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NONLINEAR OSCILLATIONS AND SOME EFFECTS DUE TO A LONGITUDINAL CURRENT IN A PLASMA

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Stationary nonlinear oscillations due to cyclotron instability of a plasma with a longitudinal current are considered. The spectrum of the oscillations is investigated and it is shown that the oscillations produce an additional resistance and heating of ions.

1. Drummond and Rosenbluth[1] have shown that an electron current along the magnetic field, with a velocity lower than the thermal velocity of the electrons, causes plasma oscillations to be built up with frequencies on the order of the ion cyclotron frequency and with a wave vector which is almost perpendicular to the magnetic field. In the present paper we consider oscillations which develop as a result of this instability. We show that these oscillations become stabilized by strong ion absorption, and this leads to considerable heating of the ions at the expense of the work done by the external electric field. With the aid of nonlinear equations previously derived by Kadomtsev and the author[2], the mechanism whereby the ions absorb the oscillation energy is clarified, the oscillation spectrum in the instability region is studied, and it is shown that although this region lies outside the strong ion absorption of the Landau type, nevertheless such an absorption is produced by the nonlinear effects.

2. The instability described by Drummond and Rosenbluth[1] should lead to random potential oscillations in the following region of wave numbers and frequencies:

$$k_\perp^2\rho_i^2 \sim 1.5, \qquad \Omega/v_D < k_z < 0.1\,\Omega/v_i, \tag{1}$$

$$\omega = \Omega\,\{1 + (T_e/T_i)\,\Gamma_1\,(k_\perp^2\rho_i^2)\}\,\mathrm{sgn}\,k_z, \tag{2}$$

where $v_i = \sqrt{T_i/m_i}$ — thermal velocity, Ω — cyclotron frequency, $\rho_i = v_i/\Omega$ — Larmor radius of the ions, v_D — velocity of electron flow along the magnetic field, and

$$\Gamma_1\,(x) = e^{-x}I_1\,(x), \tag{3}$$

where $I_1(x)$ — Bessel function of imaginary argument.

Under such conditions, owing to the smallness of the electron Larmor radius and of $k_z/k_\perp$, the nonlinear corrections for the electrons can be neglected and the quasilinear approximation employed. In this approximation, the equation for the averaged electron distribution function f_{0e} is written in the form

$$\frac{e}{m_e}E_0\frac{\partial f_{0e}}{\partial v_z} = \frac{v_e^2}{\tau_e}\frac{\partial}{\partial v_z}\left\{2\frac{\partial f_{0e}}{\partial v_z} + \frac{m_e}{T_e}(2v_z - v_D)\,f_{0e}\right\} + \frac{\partial}{\partial v_z}\left\{F(v_z)\frac{\partial f_{0e}}{\partial v_z}\right\}. \tag{4}$$

Here v_e is the thermal velocity, $\tau_e = 2\pi L e^4 n_0 \times v_e/m_e^2$ the electron collision time,

$$F\,(v_z) = \frac{\pi e^2}{m_e^2}\int E_z^2\,(\mathbf{k},\,\omega)\,\delta\,(\omega - k_z v_z)\,d\mathbf{k}, \tag{5}$$

E_0 the constant electric field, and $\mathbf{E}(\mathbf{k},\omega)$ the Fourier component of the oscillating electric field. In (4) the collisions are considered in the simplest form (see [3]), and the term $\partial f_{0e}/\partial t$ is left out [we consider the discharge to be stationary (T_e = const), assuming that the Joule heat absorbed by the electrons is carried away by the radiation].

Since $E_z(\mathbf{k},\omega)$ differs noticeably from zero only in the region $\omega/k_z < v_D$, a non-zero $F(v_z)$ exists for $v_D \ll v_e$ only in the interval $0 < v_z < v_D \ll v_e$. Then, multiplying (4) by v_z and integrating from $-\infty$ to ∞, we obtain

$$-\frac{e}{m_e}n_0E_0 + \frac{v_e^2}{\tau_e}\frac{m_e}{T_e}n_0v_D + \int_0^{v_D} F\,(v_z)\frac{\partial f_{0e}}{\partial v_z}dv_z = 0. \tag{6}$$

Integrating (4) from $-\infty$ to v_z and then eliminating E_0 from the result and from (6), and neglecting the last term of (6) because of the smallness of the integration region, we obtain an expression for $\partial f_{0e}/\partial z_z$ in the region of the plateau [non-zero $F(v_z)$]:

$$\frac{\partial f_{0e}}{\partial v_z} \approx -\frac{2}{\tau_e}\frac{(v_z - v_D)\,f_{0e}}{2v_e^2/\tau_e + F\,(v_z)}. \tag{7}$$

