NONRELATIVISTIC REGGE TRAJECTORIES. I

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Using the exact solution of the Schrödinger equation, we study the motion of Regge poles for potentials of the type $r^{2\epsilon-2}\exp[-(r\mu)^{2\tau}]$, with $\epsilon > 0$, $\tau > 0$, as a function of the coupling constant g and the energy $E = k^2/2m$.

1. INTRODUCTION

RECENTLY in high energy physics a great deal of attention has been given to the study of the analytic properties of partial wave amplitudes. The hypothesis that in quantum field theory the only singularities of the partial amplitudes are moving poles is one of the basic assumptions for deriving the relations between cross sections of different processes at high energy. For this reason a great many papers have been written about Regge poles in nonrelativistic quantum mechanics.[1-5] Although many general results have been obtained, the motion of Regge poles is well understood only for the Coulomb potential and the square well.[6] Trajectories for a Yukawa potential were found by Lovelace[7] using a machine computation, while the work of Arbuzov et al[8] and Azimov et al[9] is only to second order in perturbation theory.

In the present work we consider a potential which can be varied from a square well to a Yukawa potential. The wave function is found in the form of a triple series in powers of r. A method is proposed for transforming the equation for the Regge poles, which comes from the S matrix, to a simpler form. Using this method, in Sec. 4 we find, in first order perturbation theory, the equations of the trajectory for an arbitrary potential V(r) in a form which is very convenient for study. Section 5 treats the motion of Regge poles as a function of the coupling constant g at low energies. In a later paper we shall study the motion of the poles with varying energy.

2. SOLUTION OF THE SCHRÖDINGER EQUATION

In nonrelativistic quantum mechanics, the S matrix for a centrally symmetric potential V(r) is expressed in terms of the radial function $\psi(r)$ in the form

$$S(\nu)=\left[1+\frac{2m}{ik}\int_0^\infty dr\,\psi(r)\,V(r)\sqrt{\frac{\pi kr}{2}}H_\nu^{(2)}(kr)\right]\times\left[1-\frac{2m}{ik}\int_0^\infty dr\psi(r)\,V(r)\sqrt{\frac{\pi kr}{2}}H_\nu^{(1)}(kr)\right]^{-1},\qquad(2.1)$$

where $H_\nu^{1,2}(kr)$ is a Hankel function, $\nu = l + \frac{1}{2}$. The radial function $\psi(r)$ satisfies the following integral equation:

$$\psi(r)=\sqrt{\frac{\pi kr}{2}}J_\nu(kr)+\int_0^r G(r,\,r')\,V(r')\,\psi(r')\,dr',\qquad(2.2)$$

where $G(r, r')$ is the kernel of the Bessel equation. To study the properties of $\psi(r)$ by integration with this potential is too difficult. For example, already in second order we get an integral which is not expressible in terms of known functions. We therefore must go to the original differential equation for $\psi(r)$:

$$\psi''+k^2\psi-l(l+1)r^{-2}\psi=2mV(r)\psi.\qquad(2.3)$$

Equation (2.3) is a second order differential equation, and therefore has two linearly independent solutions. In order to construct the solution of (2.3) satisfying the integral equation (2.2), we must impose the following asymptotic condition on the solution:

$$\lim_{V\to 0}\psi(r)=\sqrt{\pi kr/2}\,J_\nu(kr).\qquad(2.4)$$

Exact solutions of (2.3) are known for only two potentials: αr and the square well potential. We shall treat a broader class of potentials of the type

$$V(r)=\frac{g}{2m}r^{2\epsilon-2}\exp[-(\mu r)^{2\tau}].\qquad(2.5)$$

The parameters ϵ, τ will be assumed to be positive. We point out that as special cases the potential (2.5) contains the well ($\tau = \infty$, $\epsilon = 1$), the Coulomb potential ($\tau = 0$, $\epsilon = \frac{1}{2}$) and the Yukawa potential ($\epsilon = \tau = \frac{1}{2}$); the last is most interesting.

