EQUILIBRIUM SHAPE OF A NUCLEUS ACCORDING TO THE DROP MODEL WITH VARIABLE SURFACE TENSION

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Results of calculations of symmetric and asymmetric equilibrium shapes of nuclei based on the drop model with a variable surface tension are presented.

1. INTRODUCTION

FOR the solution of the traditional problem of the drop model of the nucleus, viz., that of determining the equilibrium shape of the surface, an iteration method was proposed based on the weak dependence of the equilibrium shape on the Coulomb potential with the auxiliary condition that the deformation is constant [1]. Results of calculations for symmetric equilibrium shapes have been published [2], and some details referring to the calculations themselves were given. Later the same iteration method was used for finding asymmetric solutions of the problem, and the possible dependence of the surface tension on the curvature of the nuclear surface was taken into account at the same time. The iteration method enables us to obtain an exact and sufficiently complete solution of the problem also in this case.

In the case of variable surface tension the form of the differential operator in the basic equation [1,2] is somewhat altered. In the present article we describe the results of greatest interest for the usual sequence of equilibrium symmetric shapes without a neck (1;1), and with a single neck (2;2), and of asymmetric shapes with a single neck (3;1) in accordance with the classification utilized in [1] (cf. also Sec. 4). Equilibrium shapes of other types are characterized by considerably higher energies. Calculations were also carried out for some nonequilibrium shapes, in particular in order to analyze the stability of equilibrium symmetric shapes with respect to different variations of the surface, including asymmetric ones. These results will be published separately [3].

The effectiveness of the iteration method has been confirmed in the new calculations. In all the significant cases the answer could be obtained practically with any desired degree of accuracy, and the results which are described below without any doubt correspond to the true solutions of the problem.

The surface energy was assumed to have the form

$$E_s = \frac{1}{4\pi(1-\Gamma)}\int dS\,(1-\Gamma H),$$

where Γ is a constant of the order of the ratio of the thickness of the diffuse layer to the radius of the nucleus, i.e., of order $A^{-1/3}$ (A is the atomic weight of the nucleus), H is the mean curvature of the surface:

$$H = \tfrac{1}{2}(R_1^{-1} + R_2^{-1}), \tag{2}$$

where R_1 and R_2 are the principal radii of curvature at a given point of the surface (for a sphere $R_1 = R_2 = 1$). Here, as in [1,2], we have adopted for the unit of energy the surface energy of a spherical nucleus of equal volume; the lengths are expressed in terms of the radius of such a nucleus.

In those cases when an essential role is played by the deformation of the nucleus, and in particular for nuclear fission, the sum of the energy (1) and of the Coulomb energy represents the most important part of the energy. The difference between the sum of these energies and the corresponding quantity for the initial spherical nucleus can be written in the form

$$W = E_S + E_C - (1 + 2x), \tag{3}$$

where the constant x is defined as usual:

$$x = E_C^{(\mathrm{sph})}/2E_S^{(\mathrm{sph})}, \tag{4}$$

where $E_S^{(\mathrm{sph})}$ and $E_C^{(\mathrm{sph})}$ refer to a spherical nucleus of the same volume or, more accurately, with the same number of particles. The calculations have been carried out for different values of x and $\Gamma = 0, \pm 1$. The correction ΓH to the surface tension takes into account the deformation of the diffuse layer of the nucleus. The effect due to the finite thickness of the diffuse layer is taken into