From (6) we can estimate the additional resistance resulting from the presence of random oscillations, given by the last term of (6). Substituting

(7) in it and assuming that $F(v_z) \gg v_e^2/\tau_e$, we obtain

$$\int_0^{v_D} \frac{\partial f_{0e}}{\partial v_z} F(v_z)\, dv_z \approx -\frac{2}{\tau_e} \int_0^{v_D} (v_z - v_D) f_{0e}\, dv_z = \frac{v_D^2}{\tau_e} \frac{n_0}{v_e}.$$

Therefore the total resistance obtained from (6) is

$$\sigma^{-1} \approx \sigma_0^{-1} + m_e v_D / e^2 n_0 \tau_e v_e \approx \sigma_0^{-1} (1 + v_D/v_e), \quad (8)$$

where σ_0 is the usual conductance.

It is interesting to note that we have determined the additional resistance without knowing the oscillation amplitude. We shall show with the aid of the nonlinear equations that the square of the amplitude of the oscillations along z is proportional to $\sqrt{v_D}$.

We see from (8) that the additional Joule heat, released by the electrons, is $n_0 m_e v_D^3/\tau_e v_e$. In the stationary case part of this energy is transferred to the electrons and is then radiated, while approximately a third of it is transferred to the ions, as can be readily shown by multiplying the last term of (4) by $m_e v_z^2/2$ and integrating with respect to dv_z.

Since the heat transfer from the ions to the electrons is small, the ions in a stationary discharge can be heated to a temperature exceeding the electron temperature, until the instability region (1) disappears. We see from (1) that the steady-state ion temperature will be approximately equal to

$$T_i \approx 20\, (v_D/v_e)^2\, T_e. \quad (9)$$

Formula (9) is valid up to $v_D \sim v_e$.

3. We now proceed to a determination of the spectrum of the oscillations that develop in the instability region. As follows from the foregoing, the nonlinear equations mentioned in Sec. 1[2] for the spectra of the pair ($p = \langle \Phi f' \rangle$, $I = \langle \Phi\Phi' \rangle$) and triple ($Q = \langle \Phi\Phi' f'' \rangle$, $q = \langle \Phi\Phi'\Phi'' \rangle$) correlation functions of the randomly varying f and the electric potential Φ have, for electrons, the form

$$(-i\omega + ik_z v_z)\, p_e = \frac{ie}{m_e} k_z \frac{\partial f_{0e}}{\partial v_z} I, \quad (10)$$

$$(-i\omega'' + ik_z'' v_z)\, Q_e = \frac{ie}{m_e} k_z \frac{\partial f_{0e}}{\partial v_z} q. \quad (11)$$

We have confined ourselves, as in (4), to the linear terms.

In the equation for the ions we confine ourselves to nonlinear corrections which contain only electric-field components that are transverse to the magnetic field, since they are much larger than the longitudinal ones:

$$\left(-i\omega + i\mathbf{k}\mathbf{v} - \Omega \frac{\partial}{\partial \varphi}\right) p_i = \frac{ie}{T_i} \mathbf{k}\mathbf{v} f_{0i} I + \frac{ie}{m_i} \int \mathbf{k}'_\perp \frac{\partial Q_i}{\partial \mathbf{v}_\perp} d\mathbf{k}'\, d\omega', \quad (12)$$

$$\left(-i\omega'' + i\mathbf{k}''\mathbf{v} - \Omega \frac{\partial}{\partial \varphi}\right) Q_i = \frac{ie}{T_i} \mathbf{k}''\mathbf{v} f_{0i}\, q + \frac{ie}{m_i} I' \mathbf{k}'_\perp \frac{\partial p}{\partial \mathbf{v}_\perp} + \frac{ie}{m_i} I \mathbf{k}_\perp \frac{\partial p'}{\partial \mathbf{v}_\perp}. \quad (13)$$

We choose here a Maxwellian function f_{0i}, $p' = p(\mathbf{k}', \omega')$, $I' = I(\mathbf{k}', \omega')$, $\mathbf{k}'' = \mathbf{k} + \mathbf{k}'$, $\omega'' = \omega + \omega'$, and leave out the decay terms since (2) is a non-decay spectrum.