The differential equation to be solved has the form

$$\psi'' + k^2\psi - l(l+1)r^{-2}\psi = gr^{2\varepsilon-2}\psi\exp[-(\mu r)^{2\tau}]. \quad (2.6)$$

For arbitrary τ, ϵ, Eq. (2.6) is an equation with two irregular singular points: r = 0 and r = ∞. Thus the usual Laurent expansion in a single series in powers of r is not applicable. It is easy to see that we need a solution which contains all the powers of r which are contained in the potential. Thus in our case $\psi(r)$ should be assumed to have the form of a triple series:

$$\psi(r) = \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty} c(n,\ p,\ m)\, r^{n+\tau p+\varepsilon m+\gamma}.$$

Substituting ψ in (2.6) and comparing coefficients of like powers of r, we get a functional equation for c(n, p, m). From the condition that $\psi(0)$ be finite, it follows that $\gamma = \nu + \frac{1}{2}$. In solving the equation for c(n, p, m) it turns out that some of the coefficients are zero. We denote the nonzero coefficients by b. Furthermore, in order that the equation for b contain no superfluous factors and be convenient for solution, we must redefine ψ:

$$\psi(r) = \sqrt{\frac{\pi kr}{2}}\left(\frac{kr}{2}\right)^{\nu}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{m=0}^{\infty}\left[\frac{ikr}{2}\right]^{2n}$$

$$\times\,[-(\mu r)^{2\tau}]^{p}\left[\frac{1}{4}gr^{2\varepsilon}\right]^{m} b(n, p, m). \quad (2.7)$$

The functional equation for b is a multiterm equation:

$$(n+\tau p+\varepsilon m)(n+\tau p+\varepsilon m+\nu)\, b(n, p, m) = b(n-1, p, m) + \sum_{l=0}^{l=p}\frac{b(n, l, m-1)}{\Gamma(1+p-l)}. \quad (2.8)$$

If g = 0, Eq. (2.6) becomes "free," and so (2.7) must reduce to a Bessel function according to condition (2.4). The coefficients b(n, p, 0), which will then determine (2.7), satisfy the two-term functional equation

$$(n+\tau p)(n+\tau p+\nu)\, b(n, p, 0) = b(n-1, p, 0),$$

which is easily solved in terms of Γ functions:

$$b(n, p, 0) = \frac{\delta_{0,p}\, b_0}{\Gamma(1+n)\,\Gamma(1+n+\nu)},\quad \delta_{0,p} = \begin{cases}1, & p=0\\ 0, & p\neq 0\end{cases}. \quad (2.9)$$

According to (2.4) b_0 should be set equal to one.

We note one important point. Since the coupling constant g is not contained in the equation for b, we can establish a connection between the series (2.7) and standard perturbation theory: the order in perturbation theory coincides with the index m. This in turn enables us to follow the changes introduced by treating different orders in perturbation theory. This point represents one of the advantages of this method over those methods which use the integral equation for ψ.

3. THE COEFFICIENTS b(n, p, m)

The coefficients b(n, p, 1), b(n, p, 2) etc. are no longer expressible in terms of products of Γ functions, except in special cases. We shall try to find a form for b in which its properties and values are simply determined. We first establish a relation between the coefficients of order m and (m − 1). To do this we express the coefficient b(n − 1, p, m) in (2.8) in terms of b(n − 1, *l*, m − 1) and b(n − 2, p, m), then express b(n − 2, p, m) in terms of b(n − 2, *l*, m − 1) and b(n − 3, p, m), etc. As a result we get the following relation between the coefficients:

$$b(n,\ p,\ m) = \sum_{k=0}^{k=n}\sum_{l=0}^{l=p} D(m;\ np\to kl)\, b(k,\ l,\ m-1),\ m\neq 0;$$

$$D(m;\ np\to kl) = \frac{\Gamma(k+\varepsilon m+\tau p)\,\Gamma(k+\varepsilon m+\tau p+\nu)}{\Gamma(1+n+\varepsilon m+\tau p)\,\Gamma(1+n+\varepsilon m+\tau p+\nu)\,\Gamma(1+p-l)}. \quad (3.1)$$