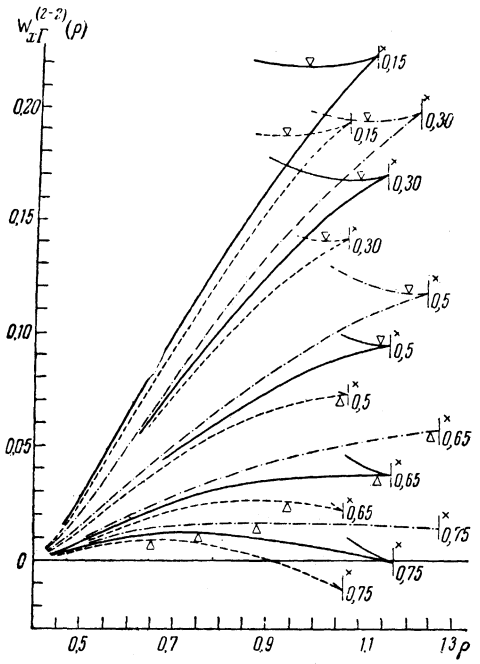

FIG. 1. Potential energy for symmetric conditional equilibrium shapes of the usual type [(1; 1) – (2, 2)] as a function of the deformation parameter $\Gamma = 0, \pm 0.1$. The values of x are shown near the vertical lines corresponding to $\rho = \rho_{crit}$. Triangles denote points corresponding to absolute equilibrium, crosses correspond to the critical deformation. Solid lines correspond to $\Gamma = 0$, dotted lines correspond to $\Gamma = +0.1$, the dash-dotted lines correspond to $\Gamma = -0.1$.

account in this case up to the second order inclusive, and this corresponds to terms of order $A^{1/3}$ in expression (3) for the nuclear energy (i.e., of order $A^{-2/3}$ with respect to the volume energy).

The quasistatic drop model of the nucleus can be obtained as a formal approximation to the more general statistical model under the condition of low compressibility of nuclear matter[4]. In the present calculations compressibility deeper inside the nucleus has not been taken into account, although, apparently, such calculations would not have been much more complicated. In such an approximation one can take for the Coulomb energy E_C an expression[2] of the same form as for a nucleus with a sharply defined surface. The value $\Gamma = 0$ corresponds to the traditional drop model with constant surface tension.

As in [1,2] we shall define the absolute equilibrium (AE) shape to be that surface which yields an extremum for the potential energy for arbitrary small variations of the shape of the surface, and we shall define the conditional equilibrium (CE) shape to be such a shape which yields an extremum only under variations of the surface which do not change the magnitude of the deformation. Additional conditions imposed on allowable variations are conditions of conservation of volume and of deformation for the "left" and "right" parts of the nucleus separately, where for the deformation parameter one takes the distances between the centers of mass of each part ρ_l and ρ_r from a certain conjugate point z = 0 between the "left" and "right" parts. The point z = 0 is determined from the condition of vanishing of the derivative of the function y(z) generating the surface:

$$(dy/dz)_{z=0} = 0, \qquad (5)$$

(the formulation of the problem is described in detail in [1,2]).

The AE shapes do not depend on the choice of the deformation parameter; the CE shapes correspond to the physical process of fission as long as the deformation parameter is the slowest degree of freedom. For symmetric shapes the point z = 0 coincides with the center of the nucleus, and the "left" part coincides with the "right" part, and in particular $\rho_l = \rho_r = \rho$.

2. SYMMETRIC SOLUTIONS (EFFECT OF THE CORRECTION ΓH)

For each conditional equilibrium shape one can calculate the potential energy $W_{x,\Gamma}(\rho_l, \rho_r)$. Figure 1 shows families of potential energy curves for symmetric conditional equilibrium shapes of ordinary type (1;1) — (2;2), i.e., the potential function $W_{x,\Gamma}(\rho)$. The absolute extremum of the functional of the potential energy (3) corresponds to the extremum of the potential function $W_{x,\Gamma}(\rho)$. More accurately the absolute extrema are determined from the condition of vanishing of the Lagrangian multipliers associated with the deformation condition (cf. [1] and also Sec. 4).

In Fig. 1 and in other figures the points on the potential energy curves corresponding to absolute extrema, and the shapes themselves are denoted by triangles. For $\Gamma = 0$ the curves $W_{x,0}(\rho)$ have been described earlier[2]. Between these curves and the curves for $\Gamma \neq 0$ there is only a quantitative difference.

The most interesting property is the existence of characteristic cusps at $\rho = \rho_{crit}$, which is practically the same for all x, but which varies slightly depending on Γ. For reasons mentioned earlier[2] one can assume that for $\rho = \rho_{crit}$ the nucleus becomes absolutely unstable with respect to fission into two fragments; the deformation ρ_{crit} can then be interpreted as the point at which fission occurs. The CE shape is characterized for $\rho = \rho_{crit}$ by a maximal deformation for which the conditional extremum of the potential energy can correspond to a continuous shape (cf. Fig. 2C).

