ANISOTROPIC SUPERCONDUCTORS WITH NONMAGNETIC IMPURITIES

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The properties of weakly anisotropic superconductors containing nonmagnetic impurities are investigated. The dependence of the critical temperature, the energy gap, and the nuclear spin relaxation time on the concentration of impurities is determined. The effect of impurities on the density of states as a function of energy and direction is investigated. It is shown that the impurities lead to an isotropic density of states at comparatively small concentrations ($l \sim \xi_0$).

RECENTLY a considerable amount of attention has been devoted to the study of superconductors containing nonmagnetic impurities. It is known[1] that the introduction of impurities into the isotropic BCS model has no effect on the thermodynamical properties and the density of states of a superconductor. However, a lowering of the critical temperature T_c with decreasing electron mean free path length has been experimentally observed.[2] A qualitative explanation of this phenomenon was given by Anderson,[3] who pointed out the necessity of accounting for the anisotropy effects. A quantitative theory, taking into account the effect of the mean free path of the electrons on T_c for an anisotropic superconductor, was given in articles by Tsuneto,[4] Caroli et al.[5] and Markowitz and Kadanoff.[6] The latter authors carried out, in addition, an empirical analysis of the contribution of the "valence" effects to T_c, i.e., the change in the characteristic parameters of the metal.

We studied a simple model of an anisotropic superconductor and investigated the change of its properties as a function of the concentration of impurities, limiting ourselves to a consideration of only the mean-free-path effects, for which a simple theoretical analysis is possible. We did not take account of the "valence" effects, since a quantitative theory does not exist for them at the present time. Special assumptions about the angular dependence of the electron-electron interaction were not made, although the scattering of the electron by impurities was assumed to be isotropic. The physical approximation, which enables us to solve the Gor'kov type equations for the Green's functions in the presence of impurities, is the smallness of the anisotropic pairing potential, which is expressed in terms of the parameter χ which appears in the final results. This assumption is experimentally well-confirmed.[7,2] The dependence of the critical temperature on the concentration of impurities is basically in agreement with the results of the articles mentioned.[3-5]. In particular, the initial drop in T_c is linear in the concentration n, but for large concentrations the dependence has the form $n^{-\chi}$.

We also calculated the dependence of the energy gap, i.e., the absorption threshold, on the impurity concentration. It initially increases linearly with n, reaches a maximum, and afterwards decreases, still being in the region of "small concentrations," where the change in T_c is linear. (This corresponds to a mean path length l larger than the correlation length ξ_0.) In the region of concentrations for which $l \ll \xi_0$ the gap slowly diminishes according to the same law $n^{-\chi}$ as for T_c.

Finally, we studied the effect of impurities on the "smearing" over energy of the density of states which is due to the anisotropy, and which manifests itself in the nuclear spin relaxation time.[8,3] We found that this smearing already vanishes for small concentrations ($l \gtrsim \xi_0$), which leads to an "isotropic gap."[3] As a result of this, the relaxation time must decrease, which has also actually been observed, although there is not enough experimental data[9] for a quantitative comparison.

1. SIMPLE MODEL OF A WEAKLY ANISOTROPIC SUPERCONDUCTOR

Since the properties of a "dirty" superconductor are expressed in terms of the characteristics of a pure superconductor, we shall briefly discuss the weakly anisotropic pure superconductor.[9]

We introduce an effective electron-electron interaction

$$V(\mathbf{p}, \mathbf{p}') \equiv \lambda [1 + \zeta(\mathbf{p}, \mathbf{p}')], \quad \int d\mathbf{p} \int d\mathbf{p}' \zeta(\mathbf{p}, \mathbf{p}') = 0, \quad (1)$$

which is constant over the energy interval of width $2\omega_D$ near the Fermi surface, but depends on the angle between the vectors $\mathbf{p}$ and $\mathbf{p}'$. All of the following calculations will be carried out in the approximation of weak anisotropy, namely

$$|\zeta(\mathbf{p}, \mathbf{p}')| \ll 1. \quad (2)$$

The interaction (1) leads to a dependence of the parameter $\Delta(\mathbf{p})$ on orientation. Near the transition temperature $\Delta(\mathbf{p})$ satisfies the linear equation

$$\Delta(\mathbf{p}) = T \sum_{\omega_n} \int \frac{d^3p}{(2\pi)^3} V(\mathbf{p}, \mathbf{p}') \mathfrak{G}_0(\mathbf{p}, \omega_n) \mathfrak{G}_0(-\mathbf{p}', -\omega_n) \Delta(\mathbf{p}'); \quad (3)$$