The further arguments and computations are conveniently done using the following scheme. Let the point M with coordinates (n, p, m) correspond to the coefficient b(n, p, m) and the line joining the points (n, p, m) and (k, *l*, m − 1) to the function D(m; np → k*l*). In this sense the function D plays the role of a propagator. The formula (3.1) connects the point M_m = (n, p, m) with the points M_{m-1} = (k, *l*, m − 1), located on the m − 1 plane in the rectangle (n, p). It is obvious that each point of the plane (m − 1) can be expressed in terms of points of the plane (m − 2), etc. Carrying this series of substitutions to completion, we get the general formula for the coefficients b(n, p, m):

$$b(n,\ p,\ m) = \frac{1}{\Gamma(1+\nu)}\sum_{M_{m-1}}^{n,\ p} D(M_m\to M_{m-1})\ldots\sum_{M_0}^{k,\ l} D(M_1\to M_0). \quad (3.2)$$

We shall write this formula in a more compact form:

$$b(M) = \frac{1}{\Gamma(1+\nu)}\sum_{L} D(L), \quad (3.3)$$

where the sum goes over all paths L joining the point M with the origin, and D(L) is the product of M propagation functions along the path L.

As a first example we shall determine the coefficient b(0, 0, m). The point M = (0, 0, m) can be joined to the origin in only one way, along the m axis. There is therefore only one term in (3.3):

$$b(0,\ 0,\ m) = D(m;\ 00 \to 00) \ldots D(1,\ 00 \to 00)\,\Gamma^{-1}(1+\nu) = \Gamma(1+\nu/\varepsilon)/\Gamma(1+\nu)\,\Gamma(1+m+\nu/\varepsilon)\,\Gamma(1+m)\,\varepsilon^{2m}. \tag{3.4}$$

The second example will be a more complicated one. Let us find the coefficients for the case of the potential well ($\epsilon = 1$, $\tau = \infty$). For all points M not lying on the plane (n, m), the functions D(L) are zero, since the denominator contains $\tau = \infty$. Thus only b(n, 0, m) is different from zero. Let us choose some path in the (n, m) plane (Fig. 1) and determine D(L) for it:

$$D(L) = \frac{1}{(m+n)(m+n+\nu)} \cdots \frac{1}{(1)(1+\nu)\,\Gamma(1+\nu)} = \frac{1}{\Gamma(1+m+n)\,\Gamma(1+m+n+\nu)}.$$

We see that D(L) is independent of the shape of the path. Therefore in (3.3) D(L) can be removed from the summation sign, and to determine the coefficient b(n, 0, m) we shall need only to determine the number of possible paths L.

It is easy to prove by induction that for the point (n, p, m) the number of different paths L is given by the following formula:

$$L(n,\ p,\ m) = \Gamma(m+p)\,\Gamma(1+m+n)/\Gamma(m) \times \Gamma(1+m)\,\Gamma(1+p)\,\Gamma(1+n). \tag{3.5}$$

Using (3.5), we find for b(n, p, m) in the case of the square well

$$b(n,\ p,\ m) = \delta_{0,p}/\Gamma(1+m)\,\Gamma(1+m+n+\nu)\,\Gamma(1+n). \tag{3.6}$$

We now turn to the general treatment of the properties of the coefficients b. Let us, for example, write out b(n, p, 1) in detail:

$$b(n,\ p,\ 1) = \Gamma^{-1}(1+\nu)\,\Gamma^{-1}(1+p) \times \sum_{k=0}^{k=n} \frac{\Gamma(k+\tau p+\varepsilon+\nu)\,\Gamma(k+\tau p+\varepsilon)}{\Gamma(1+n+\tau p+\varepsilon+\nu)\,\Gamma(1+n+\tau p+\varepsilon)\,\Gamma(1+k+\nu)\,\Gamma(1+k)}. \tag{3.7}$$

Like the coefficient b(n, p, 1), the coefficient b(n, p, m) is a finite sum of terms of similar type. As we see from (3.7), the denominators of the terms are products of terms of the type (n + τp + εm) and (n + τp + εm + ν). Therefore, because of the factors of this second type, the coefficients b(n, p, m) have singularities in the variable ν, poles of first order, located on the negative axis. Starting from formulas (3.2) and (3.3) one can assert that the coefficient b(n, p, m) has poles at the points $\nu = -\epsilon M - \tau P - N$, where, if M = m, P = p and N = 0, 1, 2, ..., n, and if M = 1, 2, ... m − 1, then P = 0, 1, ... p and N = 0, 1, ..., n.