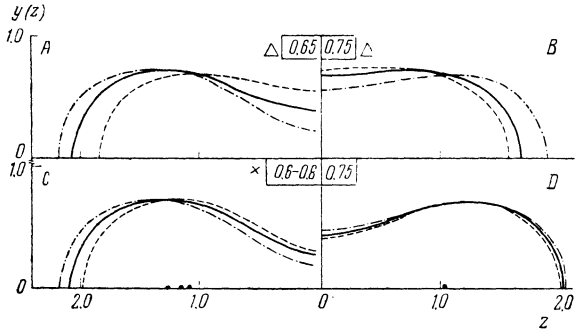

FIG. 2. Some equilibrium symmetric shapes of the nuclear surface. Solid line – $\Gamma = 0$, dash-dotted line – $\Gamma = +0.1$, dotted line – $\Gamma = -0.1$. Quarter cross sections along the symmetry axis are shown. Cases A, B show an absolute extremum (saddle) for $x = 0.65$ and 0.75; C shows critical contours for $x = 0.65$, for other values of x and corresponding Γ the critical contours practically do not differ from these; D shows contours of conditional equilibrium shapes for the same value of ρ equal to 1.02 ($x = 0.75$). The black dot on the horizontal axis in cases C and D denotes the position of the center of mass of half the figure (the quantity ρ).

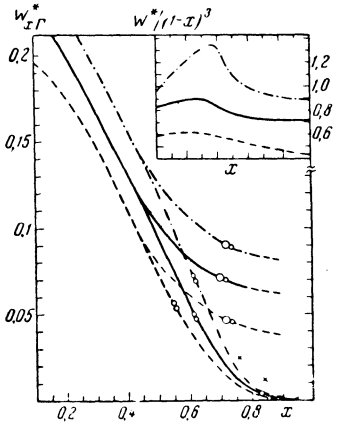

FIG. 3. The potential energy of symmetric absolute equilibrium shapes of the usual type [(1; 1) – (2; 2)] and of asymmetric shapes of type (3; 1) as a function of x. The parts of the curves corresponding to shapes without a neck (1; 1) are denoted by a single circle, those with a single neck (2; 2) are denoted by two circles. The branches corresponding to the asymmetric AE shapes are denoted by circles of different size. The insert shows a graph of the potential energy for symmetric absolute extrema (W*) divided by $(1 - x)^3$. The crosses correspond to asymmetric absolute extrema evaluated for $x = 0.74$ and 0.84 in reference[7]. The solid curve corresponds to $\Gamma = 0$, the dotted curve to $\Gamma = +0.1$, the dash-dotted line corresponds to $\Gamma = -0.1$.

The two branches of the function $W_{x,\Gamma}(\rho)$ are joined in a cusp at $\rho = \rho_{crit}$ (cf. Fig. 1). The point corresponding to absolute equilibrium denoted in Fig. 1 by triangles passes at a certain $x = x^*$ from the lower branch to the upper branch, while the corresponding extremum of the potential function $W_{x,\Gamma}(\rho)$ is converted from a maximum into a minimum. The value of ρ_{crit} is correspondingly equal to 1.07 and 1.27 for $\Gamma = +0.1$ and $\Gamma = -0.1$; $x^* = 0.5$ for $\Gamma = +0.1$ and $x^* = 0.65$ for $\Gamma = -0.1$ (for $\Gamma = 0$ $\rho_{crit} = 1.16$, $x^* = 0.55$).

The CE shape corresponding to the critical deformation for a given Γ depends on x only very weakly. As a result of this the Coulomb energy of mutual repulsion of the two "halves" of the nucleus is at critical deformation proportional to x with a high degree of accuracy:

$$E_C^{(mut)}(\rho_{crit}) = k_\Gamma x, \qquad (6)$$

where the coefficient k_Γ depends on Γ. The values of k_Γ for critical deformation have turned out to be equal respectively to 0.38; 0.43 and 0.48 for Γ equal to -0.1; 0 and $+0.1$.