here $\mathfrak{G}_0$ is the Green's function for the electrons in a normal metal, which we choose in the form

$$\mathfrak{G}_0(\mathbf{p}, \omega_n) = [i\omega_n - \xi(\mathbf{p})]^{-1}, \quad \omega_n = (2n+1)\pi T; \quad (4)$$

here $\xi(\mathbf{p})$ is the energy measured from the Fermi surface in the direction of the vector $\mathbf{p}$. We write the integral over momentum in the form

$$\int \frac{d^3p}{(2\pi)^3} = \frac{mp_0}{2\pi^2} \int d\sigma \int d\xi(\mathbf{p}), \quad (5)$$

where $d\sigma$ is a dimensionless element of area on the Fermi surface, normalized to unity, i.e., $\int d\sigma = 1$ (the integral is over the entire surface) and p_0 is the correspondingly normalized momentum of the order of the Fermi momentum.

Calculating the sum over ω_n in Eq. (3) in the usual manner, we obtain

$$\Delta(\hat{\mathbf{p}}) = \frac{mp_0}{2\pi^2} \int_0^{\omega_D} d\xi(\mathbf{p}') \frac{\tanh(\xi(\mathbf{p}')/2T)}{\xi(\mathbf{p}')} \int d\sigma' V(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \Delta(\hat{\mathbf{p}}'),$$

where $\hat{\mathbf{p}}$ denotes a vector lying on the Fermi surface. Since the dependence on ω_D in the last integral is logarithmic, one can replace ω_D by some (independent of angle) average frequency $\bar{\omega}_0$; then we obtain

$$\Delta(\hat{\mathbf{p}}) = \frac{mp_0}{2\pi^2} \ln \frac{2\bar{\omega}_0\gamma}{\pi T} \int d\sigma' V(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \Delta(\hat{\mathbf{p}}') \quad (6)$$

($\ln \gamma = C = 0.577$). We shall solve this linear homogeneous integral equation in the limit of weak anisotropy. For this purpose we introduce the following notation:

$$\Delta(\hat{\mathbf{p}}) = \bar{\Delta} + \Delta_1(\hat{\mathbf{p}}), \quad \int d\sigma \Delta_1(\hat{\mathbf{p}}) = 0, \quad \int \zeta(\hat{\mathbf{p}}, \hat{\mathbf{p}}') d\sigma' = \zeta_r(\hat{\mathbf{p}}),$$
$$\int \zeta(\hat{\mathbf{p}}, \hat{\mathbf{p}}') d\sigma = \zeta_l(\hat{\mathbf{p}}'), \quad \int \zeta_r(\hat{\mathbf{p}}) \zeta_l(\hat{\mathbf{p}}) d\sigma = \chi,$$
$$a = \frac{\lambda p_0 m}{2\pi^2}, \quad b = \frac{2\bar{\omega}_0\gamma}{\pi} \quad (7)$$

and we rewrite Eq. (6) in the form

$$\bar{\Delta} + \Delta_1(\hat{\mathbf{p}}) = a \ln \frac{b}{T} \int d\sigma' [1 + \zeta(\hat{\mathbf{p}}, \hat{\mathbf{p}}')] [\bar{\Delta} + \Delta_1(\hat{\mathbf{p}}')], \quad (8)$$

from which we obtain

$$\Delta_1(\hat{\mathbf{p}}) = a \ln \frac{b}{T} \left[\bar{\Delta} \zeta_r(\hat{\mathbf{p}}) + \int d\sigma' \{\zeta(\hat{\mathbf{p}}, \hat{\mathbf{p}}') - \zeta_l(\hat{\mathbf{p}}')\} \Delta_1(\hat{\mathbf{p}}') \right]. \quad (9)$$

We now utilize the weakness of the anisotropy, which implies, as we shall see, the condition $\Delta_1 \ll \bar{\Delta}$, and we keep only the first term on the right hand side of Eq. (9). Substituting this into (8) and integrating, we obtain

$$\bar{\Delta} = a \ln \frac{b}{T} \left[\bar{\Delta} + a \ln \frac{b}{T} \bar{\Delta} \chi \right].$$

This equation determines the transition temperature T_c of a pure superconductor to first order in the anisotropy χ:

$$a \ln \frac{b}{T_{c_0}} = 1 - \chi, \quad T_{c_0} = \frac{2\bar{\omega}_0\gamma}{\pi} \exp\left\{ -\frac{2\pi^2(1-\chi)}{\lambda p_0 m} \right\}. \quad (10)$$