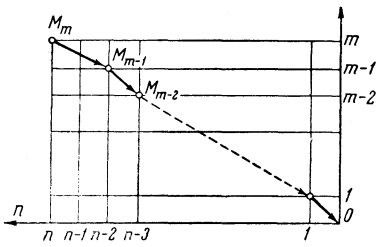

FIG. 1

In conclusion we consider the question of the convergence of the series (2.7). The only singular points of the differential equation (2.6) are 0 and ∞. From the divergence of the series at some point r_0 it would follow that there is at least one other singular point of the differential equation on the circle $|r| = |r_0|$. A rigorous proof of the convergence of the series can be given on the basis of the following approximate formula:

$$|b(n,\ p,\ m)| \leqslant L(n,\ p,\ m)\,|\overline{D(L)}|, \tag{3.8}$$

where $\overline{D(L)}$ is the largest of the D(L).

4. TRANSFORMATION OF THE EQUATION FOR THE POLES OF THE S MATRIX

As was shown in papers of Regge[1] and Mandelstam,[2] the only singularities of S(ν) in the ν plane are moving poles whose positions are determined by the zeros of the denominator in (2.1):

$$1 = \frac{2m}{ik}\int_0^\infty V(r)\sqrt{\frac{\pi kr}{2}}\,H_\nu^1(kr)\,\psi(r)\,dr. \tag{4.1}$$

The ν dependence in this integral enters in $H_\nu^{(1)}$ and ψ. The behavior of $H_\nu^{(1)}$ in the complex plane has been little studied and is nothing simple. It is therefore expedient to transform (4.1) so that the integrand contains better known and simpler functions. Among the integral representations of the Bessel functions only the Mellin transforms have these properties.

The transform F(s) and the original f(r) are related in the Mellin transformation by the following relations

$$F(s) = \int_0^\infty dr \cdot r^{s-1} f(r), \quad f(r) = \int \frac{ds}{2\pi i}\frac{F(s)}{r^s}. \tag{4.2}$$

Let F_1, F_2, F_3 be Mellin transforms of the functions f_1, f_2, f_3 respectively. Then the Mellin transform of the product $f_1 f_2 f_3$ can be written in the form (folding theorem)

$$\int_0^\infty dr \cdot r^{s-1} f_1 f_2 f_3 = \int \frac{d\sigma_1\,d\sigma_2}{(2\pi i)^2} F_1(\sigma_1)\,F_2(\sigma_2)\,F_3(s-\sigma_1-\sigma_2). \tag{4.3}$$

Relation (4.3) is fundamental for the transformation of Eq. (4.1).

We now proceed to transform Eq. (4.1). We first do the calculations in first order perturbation theory. There are two reasons for this. First, perturbation theory in first order coincides with the first term in the expansion in 1/k, since the expansion parameter is $g/k^{2\epsilon}$. Second, formula (4.4), which we shall obtain, does not contain the potential explicitly.

We introduce the notation

$$f_1 = \frac{2m\sqrt{\pi}k^2r^2}{4ik} V(r),$$

$$f_2 = \left(\frac{kr}{2}\right)^{\nu-1} H_\nu^{(1)}, \quad f_3 = \sum_\varkappa b(\varkappa) r^\varkappa, \quad f_{ik} = f_i f_k.$$

Setting $\psi = \sqrt{\pi kr/2}\, J_\nu(kr)$, using (4.1) and (4.3) we get the following equation for the Regge trajectory in the first order of perturbation theory:

$$1 = \int \frac{d\sigma_1}{2\pi i} F_1(\sigma_1) F_{23}(1-\sigma_1)$$
$$= -\frac{m}{2\sqrt{\pi}} \int_\Omega \frac{d\sigma}{2\pi i} \left(\frac{k}{i}\right)^\sigma$$
$$\times \frac{\Gamma(-\sigma/2)\Gamma(\nu-\sigma/2)\Gamma(1/2+\sigma/2)}{\Gamma(1+\nu+\sigma/2)} V(2+\sigma). \qquad (4.4)$$

As an example of the application of (4.4) we consider the potential (2.5):

$$1 = -\frac{g}{4\tau\mu^{2\varepsilon}\sqrt{\pi}} \int_\Omega \frac{d\sigma}{2\pi i} \left(\frac{k}{i\mu}\right)^{2\sigma} \Gamma\left(\frac{\varepsilon+\sigma}{\tau}\right) \Gamma(\nu-\sigma)$$
$$\times \Gamma(-\sigma)\Gamma(1/2+\sigma)\,\Gamma^{-1}(1+\nu+\sigma). \qquad (4.5)$$