For a given x the correction ΓH affects the equilibrium shape for a given ρ only very slightly (cf. Fig. 2D). In contrast to this both the AE shape and the quantity ρ corresponding to it vary quite appreciably, in particular for $x \gtrsim 0.7$ (Fig. 2A, B). One can get an idea of the effect of the correction ΓH on the value of the AE energy (the saddle for $x > x^*$) from the curves shown in Fig. 3 and from the numerical values given in the table. As should have been expected on the basis of (1) the AE energy decreases for $\Gamma > 0$ and increases for $\Gamma < 0$, but this variation in the energy is considerably smaller than the absolute value of the correction to the surface energy. If the AE shape remained unchanged, then the change in the AE energy would have been equal to the correction to the surface energy.

For the same value of the deformation parameter ρ the change in the energy is much greater. In other words, the correction ΓH strongly affects the form of the potential function $W_{x,\Gamma}(\rho)$ (Fig. 1). In the quasistatic theory of fission this function is regarded as the potential energy of a sequence of fission shapes and is utilized, in particular, in order to estimate the lifetime for spontaneous fission. It is clear that the correction ΓH would in this case be quite significant.

3. ASYMMETRIC SOLUTIONS

In contrast to the symmetric CE shapes which correspond, at least approximately, to some physical fission process, it is not clear whether it is possible to define a similar "physical" sequence for asymmetric shapes.

In practice it has turned out that when an attempt is made to obtain a second extremum by assuming $T_l \neq T_r$, but $\nu_l = \nu_r$, the standard iteration sequence diverges rapidly. At the same time

for essentially asymmetric shapes, for example of the type (3;1), where the sequence of CE shapes has only formal significance[1] it has turned out to be possible to obtain with its aid the corresponding absolute extremum.

Calculations have shown that among the shapes of the type (3;1) for a given x and Γ there is indeed one for which the AE conditions are satisfied, and, moreover, for $x < 0.7$ the iteration sequence converges well near the desired AE shape, as a result of which it was possible to obtain a solution with any desired degree of accuracy. For $x \gtrsim 0.7$ the iteration sequence diverges after a number of converging iterations. However, this is essential only from the point of view of the question of the existence of an asymmetric absolute extremum in this range of x, since for x in the range 0.70—0.75 the accuracy which is obtained for the interval of convergence is in practice quite sufficient. The results which are given below refer only to the asymmetric absolute equilibrium shapes.

The values of the energy for some values of x and Γ are given in the table. Qualitatively the dependence of the energy of the asymmetric absolute extremum on x and Γ is illustrated in Fig. 3 which also shows the corresponding curves for symmetric extrema. It can be seen that for a given Γ the symmetric and the asymmetric branches practically merge for $x \approx 0.4$, while for greater values of x the branch of the asymmetric AE shapes lies higher.

One can get an idea of the variation in the AE shape as a function of x and Γ from Fig. 4. The shape of the surface depends weakly on Γ, but its dependence on x is more pronounced. As x increases the dimensions of the smaller portion of the contour situated in Fig. 4A to the left of the thin neck diminish, while the contour itself evi-

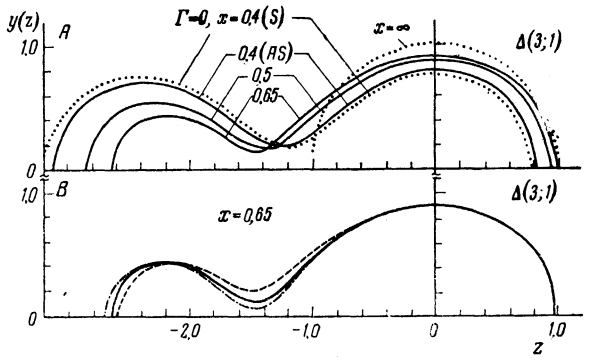

FIG. 4. Asymmetric absolute equilibrium shapes (saddles) of type (3; 1). Solid lines represent Γ = 0, dotted lines represent Γ = + 0.1, the dash-dotted lines represent Γ = – 0.1. In case A the dots indicate the contour of symmetric AE shape for x = 0.4 (Γ = 0), and also the sphere of unit radius.

