ.

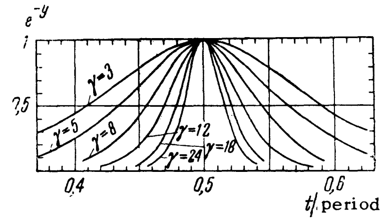

FIG. 5. The intensity of the laser output against time, expressed in units of a period, for different γ (the curves are normalized so that their maximum ordinate is unity).

Except for a constant multiplying factor, the intensity of the directional radiation is proportional to the function e^{-y}. In Fig. 5 we show graphs of this function for various values of γ. It is clear from the figure that the radiation spike occupies a small fraction of the period. The ratio of the half-width of this spike (i.e., its width at half-maximum height) to the period for large γ has the form $0.88/\widetilde{y} + o(\widetilde{y}^{-2})$.

We now turn to the solution of the second part of the problem, the determination of the dependence of the parameter γ on the number of the period n. The parameter γ is a functional, depending on the function y; its form is given by (21). The integrand in this expression is large only in the neighborhood of the points u_n, corresponding to maxima of y. Hence the integral in (21) can be expanded in a sum of integrals taken over the neighborhoods of the points u_k. In the neighborhood of these points it is simple to find y from (19), in which the exponential term can be dropped; $y_n(u) = y(u_k) - y(u_n) - (u - u_k)^2/2$. Putting this in (21) and taking the above into account, we obtain the functional (21) in the form

$$e^{-\gamma_n} = \varepsilon\sqrt{2\pi}\sum_k (\bar{u}_n - u_k)^{-1/2}\exp\,[y\,(u_k) - y\,(u_n)]. \quad (24)$$

Thus the functional γ depends only on the maximum values of the function y.

We have obtained n equations connecting the 2n unknown quantities $y(u_k)$ and γ_k ($k = 1, 2, \ldots, n$). In order to obtain the missing n equations we will find the damping of the oscillation during a period $\Delta y_k = y(u_{k+1}) - y(u_k)$. This damping is related to the "dissipative" term ξ in (20), which we have so far neglected.

We integrate (20) over a single period as an equation of mechanical vibration of a material point with unit mass located in a potential field $U = y + e^{-\gamma - y}$ and subject to a small dissipative force $\xi(u)$. The variation in the maximum value of y over a period, $\Delta y_n = y(u_{n+1}) - y(u_n)$, can be easily related to the change in the energy of vibration ΔH, caused by this dissipation. Clearly we have $H = U(y_{max}) + \text{const.} = y_{max} + \exp(-\gamma - y_{max}) + \text{const.}$ Neglecting the exponential term (for large γ), we find

$$\Delta H = \Delta y_{max} = y\,(u_{n+1}) - y\,(u_n) = \Delta y_n. \quad (25)$$

On the other hand, the variation in the energy of vibration during the period due to the effect of the small dissipative force ξ is equal to the integral over the period of $-\oint \xi\,dy$. Thus

$$\Delta y_n = y\,(u_{n+1}) - y\,(u_n) = -\oint \xi\,dy = -\int_{u_n}^{u_{n+1}} \xi\,(u)\,y'(u)\,du. \quad (26)$$

Calculating the integral with the help of (19), we have

$$\Delta y_n = -2^{3/2}\beta\left(1+\frac{1}{\zeta}\right)\varphi\,(\gamma_n) - \sqrt{\frac{\pi}{2}}\,\varepsilon b_n e^{\gamma_n}\sum_{k=1}^{n}\frac{e^{y(u_k)-y(u_n)}}{(\bar{u}_n - u_k)^{3/2}}, \quad (27)$$

where we have introduced the notation

$$\varphi\,(\gamma) = \int_{t_{min}}^{t_{max}} \sqrt{\gamma - t - e^{-t}}\,dt. \quad (28)$$

Equations (24) and (27) constitute a complete system. Combining them we obtain equations for the γ_n:

$$\Delta\gamma_n \equiv \gamma_{n+1} - \gamma_n = -2^{3/2}\beta\left(1+\frac{1}{\zeta}\right)\varphi\,(\gamma_n) - 6\varepsilon b_n^{-1/2} e^{\gamma_n}. \quad (29)$$

In this equation we have omitted terms which may be neglected because of their smallness after n has become of the order of a few units (it will be seen from what follows that for small ϵ and β the oscillations damp out over a large number of periods, so that the above restriction is unimportant).