We can, in addition, calculate the anisotropy of $\Delta(\hat{\mathbf{p}})$ near T_{c_0} from (9). To first order in χ we obtain

$$|\bar{\Delta}|^{-2} \int |\Delta(\hat{\mathbf{p}})|^2 d\sigma - 1 = |\bar{\Delta}|^{-2} \int |\Delta_1(\hat{\mathbf{p}})|^2 d\sigma = \chi. \quad (11)$$

The result contained in (11) justifies the approximation $\Delta_1 \ll \bar{\Delta}$, if $\zeta \ll 1$. For $T = 0$ it is possible to write an equation of the BCS type[10] in the anisotropic case:[11]

$$\Delta(\mathbf{p}) = \frac{1}{2} \int \frac{d^3p'}{(2\pi)^3} V(\mathbf{p}, \mathbf{p}') \frac{\Delta(\mathbf{p}')}{[\xi^2(\mathbf{p}') + \Delta^2(\mathbf{p}')]^{1/2}},$$

which to first order in χ gives

$$\Delta(\hat{\mathbf{p}}) = \frac{p_0 m}{2\pi^2} \ln \frac{2\bar{\omega}_0}{\bar{\Delta}_0} \int d\sigma' V(\hat{\mathbf{p}}, \hat{\mathbf{p}}') \Delta(\hat{\mathbf{p}}'), \quad (12)$$

where $\bar{\Delta}_0$ is the averaged gap at $T = 0$ in the pure metal. From (12) it is easy to see that the anisotropy of the gap, defined by Eq. (11), also equals χ for $T = 0$ and, in addition, $T_{c_0} = \pi\bar{\Delta}_0/\gamma$ to first order in χ.

The results obtained are in agreement with the more general theory.[9] The method of solving Eq. (6) used here, based only on the smallness of the parameter of anisotropy χ, is identical to the method which we shall apply for the solution of the problem in the presence of impurities.

2. DEPENDENCE OF THE CRITICAL TEMPERATURE ON THE IMPURITY CONCENTRATION

As shown in the articles by Abrikosov and Gor'kov,[1,12] in the presence of impurities there

again exists an equation of type (3), which must be averaged over the positions of the impurities. This leads to the equation

$$\Delta(\mathbf{p}) = T\sum_{\omega_n}\int \frac{d^3p'}{(2\pi)^3} \times \tilde{V}(\mathbf{p},\mathbf{p}')\,\mathfrak{G}_0^{\tau}(\mathbf{p}',\omega_n)\,\mathfrak{G}_0^{\tau}(-\mathbf{p}',-\omega_n)\,\Delta(\mathbf{p}'); \quad (13)$$

$$\mathfrak{G}_0^{\tau}(\mathbf{p},\omega_n) = \frac{1}{i\omega_n\eta - \xi(\mathbf{p})},$$

$$\eta \equiv 1 + \frac{1}{|\omega_n|}\frac{np_0m}{2\pi}\int d\sigma' |u(\hat{\mathbf{p}},\hat{\mathbf{p}}')|^2 \equiv 1 + \frac{1}{2|\omega_n|\tau},$$

$$\tilde{V}(\mathbf{p},\mathbf{p}') = V(\mathbf{p},\mathbf{p}') + n\int \frac{d^3q}{(2\pi)^3}\tilde{V}(\mathbf{p},\mathbf{q})\,\mathfrak{G}_0^{\tau}(\mathbf{q},\omega_n)\,\mathfrak{G}_0^{\tau}(-\mathbf{q},-\omega_n)\,|u(\mathbf{q},\mathbf{p})|^2; \quad (14)$$

n is the concentration of impurities, u is the interaction potential of an electron with an impurity atom.

In the case of isotropic scattering, the equation for $\tilde{V}$ is solved in the form

$$\tilde{V}(\hat{\mathbf{p}},\hat{\mathbf{p}}') = V(\hat{\mathbf{p}},\hat{\mathbf{p}}') + \frac{V_r(\hat{\mathbf{p}})}{2\tau|\omega_n|}, \quad V_r(\hat{\mathbf{p}}) = \int V(\hat{\mathbf{p}},\hat{\mathbf{p}}')\,d\sigma'. \quad (15)$$