dently tends to a sphere of unit radius which is shown by dots in Fig. 4A. Conversely, as x is decreased the asymmetry of the contour becomes less pronounced, and the contour approaches the symmetric AE shape for the same values of x and Γ. As can be seen from Fig. 4A, already for x = 0.4 the symmetric and asymmetric shapes differ very little from each other, and the difference in the energy amounts to only approximately 0.1% (cf. the table). Nevertheless, the symmetric and the asymmetric extrema coincide exactly only for x = 0, when the absolute extremum corresponds to the shape consisting of two identical spheres in contact with each other which is common to both sequences, the symmetric one (2;2) and the asymmetric one (3;1) (cf. [1]). For $x < 0.4$ the symmetric extremum corresponds to an energy which is higher, but not by very much (cf. the table). This explains the instability of the latter with respect to small asymmetric deformations of octopole type for $x < 0.4$ which was

The energy of the absolute extremum in units of the surface energy of the initial spherical nucleus (multiplied by 100)

x	Γ			x	Γ		
	+ 0,1	0	– 0,1		+ 0,1	0	– 0,1
0.15		21,775 21,680		0.65	2,619 6,143	3,822 7,640	5.677 9,947
0.3		16.76 16.76		0,70	1,580 4,82	2,235 7,00	3,600
0,4	10.74	13.17 13.18	15,66	0.75	0.8772	1,205 6,4	1.683
0.5	7,301 7.937	9.458 10.31	11.76 12.73	0,80		0,5906	0,792
0,6	4.009 6.143	5,672 8.386	7,737	0.90	0,0511	0,0703	0,0910

Note: The upper line in each corresponds to symmetric absolute extrema of type (2; 2) [(1; 1) for x = 0.90], the lower line corresponds to asymmetric absolute extrema of type (3; 1).

discovered in a number of papers [5,6] (cf. also [3]).

The potential energy curves for asymmetric absolute extrema calculated for $x \lesssim 0.65$ can be extrapolated into the region of large x (the extrapolated segments are shown by dotted lines in Fig. 3). It is quite evident that for $x \approx 1$ the asymmetric branch corresponds to a finite energy, while the shape itself still differs considerably from a sphere.

Asymmetric absolute extrema have been evaluated by Present and Knipp [7] for two values of x utilizing an expansion of the surface in a restricted series in terms of Legendre polynomials. The two corresponding values of the potential energy are shown by crosses in Fig. 3. The asymmetric extremum obtained by them belongs apparently to the sequence (3;1), but there is a considerable discrepancy with the results of the present paper both with respect to the value of the energy ($\Gamma = 0$), and to the shape (cf. Fig. 4A and [5,7]). This discrepancy can be explained by an insufficient accuracy of the calculations of [5,7] due to the fact that in those references the shape of the desired extremum is strongly restricted in advance.

It is, of course, impossible to say on the basis of only the absolute extremum to what fission process it might correspond. It is clear that the asymmetric solutions have nothing to do with ordinary fission. For $x > 0.6$ their energy is considerably higher than the energy of the symmetric absolute extremum, and, moreover, the difference in the energies increases with increasing x, which is rather contrary to the observed tendency in the dependence of the asymmetry of fission on Z^2/A. For the explanation of asymmetric fission sometimes the hypothesis is invoked of the existence of two fission barriers, a symmetric one and an asymmetric one. In the present calculations no asymmetric extremum has been found which would lie sufficiently low. Apparently, the symmetric shape corresponds to the true fission threshold. This question is discussed in greater detail in [3] in connection with the problem of stability.