We confine ourselves to the case when p and p′ are taken from the instability region, for it is precisely then that the nonlinear corrections (12) and (13) will be sufficiently large. Then, according to (2), ω'' is of the order of $2.5\,\Omega$ for $\omega' \sim \omega$, and of the order of zero for $\omega' \sim -\omega$. The meaning of Eq. (13) can be readily clarified by considering the Fourier transform of Vlasov's equation in the vicinity of $\omega' \sim \omega + \omega'$ (see [2]):

$$\left(-i\omega'' + i\mathbf{k}''\mathbf{v} - \Omega \frac{\partial}{\partial \varphi}\right) f_i'' = \frac{ie}{T_i} \mathbf{k}''\mathbf{v} f_{0i} \Phi'' + \frac{ie}{m_i} \int \mathbf{k}'_\perp \frac{\partial f_i}{\partial \mathbf{v}_\perp} \Phi' \delta(\mathbf{k} + \mathbf{k}' - \mathbf{k}'')\, \delta \times (\omega + \omega' - \omega'')\, d\mathbf{k}\, d\omega\, d\mathbf{k}'\, d\omega'. \quad (14)$$

Equation (14) describes the forced oscillations in a region where the oscillations are damped ($\omega'' \sim 2.5\Omega$ or $\omega'' \ll \Omega$), due to beats of neighboring frequencies in the region $|\omega|$, $|\omega'| \sim \Omega$. When $\omega'' \ll \Omega''$ there is a particularly strong ion Landau damping, for in this case $\omega''/k_z'' \sim v_i$.

We see therefore that the nonlinear term in (12) describes the pumping of energy from the region of the instability into the region of ion Landau damping (nonlinear Landau damping). This damping is described by (13) and (14). It is therefore sufficient to consider Eq. (13) only and to integrate in (12) over the region $\omega'' = \omega + \omega' \ll \Omega$.

Using (13) and (11), substituting the linear approximations of p_i and p_i' in (13), and taking the quasi-neutrality condition into account, we obtain approximately

$$q \approx \frac{e}{4m_i\Omega} k_\perp k'_\perp I I' \frac{1}{B} \left(\frac{C}{\omega - \Omega} + \frac{C'}{\omega' - \Omega}\right), \quad (15)$$

where only the most essential terms C and C′, which make the main contribution to (12), are left in the expression for q. These terms are

$$C = \int v_\perp J_1(\lambda') J_0(\lambda) J_0(\lambda'') f_{0i} d\mathbf{v},$$
$$C' = \int v_\perp J_1(\lambda) J_0(\lambda') J_0(\lambda'') f_{0i} d\mathbf{v},$$
$$B = \int v_\perp J_0(\lambda'') J_1(\lambda'') f_{0i}\, d\mathbf{v}. \quad (16)$$

Here J — Bessel function, $\lambda' = k'_\perp v_\perp/\Omega$, and $\lambda'' = |\mathbf{k}_\perp + \mathbf{k}'_\perp| v_\perp/\Omega$.

Substituting in (12) the expression obtained from (13) for Q_i and eliminating p_i and p_e from (10) and (12), we obtain with account of (2)

$$DI \approx \frac{e^3}{4m_iT_e}\Omega^{-1}\frac{k_\perp}{\Gamma_1}$$
$$\times \int d\mathbf{v}J_1(\lambda)J_0(\lambda)\frac{\partial}{\partial v_\perp}\left\{v_\perp J_1(\lambda)\int d\mathbf{k}'\frac{k'_\perp q f_{0i}}{\omega'' - k''_z v_z}\right\}$$
$$+\frac{e^4T_i}{8m_i^2T_e^2\Omega^3}\frac{k_\perp^2 I}{\Gamma_1^2}\int d\mathbf{v}J_1(\lambda)J_0(\lambda)\frac{\partial}{\partial v_\perp}$$
$$\times\left\{v_\perp J_0(\lambda)J_1(\lambda)\int d\mathbf{k}'\frac{k'^2_\perp I' f_{0i}}{\omega'' - k''_z v_z}\right\}. \tag{17}$$

The Γ_i in (17) is the same as in (2). In some places we have replaced $k'_\perp$ in (17) by $k_\perp$, since the presence of the factor

$$[|\omega| - |\omega'| - (k_z + k'_z)v_z]^{-1}$$

with allowance for the smallness of $k_z v_i$ and $k'_z v_i$, makes only the region $k'_\perp \sim k_\perp$ of importance in the integral. We have also approximated $J(\lambda'')$ by its mean value $J(\lambda)$. The left half of (17) contains the same expression as the usual linear dispersion equation, so that by setting the right half equal to zero we obtain the usual dispersion equation.