The integration contour Ω in (4.5) is chosen so that the poles of $\Gamma(\nu-\sigma)$, $\Gamma(\sigma)$ are to the right and the poles of $\Gamma((\epsilon+\sigma)/\tau)$, $\Gamma(1/2+\sigma)$ are to the left (Fig. 2). The behavior of the integrand on the circle $R \to \infty$ is determined by the value of τ. For $\tau > 1/2$, the contour Ω can be closed to the right, while for $\tau < 1/2$, it can be closed to the left. When $\tau = 1/2$, everything depends on the ratio $|k/\mu| \lesssim 1$. Using the Cauchy residue theorem gives an expansion of the integral (4.5) in powers of k or k^{-1}. In the intermediate case of $|k| \sim |\mu|$, it is convenient to use the saddle point method in (4.5)

We give some more details of the results for the case of $|k| \to \infty$. Closing the contour to the left for $\tau \le 1/2$, we get

$$1 = -\frac{g}{4\tau\mu^{2\varepsilon}\sqrt{\pi}}$$
$$\times \left[\tau \sum \left(\frac{i\mu}{k}\right)^{2\varepsilon+2\tau n} \frac{(-)^n \Gamma(\nu+\varepsilon+\tau n)\,\Gamma(\varepsilon+\tau n)\,\Gamma(1/2-\varepsilon-\tau n)}{\Gamma(1+n)\,\Gamma(1+\nu-\varepsilon-\tau n)}\right.$$
$$\left. + \sum \left(\frac{i\mu}{k}\right)^{1+2n} \frac{(-)^n \Gamma(1/2+n)\,\Gamma(\nu+1/2+n)\,\Gamma((\varepsilon-n-1/2)/\tau)}{\Gamma(1+n)\,\Gamma(1/2+\nu-n)}\right]. \qquad (4.6)$$

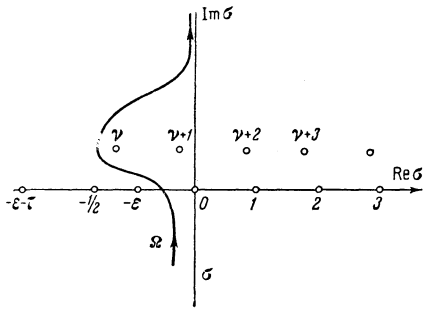

FIG. 2

Let us determine the finite roots ν of this equation when $|k| = \infty$. If ν does not make any of the Γ functions in the numerator become infinite, then when $|k| = \infty$ each term of the series is zero, and the particular value of ν is not a root of (4.6). A different situation arises when ν can make one of the Γ functions infinite. Let $\nu = -\epsilon - \tau P - N + \epsilon_1$ where $|\epsilon_1| \ll 1$, while P and N are integers. Expanding (4.6) in powers of ϵ_1, we get

$$\nu = -\varepsilon - \tau P - N$$
$$- \frac{g(-)^{P+N}\Gamma(\varepsilon+\tau P)\,\Gamma(1/2-\varepsilon-\tau P)\,(i\mu/k)^{2\varepsilon+2\tau P}}{4\mu^{2\varepsilon}\sqrt{\pi}\,\Gamma(1-2\varepsilon-2\tau P-N)\,\Gamma(1+N)\,\Gamma(1+P)}. \qquad (4.7)$$

In deriving (4.7) it is essential to assume that ϵ and τ are irrational numbers. If ϵ and τ are rational, then ϵ_1 will be given by a finite polynomial in k^{-1}. For example, for the Yukawa potential, $\epsilon = \tau = 1/2$, these will be the Legendre polynomials.