We add and subtract the term $V\mathfrak{G}_0\mathfrak{G}_0$ to the kernel of Eq. (13). Then the sum over ω_n and the integral over ξ are calculated in the added term as in a normal metal. One can first integrate the remaining difference over ξ, and express the series over ω_n in terms of the logarithmic derivative of the Γ-function. This gives

$$\Delta(\hat{\mathbf{p}}) = \frac{a}{\lambda}\ln\frac{b}{T_c}\int V(\hat{\mathbf{p}},\hat{\mathbf{p}}')\,\Delta(\hat{\mathbf{p}}')\,d\sigma' - \frac{a}{\lambda}K(2\pi T_c\tau)\int\{V(\hat{\mathbf{p}},\hat{\mathbf{p}}') - V_r(\hat{\mathbf{p}})\}\,\Delta(\hat{\mathbf{p}}')\,d\sigma', \quad (16)$$

where[12]

$$K(x) \equiv 2\sum_{n>0}\frac{1}{(2n+1)[1+x(2n+1)]} = \psi\left(\frac{1+x}{2}\right) - \psi\left(\frac{1}{2}\right),$$

$$K(x) \sim \frac{\pi^2}{4x}, \quad x \gg 1, \quad K(x) \sim \ln\frac{2\gamma}{x}, \quad x \ll 1. \quad (17)$$

Equation (16) was also obtained in the articles by Tsuneto[4] and Caroli et al.[5]

Using the same method as was used for the solution of Eq. (6), for small values of χ we obtain

$$\bar{\Delta} = \bar{\Delta}\,a\ln\frac{b}{T_c} + \bar{\Delta}\,a\ln\frac{b}{T_c}\left[a\ln\frac{b}{T_c}\chi - aK(x)\chi\right],$$

which to first order in χ, with account of (10), gives

$$(1+\chi)\ln\frac{T_{c_0}}{T_c} = \chi K(2\pi T_c\tau). \quad (18)$$

For small concentrations $\tau T_c \gg 1$ and $T_c \sim T_{c_0}$, it follows from (17) and (18) that

$$\frac{T_c - T_{c_0}}{T_{c_0}} = -\frac{\pi}{8}\frac{1}{T_{c_0}\tau}\chi, \quad (19)$$

and for large concentrations $\tau T_c \ll 1$

$$T_c/T_{c_0} = (\gamma/\pi T_{c_0}\tau)^{-\chi}. \quad (20)$$

In the derivation of Eq. (18), we neglected terms containing the unknown temperature (T_c); now we note that in both limiting cases they are actually small for small values of χ.

The results contained in Eqs. (19), (20) were independently obtained by Caroli et al[5] under the same assumptions. Unlike our result (20), saturation in the dependence of T_c is obtained in the article by Markowitz and Kadanoff[6] for very large concentrations, $\tau\omega_D \ll 1$. This difference results from the fact that they made a physically more correct cutoff in frequency. However, in the region of applicability of the theory ($\tau\omega_D \gg 1$) the results agree, as was to be expected. Similar results, but with different numerical coefficients, were obtained for a pairing potential of the special form used by Tsuneto.[4]

3. ENERGY GAP AS A FUNCTION OF CONCENTRATION

As noted by Abrikosov and Gor'kov,[12] the one-particle spectrum, determined from the poles of the function $\mathfrak{G}$, does not exist. However, the concept of an energy gap remains as the value of the frequency ω_0 at which an imaginary part of the retarded Green's function G^R for T = 0 appears for the first time. For the determination of the gap, we write the solution of the Gor'kov equations in the form[12]

$$\mathfrak{G}(\mathbf{p},\omega_n) = -\frac{i\tilde{\omega}_n + \xi(\mathbf{p})}{\tilde{\omega}_n^2 + \xi^2(\mathbf{p}) + \tilde{\Delta}^2(\mathbf{p})}, \quad (21)$$

$$\mathfrak{F}(\mathbf{p},\omega_n) = \frac{\tilde{\Delta}(\mathbf{p})}{\tilde{\omega}_n^2 + \xi^2(\mathbf{p}) + \tilde{\Delta}^2(\mathbf{p})};$$

$$\tilde{\omega}_n = \omega_n\left[1 + \frac{1}{2\tau}\int\frac{d\sigma'}{[\tilde{\omega}_n^2 + \tilde{\Delta}^2(\hat{\mathbf{p}})]^{1/2}}\right],$$

$$\tilde{\Delta}(\hat{\mathbf{p}}) = \Delta(\hat{\mathbf{p}}) + \frac{1}{2\tau}\int\frac{\tilde{\Delta}(\hat{\mathbf{p}}')\,d\sigma'}{[\tilde{\omega}_n^2 + \tilde{\Delta}^2(\hat{\mathbf{p}})]^{1/2}}. \quad (22)$$