As has been shown earlier [1], conditional equilibrium shapes can exist which are more complicated compared to the simplest ones described above. Calculations of such shapes, and also the determination of the corresponding absolute extrema, if the latter exist, can be carried out with the same universal program for an electronic computer. Systematic calculations of such shapes, however, have not been carried out, since trial calculations of asymmetric figures of the type (2;4) ($\nu_l = 2$, $\nu_r = 4$, cf. [1]) with two necks have shown that they correspond to a considerably higher energy than the shapes (3;1). Moreover, we did not succeed in finding a domain of parameters for which the iteration procedure would converge.

4. SOME DETAILS OF CALCULATIONS

The present calculations are a direct development of results presented earlier [1,2]. Below are given only certain relations which have been altered as a result of the introduction of the correction ΓH. The definition of A adopted here coincides with $\hat{A}$ in [1,2], while β corresponds to $\beta + 1$ from [2].

In cylindrical coordinates we have

$$R_1 = y\,(1 + y'^2)^{1/2}, \qquad R_2 = -(1 + y'^2)^{3/2} y''^{-2}, \tag{7}$$

$$E_S = \frac{1}{2(1-\Gamma)} \int_{z_l^*}^{z_r^*} dz\,\{y\sqrt{1+y'^2} - \Gamma\,(1 + y'\,\mathrm{arc\,tg}\,y')\}. \tag{8}$$*

From the condition that the variation of the energy (3) vanishes subject to the additional conditions that the volume and the deformation remain constant we obtain the following equation for the function y(z) which generates the surface of the nucleus:

$$H - \Gamma/R_1R_2 + [10x\,(1-\Gamma)\,\Phi_S + \lambda_{V_l} + \lambda_{\rho_l}|z|] = 0. \tag{9}$$

The condition that the variation of the functional should vanish when the end points z_l^* and z_r^* are varied (the condition of transversality) yields boundary conditions at these points. For $z = z_l^*$ we have

$$(y|y'|)_{z=z_l^*} = -f_l^*\,(1 + \sqrt{1 + f_l^*}), \tag{10}$$

where f_l^* is the value of the expression in square brackets in formula (9) at the point $z = z_l^*$.

Just as in the case $\Gamma = 0$, condition (10) coincides with the condition for the existence of a solution of equation (9) with continuous curvature at the end point, where the function y(z) vanishes. In fact we have solved the equation for the transformed function (cf. also [1,2])

$$\zeta_l(\tau) = A_l y_l\,[A_l^{-1}(\tau_l^* - \tau)], \tag{11}$$

which in the present case has the form

$$\zeta_l'' - [1 - 2A_l\Gamma\,(4\zeta_l + \zeta_l'^2)^{-1/2}]^{-1}\{2 + \zeta_l^{-1}\zeta_l'^2 \times [1 - A_l\Gamma\,(4\zeta_l + \zeta_l'^2)^{-1/2}] - [T_l - \tau - \tfrac{5}{2}x\,(1-\Gamma)\,\Phi_S^{(l)}]\,(4\zeta_l + \zeta_l'^2)^{3/2}\} = 0 \tag{12}$$

*arc tg = $\tan^{-1}$.

with the boundary conditions

$$\zeta_{0l} = (\zeta_l)_{t=0} = 0, \tag{13}$$

$$\zeta'_{0л} = (\zeta'_l)_{\tau=0} = (2T_l)^{-1}\,(1 + \sqrt{1 - 2A_l\Gamma T_l})\,. \tag{14}$$

The value $\zeta''_l(0)$ can be found directly from (12) by evaluating the indeterminate expression at the point $\tau = 0$:

$$(\zeta''_л)_{\tau=0} = -2 + \tfrac{1}{2}\,(1 + g_l)\,(1 - 4A_l\Gamma\zeta'^{-1}_{0l})\,\zeta'^{2}_{0l}, \tag{15}$$

where g_l is the value of the derivative of the term containing Φ_S in (12) at the point $\tau = 0$. Some absolute values are given below

$$\beta_l = \tfrac{5}{2}\,x\,(1 - \Gamma)\,(A_l^2 z_l^*)^{-1}\,[\Phi_S\,(z = 0) - \Phi_S\,(z_l^*)]. \tag{16}$$