We solve (17) by successive approximations, equating in the zeroth approximation the larger quantity Re D to zero, from which we get (2). (We are considering steady-state oscillations, so that the imaginary part of ω is equal to zero).

In the next approximation, we equate the imaginary parts of (17), neglecting the real part in the right half. Substituting here (15) and the solution of the zeroth approximation (2), we get

$$\operatorname{Im} DI \equiv I\,\operatorname{Im}\frac{e^2}{m_e}\int\frac{k_z\partial f_{0e}/\partial v_z}{\omega - k_z v_z}\,d\mathbf{v}$$
$$= \frac{\pi e^4 R}{8m_i^2T_e v_i\Omega^3}\frac{k_\perp^4}{\Gamma_1^2}I\int\frac{I'}{|k_z + k'_z|}\exp\left\{-\left(\frac{\Omega T_e}{\sqrt{2}v_iT_i}\right)^2\right.$$
$$\left.\times\left(\frac{\Gamma_1(k_\perp^2\rho_i^2) - \Gamma_1(k'^2_\perp\rho_i^2)}{k_z + k'_z}\right)^2\right\}d\mathbf{k}'. \tag{18}$$

Here R is the numerical estimate of the integral

$$\left\{\frac{1}{n_0}\frac{C}{B}\int\left(J_0J_1^2 + \frac{v_\perp}{2}\frac{\partial J_1^2}{\partial v_\perp} - \frac{m_iv_\perp^2}{T_i}J_0J_1^2\right)f_{0i}d\mathbf{v}\right.$$
$$\left. - \int\frac{m_iv_\perp^2}{2T_i}J_1^2J_0^2f_{0i}d\mathbf{v}\right\}$$

at the point of the maximum of Γ_1, i.e., for $k_\perp^2\rho_i^2 \sim 1.5$, neglecting its variation compared with the other more strongly varying functions. R is of the order of the maximum value of Γ_1, i.e., $\sim 10^{-1}$.

Owing to the smallness of the region of instability with respect to k_z, we neglect its width compared with k and integrate (18) with respect to dk_z. The integral with respect to $d\mathbf{k}'$ in (17) is then approximately equal to

$$\frac{v_iT_i}{\Omega T_e}\int d\mathbf{k}'\delta\left(\Gamma_1(k_\perp^2\rho_i^2) - \Gamma_1(k'^2_\perp\rho_i^2)\right)I(k')$$
$$= \frac{2\pi T_i\Omega}{T_e v_i|\Gamma'_1|}I(k_\perp). \tag{19}$$

Here Γ'_1 denotes the derivative of Γ_1 with respect to the argument, and $I(k_\perp) = \int I(\mathbf{k})dk_z$.

After integration with respect to dk_z we can represent the left half of (18) in the form

$$\frac{\pi e^2}{m_e}\int d\mathbf{v}\,dk_z\frac{\partial f_{0e}}{\partial v_z}I(\mathbf{k})k_z\delta(\omega - k_zv_z)$$
$$= m_e\int_0^{v_D}\frac{v_z}{\Omega}F(v_z, k_\perp)\frac{\partial f_{0e}}{\partial v_z}dv_z \approx \frac{2m_e}{\tau_e}\int_0^{v_D}\frac{v_z}{\Omega}(v_z - v_D)$$
$$\times f_{0e}\frac{F(v_z, k_\perp)}{F(v_z)}dv_z \approx \frac{2}{3}\frac{m_en_0}{\tau_e}\frac{v_D^3}{v_e}\frac{I(k_\perp)}{\int I(\mathbf{k})dk}. \tag{20}$$

We have substituted (5) and (7) in (20) and replaced ω by Ω. In analogy with (5), we have:

$$F(v_z, k_\perp) = \frac{\pi e^2}{m_e^2}\int k_z^2 I(\mathbf{k})\,\delta(\omega - k_zv_z)\,dk_z.$$

We have therefore made in (20) the approximate substitution

$$\frac{F(v_z, k_\perp)}{F(v_z)} \approx \frac{I(k_\perp)}{\int I(\mathbf{k})dk}.$$