For the determination of G^R at T = 0, we make the replacement $\omega_n \to -i\omega$, defining the root in (22) as the analytic continuation of $+\sqrt{\tilde{\Delta}^2 - \tilde{\omega}^2}$ for real values of $\tilde{\omega}$, smaller values of $\tilde{\Delta}$, into the upper half of the complex ω plane. We introduce the function

$$\varphi(\mathbf{p},\omega)=\omega\tilde{\Delta}(\mathbf{p})/\tilde{\omega}, \tag{23}$$

which satisfies the equation

$$\varphi(\hat{\mathbf{p}})=\Delta(\hat{\mathbf{p}})+\frac{1}{2\tau}\int\frac{\{\varphi(\hat{\mathbf{p}}')-\varphi(\hat{\mathbf{p}})\}\,d\sigma'}{\sqrt{\varphi^2(\hat{\mathbf{p}}')-\omega^2}}. \tag{24}$$

It is obvious that the imaginary parts of φ and G^R appear simultaneously. We introduce the notation

$$\varphi(\hat{\mathbf{p}})=\bar{\varphi}+\Delta_1(\hat{\mathbf{p}})/\alpha, \tag{25}$$

where Δ_1 is defined in Eq. (7), and $\bar{\varphi}$ and α only depend on ω and satisfy the equations

$$\bar{\varphi}=\bar{\Delta}+\frac{1}{2\tau\alpha}\int\frac{\Delta_1(\hat{\mathbf{p}}')\,d\sigma'}{\sqrt{\varphi^2(\hat{\mathbf{p}}')-\omega^2}}, \tag{26}$$

$$\alpha=1+\frac{1}{2\tau}\int\frac{d\sigma'}{\sqrt{\varphi^2(\hat{\mathbf{p}}')-\omega^2}}. \tag{27}$$

It is clear from the structure of Eqs. (25)—(27) that $\varphi(\hat{\mathbf{p}})$ becomes complex at a value ω_0 which is the same for all directions of $\hat{\mathbf{p}}$. The gap is defined as the minimum frequency for which a direction $\hat{\mathbf{p}}_1$ exists such that the root in (26) and (27) vanishes. Then one can write

$$\omega_0=\mathrm{Min}_{\hat{\mathbf{p}}'}\varphi(\hat{\mathbf{p}}',\omega_0). \tag{28}$$

The right side of this expression is minimized with respect to all directions on the Fermi surface.

For small anisotropy we again have the condition $\Delta_1/\alpha\ll\bar{\varphi}$ and for $\omega=\omega_0$ we can make the approximation

$$\varphi^2(\hat{\mathbf{p}}')-\omega_0^2\approx 2\bar{\varphi}\Delta_2(\hat{\mathbf{p}}')/\alpha,\quad \Delta_2\equiv\Delta_1+\psi, \tag{29}$$

where $\psi=-|\Delta_{1\min}|$, i.e., Δ_2 is the anisotropy of Δ, measured from the minimum value ($\Delta_2\geq 0$). Substituting the approximation (29) into (26) and (27), we obtain

$$\bar{\varphi}=\bar{\Delta}+\frac{\sqrt{\alpha}}{2\tau\alpha\sqrt{2\bar{\varphi}}}\int\frac{[\Delta_2(\hat{\mathbf{p}}')-\psi]\,d\sigma'}{\sqrt{\Delta_2(\hat{\mathbf{p}}')}},$$

$$\alpha=1+\frac{\sqrt{\alpha}}{2\tau\sqrt{2\bar{\varphi}}}\int\frac{d\sigma'}{\sqrt{\Delta_2(\hat{\mathbf{p}}')}}. \tag{30}$$

One can express the solution of these equations in the two limiting cases in terms of two quantities:

$$D=\sqrt{\bar{\Delta}}\int\frac{d\sigma'}{\sqrt{\Delta_2(\hat{\mathbf{p}}')}}\sim\chi^{-1/4}\text{ and }\delta=\frac{1}{\sqrt{\bar{\Delta}}}\int d\sigma'\sqrt{\Delta_2(\hat{\mathbf{p}}')}\sim\chi^{1/4}.$$

When $2\tau\bar{\Delta}\gg D$, i.e., $2\tau\bar{\Delta}\chi^{1/4}\gg 1$ (small concentrations), we have $\alpha\sim 1$ and

$$\omega_0=\Delta_{min}+\delta/2\tau\approx\omega_{00}[1+\chi^{1/4}/2\tau\bar{\Delta}]. \tag{31}$$