$$|\lambda_{V_l}| = A_l\,(T_l - \tau_l^*) + \tfrac{5}{2}\,x\,(1 - \Gamma)\,\Phi_S\,(z = 0), \tag{17}$$

$$|\lambda_{\rho_l}| = A_l^2\beta_l. \tag{18}$$

Relations analogous to (9)—(18) also hold for the part of the function $y(z)$ situated to the right of the point $z = 0$ ($y_r(z)$; $l \to r$). Other notation, and also expressions for the remaining quantities do not differ from those given in [2]. The constants τ_l^* and τ_r^* are determined from the following conditions at the conjugate point for the left and the right hand parts:

$$\zeta'_l\,(\tau_l^*) = \zeta'_r\,(\tau_l^*) = 0; \tag{19}$$

A_l and A_r are defined as in [1].

The parameters T_l and T_r are fundamental, and are given as the starting point in place of the deformation parameters [2] ρ_l and ρ_r. In addition to this we specify the ordinal numbers of the extremum at which two parts of the figure are joined [cf. Eq. (19)], counting from the left hand side (ν_l) and from the right hand side (ν_r) of the figure. The combination (ν_l; ν_r) qualitatively determines the type of the figure [1].

In the new program we have succeeded in eliminating the restriction on the size of the increment [2], but this did not lead to any significant improvement of convergence in the domain of symmetric transitional cylinder-like figures. In the important case of transition figures of the type (2;2) we have succeeded in eliminating the divergence of the iteration process by special extrapolation of Φ_S into the "unphysical" domain of the values of the running variable $\tau > \tau^{*(n-1)}$ where $\tau^{*(n-1)}$ is the maximum value of τ in the preceding, $(n-1)$st iteration. In this way we have artificially limited the divergence of the iterations.

The correction ΓH did not affect the convergence of the iteration process which, as a rule, was sufficiently rapid also for asymmetric figures. For a more accurate determination of the limiting values of the essential quantities (W, β_l, β_r and others) Aitken's δ^2-process was utilized (cf. [8]) with the aid of which the limiting value of a quantity, for example, W was found in terms of the last three iterations.

A stronger criterion of convergence was established than in [2]. Since in the course of calculations it turned out that the quantities β_l and β_r converge slowest of all, while the absolute extremum is determined from the condition that these particular quantities vanish, we have adopted for the criterion of an "exact solution" the condition

$$(\Delta\beta) < \varepsilon, \tag{20}$$

where $(\Delta\beta)$ was defined as the square root of the sum of the squares of the differences of β in three successive iterations and the square of the difference between the value of β extrapolated according to Aitken ($\tilde{\beta}$) and the value of β for the last iteration (the sum for the left hand and the right hand parts is also taken). Moreover, control calculations were carried out with a large number of iterations. For ϵ we have usually taken the value 10^{-3}; the relative accuracy of all the other quantities was considerably greater; for example, for the potential energy W it was of the order of 10^{-5} -10^{-6}. If necessary, the accuracy could always be improved considerably without any significant increase in the time of the computation.

As can be seen from (18) the absolute extremum corresponds to the vanishing of β_l and β_r. The determination of the absolute extremum in accordance with these conditions was carried out automatically by successive selection, using the method of chords, of such values of T_l and T_r which would make both $|\tilde{\beta}_l|$ and $|\tilde{\beta}_r|$ smaller than ϵ. The fact that the third condition for AE was satisfied [1]

$$\lambda_{V_l} = \lambda_{V_r}, \tag{21}$$

was checked by means of the equivalent condition of the equality of radii of curvature at the point at which the solutions are joined. For absolute extrema the magnitude of the quantity $|\eta_R|$ where

$$\eta_R = 2\,\{[R_2^{(l)} - R_2^{(r)}]/[R_2^{(l)} + R_2^{(r)}]\}, \tag{22}$$

did not exceed ϵ, while control calculations showed that η_R tends to zero together with β_l and β_r. Three conditions can be satisfied with the aid of only two parameters T_l and T_r because one of the relations utilized for the elimination of the other constants [1], viz., the condition that the first derivatives vanish at the point at which the solutions are joined, is not necessary.

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