The further motion of the poles of (4.7), i.e., when $|\epsilon_1| \ge 1$, is conveniently studied by using the integral (4.5) directly. But first we shall establish how one gets (4.7) from (4.5). When $|k| \to \infty$ the integrand also tends to infinity in the right half plane of σ, while it goes to zero in the left halfplane, except at the poles of the Γ functions. Therefore if the contour Ω runs in the left halfplane, the integral will be zero. However, if $\nu \to -\epsilon - \tau P - N$, then one pole of $\Gamma(\nu-\sigma)$ squeezes the contour onto a pole of $\Gamma((\epsilon+\sigma)/\tau)$. A saddle develops between the poles. The value of the integral at the saddle point depends on two factors: the value of $|k|$ and the distance between the poles, which is of order ϵ_1. With increasing $|k|$ the integrand decreases and the decreased separation between the poles leads to an increase of the modulus only in the neighborhood of the saddle point. Since we can neglect the integrand everywhere on the contour Ω except near the saddle point, it is clear that there is some relation between $|k|$ and ϵ_1 for which (3.5) is satisfied. Taking the residue at the point $\sigma = \nu$ and dropping the other terms, we get relation (4.7). From these arguments it follows that the motion of the pole

($\nu = \sigma$) is the motion of the Regge pole. The motion of the poles (4.7) for the case of the Yukawa potential in first order in g was studied in [8,9].

The transformation of (3.1) with the exact ψ differs from that given above in the order of integration in (4.3). After some minor calculations we get Eq. (4.7) in the form

$$1 = \int \frac{d\sigma_2}{2\pi i} F_2(1-\sigma_2) F_{13}(\sigma_2) = -\frac{1}{\tau}\int_\Omega \frac{d\sigma}{2\pi i} \varkappa^{2\sigma} \Gamma(\nu-\sigma)\Gamma(-\sigma)\sum b(n,p,m) \times \Gamma\left(\frac{\varepsilon+\sigma+n+\tau p+\varepsilon m}{\tau}\right)\varkappa^{2n}(-)^p G^{1+m}, \quad (4.8)$$

where $\kappa = k/2i\mu$, $G = g/4\mu^{2\epsilon}$. The contour of integration is the same as in (4.5). It is true that now there are many more singular points on the negative σ axis. Using (4.8) we shall study the trajectories of the Regge poles.

The series under the integral sign converges for all σ, if $\tau > \frac{1}{2}$. For example, in first order in G it is equal to

$$\sum \varkappa^{2n}\Gamma\left(\frac{\varepsilon+\sigma+n}{\tau}\right)\Big/\Gamma(1+n)\Gamma(1+\nu+n).$$

A general proof follows from the properties of the Mellin integral:

$$\sum \infty \int dr\psi(r) V(r) r^{s-\nu+1/2}.$$

We see that the integral can diverge only at the lower limit, which is caused by the Γ function; there are no other points of divergence so long as $\tau > \frac{1}{2}$. The integration contour in (4.8) when $\tau \geq \frac{1}{2}$ can be closed to the right and the integral calculated using residues:

$$1 = \frac{\pi}{\tau \sin \pi\nu}\sum \frac{(-)^p G^{1+m}\varkappa^{2n+2r}}{\Gamma(1+r)}\left[\frac{\varkappa^{2\nu}\Gamma(p+(\nu+\varepsilon+\varepsilon m+n+r)/\tau)}{\Gamma(1+\nu+r)} - \frac{\Gamma(p+(\varepsilon+\varepsilon m+n+r)/\tau)}{\Gamma(1-\nu+r)}\right] b(n,p,m). \quad (4.9)$$

This series converges for all κ when $\tau \geq \frac{1}{2}$. But the formulas obtained when $\tau \geq \frac{1}{2}$ from (4.9) can be continued analytically to any τ. This statement is directly verified in first order in g.

The roots of Eq. (4.8) are certain functions ν of the two variables k and G. The dependence of ν on G is important for problems of bound states. We shall therefore begin the analysis of (4.8) with a study of the dependence of the roots ν on G when $k \to 0$.

5. REGGE TRAJECTORIES AT LOW ENERGIES

In the preceding sections we treated general questions related to the solution of the Schrödinger equation and the transformation of the S matrix. Now we proceed to a specific study of the motion of Regge poles. We start with the determination of the trajectories at low energies. To get the equations of the trajectories in this case, we use the relation (4.9). Because of the presence of the factor $\kappa^{2\nu}$, Eq. (4.9) for $\kappa \to 0$ will have three different limits depending on the magnitude and sign of Re ν.