$\omega_{00}=(\Delta_0)_{\min}$ denotes the minimum value of the gap in the absence of impurities. And in the case of large concentrations we obtain

$$\sqrt{\alpha(\omega_0)}\approx 1/2\,\tau\bar{\Delta}\chi^{1/4}\gg 1,$$

$$\omega_0=\bar{\Delta}-\psi+\bar{\Delta}\delta/D\approx\Delta_{min}+\bar{\Delta}\chi^{1/2}\approx\bar{\Delta}, \tag{32}$$

i.e., ω_0 experiences saturation and becomes equal to $\bar{\Delta}$. However, it is necessary to keep in mind that $\bar{\Delta}$ is also a function of the concentration and is determined from an equation for the energy gap which, at T = 0, can be written in the form

$$\Delta(\mathbf{p})=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int\frac{d^3p'}{(2\pi)^3}\frac{\tilde{\Delta}(\mathbf{p}')}{\tilde{\omega}^2+\xi^2(\mathbf{p}')+\tilde{\Delta}^2(\mathbf{p}')}V(\mathbf{p},\mathbf{p}') \tag{33}$$

(we made the substitutions $\omega_n\to\omega$ and $T\sum_{\omega_n}\to\int d\omega/2\pi$).

In order to solve Eq. (33) for small values of χ, we add and subtract values of the integrand in which Δ and ω are substituted in place of $\tilde{\Delta}$ and $\tilde{\omega}$, respectively. Then Eq. (33) is transformed to the form

$$\Delta(\hat{\mathbf{p}})=\frac{p_0 m}{2\pi^2}\int d\sigma' V(\hat{\mathbf{p}},\hat{\mathbf{p}}')\left[\ln\frac{2\bar{\omega}_0}{\Delta(\hat{\mathbf{p}}')}\right]\Delta(\hat{\mathbf{p}}')$$

$$+\frac{p_0 m}{2\pi^2}\int d\sigma' V(\hat{\mathbf{p}},\hat{\mathbf{p}}')J(\hat{\mathbf{p}}';2\tau\bar{\Delta}); \tag{34}$$

here

$$J(\hat{\mathbf{p}};2\tau\bar{\Delta})=\frac{1}{2}\int_{-\infty}^{\infty}d\omega\left\{\frac{\varphi(\hat{\mathbf{p}})}{\sqrt{\varphi^2(\hat{\mathbf{p}})+\omega^2}}-\frac{\Delta(\hat{\mathbf{p}})}{\sqrt{\Delta^2(\hat{\mathbf{p}})+\omega^2}}\right\}. \tag{35}$$

We note that the integral (35) vanishes in the isotropic case, since $\varphi=\Delta$ then. Therefore it is legitimate for small values of χ to expand φ and Δ in a series with respect to χ, using (26) and (27) (with $\omega\to i\omega$). Limiting ourselves to terms of order $\Delta_1/\bar{\Delta}\sim\chi^{1/2}$ and $\Delta_1^2/\bar{\Delta}_1^2\sim\chi$, after rather lengthy calculations we obtain

$$J(\hat{\mathbf{p}},x')=\bar{\Delta}\chi I_1(x')+\Delta_1(\hat{\mathbf{p}})I_2(x')+\Delta_1^2(\hat{\mathbf{p}})I_3(x')/\bar{\Delta}; \tag{36}$$

$$I_1(x')=-\int_0^{\infty}dt\frac{x't^2}{(1+t^2)^2[1+x'\sqrt{1+t^2}]^2},$$

$$I_2(x')=-\int_0^{\infty}dt\frac{t^2}{(1+t^2)^{3/2}[1+x'\sqrt{1+t^2}]},$$

$$I_3(x')=\frac{3}{2}\int_0^{\infty}\frac{t^2[1+2x'\sqrt{1+t^2}]}{(1+t^2)^{5/2}[1+x'\sqrt{1+t^2}]^2},\quad x'=2\tau\bar{\Delta}. \tag{37}$$

For small concentrations $x'\gg 1$ all three integrals give contributions of the form a/x', and after substitution of (36) into (34) we obtain an equation, similar to (16) for small concentrations, which can be solved by the same method. Using expression (12) for the pure metal, we find

$$\bar{\Delta}-\bar{\Delta}_0)/\bar{\Delta}_0=-\pi\chi/16\tau\bar{\Delta}_0,\quad 2\tau\bar{\Delta}_0\gg 1. \tag{38}$$