Substituting (19) and (20) in (18) we obtain

$$\frac{2m_e}{3\Omega\tau_e}\frac{v_D^3}{v_e}\frac{I(k_\perp)}{\int I(\mathbf{k})dk} = \frac{\pi^2e^4R}{4m_iT_e^2\Omega^2}\frac{k_\perp^4}{\Gamma_1^2|\Gamma'_1|}I^2(k_\perp), \tag{21}$$

hence

$$k_\perp^2 I(k_\perp) \approx \frac{T_iT_e}{2e^2}\left[\frac{v_D^3}{\tau_e\Omega_eR_1v_ev_i^2}\right]^{1/2}\frac{\Gamma_1(k_\perp^2\rho_i^2)\,|\Gamma'_1(k_\perp^2\rho_i^2)|}{k_\perp^2\rho_i^2}, \tag{22}$$

where

$$R_1 = R\int_0^\infty\Gamma_1^2(x)\,|\Gamma'_1(x)|\,\frac{dx}{x^2} \sim 0.1, \qquad \Omega_e = \frac{m_i}{m_e}\Omega.$$

By way of comparison we note that the spectral intensity of the equilibrium fluctuations in this frequency region is of the order of $T_e^2/e^2n_0\rho_i^3$, i.e., very small compared with (22).

We see from (22) that the amplitude, as a function of $k_\perp\rho_i$, first increases like $k_\perp^2\rho_i^2$, reaches a maximum, and then decreases toward the point of maximum increment, where it vanishes. It then increases again after which it tends rapidly to zero. We note that its second maximum is much smaller than the first and is therefore more difficult to observe. This is in qualitative agreement with an experiment[4] in which the oscillations were observed for smaller $k_\perp \rho_i$ than the value corresponding to the maximum increment.

4. We have thus shown that cyclotron instability of a plasma with longitudinal current gives rise to an additional anomalous resistance, which can lead to noticeable heating of the ions.

An investigation of the spectral function of the oscillations in the instability region shows that it has in wave-number space the form of a double-humped curve, vanishing when the wave number corresponds to the maximum increment in the linear approximation. The second hump turns out to be much lower than the first and is more difficult to observe in experiment.

Along with the natural oscillations, forced oscillations and zeros due to beats of the waves in the instability region appear at frequencies on the order of double the ion cyclotron frequency. The forced oscillations of lower frequency are absorbed by the ions; this in turn heats the ions and eliminates the instability.

APPENDIX (received 1 July 1963)

If the plasma is inhomogeneous in a direction transverse to the magnetic field, then the oscillations considered above cause diffusion of the particle flow along this inhomogeneity. As shown in [5], when the electron and ion density gradients are equal their diffusion currents transverse to the magnetic fields, due to the potential oscillations of the plasma, are also equal and are given by the formula

$$NV_x = \frac{ic}{H}\int k_y \langle \Phi(-\mathbf{k}, -\omega)\, n(\mathbf{k}, \omega) \rangle \, d\mathbf{k}\, d\omega. \qquad (23)$$

The x axis is chosen here in the direction of the plasma inhomogeneity, and $n(\mathbf{k}, \omega)$ is the Fourier component of the alternating part of the electron or ion number density.

Since we are considering the electron motion in the linear approximation, it is convenient to substitute the electron density n_e in (23). As shown in [5], we have for $\omega \ll \Omega_e$

$$n_e(\mathbf{k}, \omega) \approx \frac{e}{m_e}\int \frac{k_z \partial f_{0e}/\partial v_z}{k_z v_z - \omega}\, d\mathbf{v}\Phi(\mathbf{k}, \omega)$$

$$+ \frac{\partial}{\partial x}\frac{ck_y}{H}\int \frac{f_{0e}\, d\mathbf{v}}{k_z v_z - \omega}\Phi(\mathbf{k}, \omega). \qquad (24)$$

Substituting (24) in (23), we readily obtain

$$NV_x \approx \frac{\pi c^2}{H^2}\frac{\partial}{\partial x}\int k_y^2\, I(\mathbf{k})\left\{\int f_{0e}\delta(\omega - k_z v_z)\, d\mathbf{v}\right\} d\mathbf{k}. \qquad (25)$$

Replacing in (25) k_y^2 by $k_\perp^2/2$ and $k_\perp/k_z$ by v_D/v_i, we obtain, substituting $I(k_\perp)$ from (22),

$$NV_x \sim \frac{v_e^2}{\Omega_e}\left(\frac{v_D^5}{v_e^3 v_i^2 \Omega_e \tau_e}\right)^{1/2}\frac{\partial N}{\partial x}. \qquad (26)$$

We have thus obtained a diffusion coefficient proportional to $H^{-3/2}$. It is interesting to note that were it not for the quasilinear corrections, $I(k_\perp)$ would be proportional to H^2 and the diffusion coefficient would be proportional to H^{-1}, like the Bohm diffusion coefficient. Thus, the quasilinear effect of plateau formation greatly decreases the diffusion.

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