If Re $\nu > 0$, it is obvious from (4.8) that we must keep the term in (4.9) that corresponds to the residue at the point $\sigma = 0$:

$$1 = -\frac{\pi}{\tau\Gamma(1-\nu)\sin\nu\pi}\sum(-)^p G^{1+m} b(0,p,m)\Gamma\left(p+\frac{\varepsilon+\varepsilon m}{\tau}\right). \quad (5.1)$$

For the second case, Re $\nu < 0$, we get from (4.9)

$$0 = \frac{\pi}{\tau\Gamma(1+\nu)\sin\pi\nu}\sum(-)^p G^{1+m} b(0,p,m)\Gamma\left(p+\frac{\nu+\varepsilon+\varepsilon m}{\tau}\right). \quad (5.2)$$

If we write Eqs. (5.1) and (5.2) in the form

$$0 = \frac{1}{\Gamma(\nu)} + \frac{1}{\tau}\sum(-)^p G^{1+m} b(0,p,m)\Gamma\left(p+\frac{\varepsilon+\varepsilon m}{\tau}\right), \quad (5.3)$$

$$0 = \frac{1}{\tau}\sum(-)^p G^{1+m} b(0,p,m)\Gamma\left(p+\frac{\nu+\varepsilon+\varepsilon m}{\tau}\right), \quad (5.4)$$

it is clear that for $\nu = 0$ they coincide, i.e., the Regge trajectories pass continuously from the left to the right halfplane of ν through the origin.

Let $f_1(G)$ and $f_2(G)$ be the analytic functions which describe the motion of the trajectory in the right and left ν half plane, respectively. We have shown that if $f_1(G_0) = 0$, then $f_2(G_0) = 0$. The question arises whether the functions f_1 and f_2 are not the same analytic function. For the example of the square well potential, we shall show that $f_1'(G_0) \neq f_2'(G_0)$. This means that the Regge trajectory $\nu = \nu(G)$ when $k = 0$ is not an analytic function. We shall explain the reason for this later. Finally, in the third case, when $\nu \sim 0$, we rewrite (4.9) in the form

$$1 + \frac{\tau\sin\pi\nu}{\pi}\left[\sum\ldots\Gamma\left(p+\frac{\varepsilon+\varepsilon m+n+r}{\tau}\right)\right]^{-1} = \varkappa^{2\nu}\frac{\sum\ldots\Gamma\left(p+\frac{\nu+\varepsilon+\varepsilon m+n+r}{\tau}\right)}{\sum\ldots\Gamma\left(p+\frac{\varepsilon+\varepsilon m+n+r}{\tau}\right)}. \quad (5.5)$$

Let us expand (5.5) in the neighborhood of the point $\kappa = \nu = 0$:

$$1 + \rho_1\nu + \rho_2\nu^2 \approx \varkappa^{2\nu}(1+\gamma_1\nu+\gamma_2\nu^2). \quad (5.6)$$

Its solution determines the bundle of trajectories emerging from the origin:

$$\nu \approx \frac{2\pi i n}{(2\ln\varkappa + \gamma_1 - \rho_1)} + \frac{4\pi^2 n^2 (\gamma_2 - \rho_2 + {}^1/_2\rho_1^2 - {}^1/_2\gamma_1^2)}{(2\ln\varkappa + \gamma_1 - \rho_1)^3},$$

$$n = \pm 1, \pm 2, \ldots \qquad (5.7)$$

Trajectories of this type were first obtained in the work of Gribov and Pomeranchuk.[10]

So at low energies there are three families of trajectories which satisfy respectively Eqs. (5.1), (5.2), and (5.6). We note that we have not considered the neighborhoods of the points $\nu = -\epsilon M - \tau P - N$, where M, P, and N are integers. The behavior of Eq. (4.9) at these points will be studied later.