For large concentrations $x'\ll 1$, $I_2\sim\ln x'$ and we shall only consider the contribution of this integral to lowest order in x' (I_1, $I_3\sim x'$). Equation

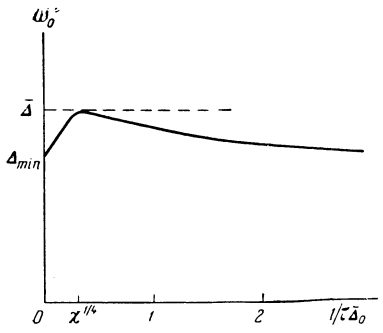

FIG. 1. Dependence of the threshold ω_0 on the quantity $1/\tau\bar{\Delta}_0$, which is proportional to the concentration n.

(34) again becomes similar to (16) for large concentrations. The solution has the form

$$\bar{\Delta}/\bar{\Delta}_0 = (2\tau\bar{\Delta}_0)^{\chi}, \qquad 2\tau\bar{\Delta}_0 \ll 1. \tag{39}$$

Thus, the dependence of ω_0 on n for small concentrations ($2\tau\Delta_0\chi^{1/4} \gg 1$) is primarily determined by the sharp increase of ω_0 with $\bar{\Delta}$ (with a slope proportional to $\chi^{1/4}$). This effect saturates at intermediate concentrations ($2\tau\bar{\Delta}_0\chi^{1/4} \sim 1$), and the slower decrease (with slope proportional to χ) dominates. The decrease of ω_0 becomes very slow ($\sim n^{-\chi}$) for large concentrations ($2\tau\bar{\Delta}_0 \ll 1$). This is schematically illustrated in Fig. 1 and can be observed in experiments which measure the threshold ω_0, for example, in tunneling experiments.

The quantity ω_0, which is the absorption threshold, is completely isotropic for any finite concentration of impurities. This is a consequence of the fact that if absorption takes place in a certain direction, then the impurities, by mixing states with different momenta, ensure absorption in any other direction, since we assumed that the scattering by impurities is isotropic. However, this does not mean that the effects of anisotropy vanish completely for arbitrary concentrations, since the intensity of absorption depends on direction. In particular, for small concentrations the absorption is negligibly small for those directions in which the magnitude of Δ in the absence of impurities differs markedly from its minimum value $(\Delta_0)_{min}$.

In order to investigate this phenomenon, it is necessary to determine the density of states as a function of energy and direction.

4. DENSITY OF STATES

The results of certain experiments with superconductors (tunnel effect, nuclear spin-lattice relaxation) are expressed in terms of the density of states, defined as [13]

$$\mathfrak{N}(\omega, \hat{\mathbf{p}}) = -\frac{\mathfrak{N}(0)}{\pi}\int d\xi\,(\mathbf{p})\,\mathrm{Im}\,G^R(\mathbf{p}, \omega) \tag{40}$$

for $\omega > \epsilon_F$, where $\mathfrak{N}(0)$ is the density of Bloch states at the Fermi surface. From (21) for T = 0, it follows that

$$\mathfrak{N}(\omega, \hat{\mathbf{p}})/\mathfrak{N}(0) = \mathrm{Re}\,\{\omega/\sqrt{\omega^2 - \varphi^2(\omega, \hat{\mathbf{p}})}\}. \tag{41}$$

In Section 3 we investigated the dependence of the absorption threshold, which is studied in tunneling experiments, on the concentration of impurities. For a calculation of the nuclear relaxation time it is necessary to have more detailed information with regard to the energy and angular dependence of the function $\mathfrak{N}$. In particular, as was first noted by Anderson, [14] in a pure superconductor the anisotropy of $\Delta(\mathbf{p})$ leads to the "smearing" of the density of states over an energy interval of order $\Delta_1(\hat{\mathbf{p}}) \sim \bar{\Delta}\chi^{1/2}$ about $\bar{\Delta}$, as shown in Fig. 2(a).

We consider the effect of impurities on this "smearing." We shall start from Eqs. (25)—(27). For values of ω satisfying the inequality $|\bar{\varphi} - \omega| \gg \Delta_1/\alpha$, one can neglect the angular dependence of the integrands in (26), (27). (We remark that $|\alpha(\omega)| \geq 1$.) As a result, we obtain

$$\mathfrak{N}(\omega, \hat{\mathbf{p}}) \sim \mathrm{Re}\,\{\omega/\sqrt{(\omega + \bar{\varphi})(\omega - \bar{\varphi})}\} \sim \mathrm{Re}\,\{\sqrt{\bar{\varphi}}/\sqrt{\omega - \bar{\varphi}}\} \tag{42}$$

(the last relation is valid for $\omega \sim \bar{\varphi}$).