As will be clear from what follows, for n exceeding several units, $\Delta\gamma_n \ll \gamma_n$; using this, we put $\Delta\gamma_n = d\gamma/dn$ and put (29) in the form

$$d\gamma/dn = -2^{3/2}\beta\,(1+1/\zeta)\,\varphi(\gamma) - 6\varepsilon e^{\gamma}/\sqrt{\tilde{b}(\gamma)}. \quad (30)$$

The solution of this equation has the form

$$n = \int\limits_{\gamma}^{\gamma_i} \left\{2^{3/2}\beta\left(1+\frac{1}{\zeta}\right)\varphi(\gamma) + 3\left[\frac{b_i}{\pi b(\gamma)}\right]^{1/2} e^{-\gamma_i+\gamma}\right\}^{-1} d\gamma \quad (31)$$

(γ_i and b_i are the initial values of the parameter γ and the period b, connected by the relation $2\gamma_i = \ln(b_i/4\pi\epsilon^2)$).

In order to investigate the damping of the oscillations we make use of the asymptotic expression for $\varphi(\gamma)$, applicable for large values of γ: $\varphi(\gamma) \cong 2\tilde{y}^{3/2}/3$ ($\tilde{y} = \gamma + \ln\tilde{y} \cong \gamma + \ln\gamma$). We consider two limiting cases.

1. $a\beta \lesssim 5e^{-\gamma_i}$ where $a = 2^{5/2}3^{-1}(1+1/\zeta)$ (usually $a \sim 2$—3). In this case one may neglect the first term in the denominator of the integrand of (31), and the dependence of γ on n takes the form

$$\gamma = \gamma_i - \ln(1 + 3n/\sqrt{\pi}). \quad (32)$$

We define (arbitrarily) the number of periods n^* over which the oscillations damp out such as to make $\gamma(n^*) = 1$. We find

$$n^* \approx 0.2\,e^{\gamma_i}. \quad (33)$$

2. $a\beta \gtrsim \tilde{y}_i^{-3/2}$. Neglecting the second term in the denominator of the integrand, we put (31) in the form

$$\tilde{y} = [\tilde{y}_i^{-1/2} + a\beta n/2]^{-2}, \qquad n^* \approx 2/a\beta = 1.06/\beta\,(1+1/\zeta). \quad (34)$$

In the intermediate case, when $5e^{-\gamma_i} \ll a\beta \ll \tilde{y}_i^{-3/2}$ the dependence expressed in (32) holds for small n, and the one shown in (34) holds for large n, while n^* is always given by expression (34). For $a\beta = 5e^{-\gamma_i}$ the expressions for n^* in the first and second cases coincide, apart from a factor of order unity.

A comparison of (33) and (34) with the results of the preceding section indicates that the oscillations with large amplitude damp out at about the same rate as the small oscillations; however, in the case of strong oscillations the amplitude decreases with time according to a different law than the amplitude of the small oscillations.

Using (6), it is not difficult to estimate the spectral width of the output radiation:

$$\delta\omega \sim A/\sqrt{v\varkappa_1 t}.$$

In this and the preceding section we have considered case A, where the absorption and luminescence spectra $P_1(\omega)$ and $P_2(\omega)$ do not overlap. However, the results of these sections remain valid also for case B [where $P_1(\omega) \equiv P_2(\omega)$], if we make the following changes in the parameters in all the formulas:

$$\zeta \to \zeta_1 = \zeta\,(1 + \mu(\omega_0)/\varkappa_1), \qquad \beta \to \beta_1 = \sqrt{1 + \mu(\omega_0)/\varkappa_1},$$
$$\varepsilon \to \varepsilon_1 = AP_2(\omega_0)\delta\Omega\,(v\varkappa_1 T)^{-1/4}\,2\sqrt{\pi}\,\zeta_1^{1/4}\zeta.$$

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