Now for an example of the potential well type (the power law well):

$$V(r) = \begin{cases} (g/2m) r^{2\varepsilon-2}, & r \leqslant \mu^{-1} \\ 0, & r \geqslant \mu^{-1}, \end{cases} \qquad (5.8)$$

we shall explain the general character of the location and behavior of the roots. For this potential it is sufficient to know just the one coefficient b(0, 0, m) [cf. (3.4)]. In the first approximation in κ, Eq. (5.5) has the form

$$\varkappa^{2\nu} \approx \Gamma(\nu) J_{\nu/\varepsilon-1}\left(\frac{2\sqrt{G}}{i\varepsilon}\right) \Big/ \Gamma(-\nu) J_{\nu/\varepsilon+1}\left(\frac{2\sqrt{G}}{i\varepsilon}\right). \qquad (5.9)$$

It clearly shows the reason for the breakdown of analyticity of the Regge poles when we go through $\nu = 0$. The point $\nu = \kappa = 0$ is an essentially singular point of the second kind for the function $\kappa^{2\nu}$ (this is a function of two complex variables!), i.e., the value of $\kappa^{2\nu}$ depends on how κ and ν tend to zero. We therefore get different equations from (5.9) for Re $\nu > 0$ and Re $\nu < 0$:

$$J_{\nu/\varepsilon-1}(2\sqrt{G}/i\varepsilon) = 0, \ \mathrm{Re}\,\nu > 0,$$

$$J_{\nu/\varepsilon+1}(2\sqrt{G}/i\varepsilon) = 0, \ \mathrm{Re}\,\nu < 0,$$

which coincide respectively with (5.1) and (5.2).

From the theory of Bessel functions it is known that the equation $J_\lambda(z) = 0$ defines an infinite but countable set of roots $\lambda = \lambda_n(z)$. Each root can be regarded as some analytic function of z or as the branch of an analytic function $\Lambda(z)$ given implicitly by $J_\Lambda(z) = 0$. A separation of the different branches is conveniently made when $z \to 0$. The coefficients in the expansion of the Bessel function

$$J_\lambda(z) = \frac{(z/2)^\lambda}{\Gamma(1+\lambda)} \sum \frac{(-)^n (z/2)^{2n}}{\Gamma(1+n)(1+\lambda)(2+\lambda)\ldots(n-1+\lambda)(n+\lambda)} \qquad (5.10)$$

have poles at the points $\lambda = -1, -2, \ldots$. Thus the point $z = 0$ and $\lambda = -n$ is an essentially singular point, and therefore satisfies the equation $J_\lambda(z) = 0$. Consequently the roots for $z \to 0$ can be written in the following form: $\lambda = -n + \epsilon_1$,

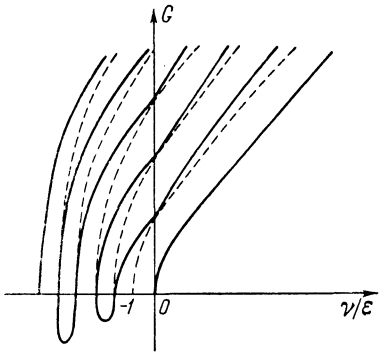

FIG. 3

where $|\epsilon_1| \ll 1$. The value of ϵ_1 can be easily determined from the series (5.10):

$$\lambda_n(z) \approx -n + \left(\frac{z}{2}\right)^{2n} \sum_{k=0}^{\infty} \left(\frac{iz}{2}\right)^{2k} \times \frac{1}{\Gamma(1+k)\Gamma(1+k+n)} \Big/ \sum_{k=0}^{n-1} \frac{\Gamma(n-k)}{\Gamma(1+k)} \left(\frac{z}{2}\right)^{2k}. \qquad (5.11)$$

If we were interested in the further terms in the expansion (5.11), we would have to write λ in the form $\lambda = -n + \epsilon_1 + \epsilon_2$ and, noting that $|\epsilon_2| < |\epsilon_1|$, expand (5.10) in ϵ_2. We note that the expansion (5.11) is valid not only when $z \to 0$, but along the whole negative λ axis. It then follows that the trajectories of the Regge poles in the case of a power law well are representable in the form

$$\nu_n = \varepsilon(1 + \lambda_n), \quad \mathrm{Re}\,\nu > 0;$$
$$\nu_n = \varepsilon(-1 + \lambda_n), \quad \mathrm{Re}\,\nu < 0. \qquad (5.12)$$

Graphs of these functions for an attractive potential (G < 0) are shown in Fig. 3. The picture changes drastically for a repulsive potential (G > 0). As expected, there are no positive solutions ν. It follows from (5.12) and (5.11) that the negative roots ν for small G move from the even points toward the odd points, collide and diverge along complex-conjugate trajectories (cf. Fig. 3). The last statement follows from the reality of the coefficients of the Bessel function.

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