For values of ω such that $|\omega - \bar{\varphi}| \lesssim \Delta_1/|\alpha|$, $\mathfrak{N}$ has a complicated dependence on energy and direction, characteristic for anisotropic supercon-

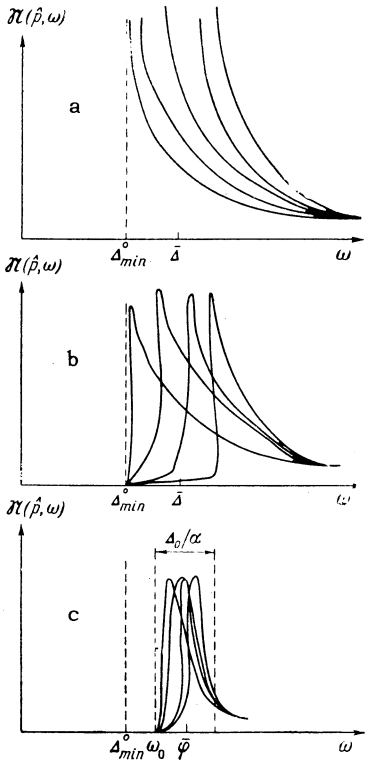

FIG. 2. Dependence of the density of states on energy for various directions: (a) pure superconductor; (b) very small concentrations ($2\tau\bar{\Delta}_0\chi^{1/4} \gg 1$); the distinction between (b) and (a) consists only in the fact that in case (b) the threshold has one and the same value for all directions; (c) intermediate concentrations ($2\tau\bar{\Delta}_0\chi^{1/4} \lesssim 1$). The "smearing" occurs over an energy interval of width Δ_1/α around $\bar{\varphi}$.

ductors. Since the "smearing" occurs only for values of ω inside this interval, it is evident from (32) that the region of "smearing" contracts to zero as the concentration of impurities increases. This means that the superconductor becomes effectively isotropic for $2\tau\bar{\Delta}_0\chi^{1/4} \ll 1$, i.e., at comparatively small concentrations ($l \gtrsim \xi_0$). The function $\Re(\omega, \hat{\mathbf{p}})$ is shown in Fig. 2(b), (c) for various impurity concentrations.

At low temperatures the nuclear spin relaxation time T_1 is proportional to $\ln(1/r)$, where $1/r = \delta\Delta/\bar{\Delta}$ is the relative deviation of Δ from its average value (see, for example, [14]). From the preceding analysis one can see that

$$T_1 \sim \ln\frac{1}{r} = \ln\frac{1}{\bar{\Delta}}\frac{\Delta_1}{\alpha(\omega_0)} \sim \ln\frac{\chi^{1/2}}{\alpha(\omega_0)} \tag{43}$$

From Eq. (30) we see that for very small concentrations ($2\tau\bar{\Delta}_0\chi^{1/4} \gg 1$)

$$1/r \sim \chi^{1/2}[1 - 1/2\tau\bar{\Delta}_0\chi^{1/4}], \tag{44}$$

i.e., 1/r decreases linearly with the concentration.

For large concentrations ($2\tau\bar{\Delta}_0\chi^{1/4} \gg 1$, here l can be even larger than ξ_0) it follows from (32) that $\alpha \gg 1$ and

$$1/r \sim 4\chi\tau^2\bar{\Delta}_0^2, \tag{45}$$

i.e., 1/r decreases quadratically with the concentration.

Using the values given by Masuda [14] for the critical temperatures of Al–Zn alloys, we find from formula (19) the value $\chi \sim 0.01$ for the anisotropy, which justifies our basic assumption. Experiments on the nuclear spin relaxation obtained for those same samples give only two points corresponding to the values $\alpha \sim 5$ and $l \sim \xi_0$. They are situated in the region of "large" concentrations, where r is proportional to the square of n. Plotting ln r as a function of ln n, one can see that the slope is closer to unity than to the value 2 predicted by our theory. However, two points are completely inadequate for a precise determination of the behavior of the curve, especially if one considers the fact that the precision in measuring r is very low. It would be of interest to carry out a more complete and accurate measurement of T_1 in order to verify the quadratic dependence for $l \lesssim \xi_0$.

We remark that the complicated quadrupole effects [14] are apparently unimportant at these concentrations.

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