INSTABILITY OF ONE-DIMENSIONAL PACKETS AND ABSORPTION OF ELECTROMAGNETIC WAVES IN A PLASMA

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It is shown that when an energetic packet of electromagnetic waves moves in a magnetoactive plasma the conditions may be such that the distribution-function plateau formed by the packet may be unstable. The transfer of electromagnetic energy to growing waves is considered. It is found that the energy of the initial packet changes into energy of both the growing waves and the plasma. The nonlinear damping decrement of a weakly non-one-dimensional electromagnetic wave packet is evaluated.

NONLINEAR interactions of electromagnetic waves in a collisionless plasma can be divided roughly into two types—nonlinear effects that can be described hydrodynamically (or quasi-hydrodynamically),[1] and nonlinear kinetic effects associated with "finer" deformations of the distribution functions. The first type of nonlinear interactions involves essentially the scattering of one wave by another; for plane waves the energy and momentum of quasiparticles (photons and plasmons) is then conserved. Parametric amplification[2] and instability against decay[3] are nonlinear interactions of this type (see also [4]). Nonlinear effects of the second (kinetic) type are associated with changes in the populations of plasma particle energy levels, similar to the interactions of electromagnetic waves in three-level masers.[1)] The interaction of electromagnetic wave packets in a magneto-active plasma, which is investigated in the present work, is an example of the second type.

We shall show that the plateau of the distribution function[6] in several cases does not lead to a disappearance of electromagnetic wave absorption, because of the instability of the distribution function having the plateau.

1. Let us consider the kinetic nonlinear interactions of electromagnetic waves in a magneto-active plasma in the case of random phases, when the equations describing interactions between plasma particles and the electromagnetic field can be derived from the equations of level population balance in the classical limit $\hbar \to 0$.[6-10] These "quasi-linear equations" (in the terminology of [6]) coincide with the Fokker-Planck equations for the Brownian motion of particles acted on by random electromagnetic fields,[8] and are valid (given the condition of random phases) when the correlation time of the electromagnetic field (for a random force) obeys the inequality $\tau \ll t_0$ (where t_0 is the time required for an appreciable change of the distribution function). If $\Delta\omega$ is the width of the wave packet frequency spectrum the condition for applying the indicated equations[2)] is

$$1/t_0 \lll \Delta\omega. \tag{1}$$

Although the quasi-linear equations do not contain the Planck constant, the use of quantum concepts simplifies the derivation of these equations and furnishes an intuitive interpretation of the results.[11]

2. Let the energy emitted per second by a particle of kind α in the S-th harmonic of the cyclotron frequency $\Omega_H^\alpha = e^\alpha H_0/m_\alpha c$ into the j-th normal wave be

$$dW^\alpha = \pi^j_{s,\alpha}\,\delta(\omega - \omega^j_{s,\alpha})\,d^3\mathbf{k}, \tag{3}$$

where $\omega^j_{s,\alpha}$ is the frequency determined from the Doppler condition

$$\omega^j = s\Omega_H^\alpha + \mathbf{k}^j\mathbf{p}/m_\alpha, \tag{4}$$

[1)]In [5] equations for nonlinear interactions of both types are given in terms of the deformation of the distribution functions.

[2)]The condition for applying quasi-linear equations in the case of plasma waves, derived from (1) is

$$\frac{\Delta\omega}{k} \gg \sqrt{\frac{e\varphi_0}{m}}\left(\frac{\omega}{k}\frac{\partial k}{\partial\omega}\right)^{3/4}, \tag{2}$$

which differs somewhat from the condition given in [6,9] (using the same notation).

$\mathbf{k}^j$ is the wave vector, $\mathbf{p} = \{p_\perp, p_z\}$ is the particle momentum, $z > 0$ is the direction of the magnetic field $\mathbf{H}_0$, e^α and m_α are the charge and relativistic mass of the particles, and c is the velocity of light.

The quasi-linear equations for the particle distribution functions f^α and the spectral density of the electromagnetic wave energy $\epsilon_\mathbf{k}$ ($\sum_j \epsilon^j_\mathbf{k} d^3k$ is the electromagnetic field energy per unit volume) are easily obtained, as in [12], from the balance equations for $\hbar \to 0$. Neglecting spontaneous emission, the equations are

$$\frac{\partial f^\alpha}{\partial t} = \sum_{s,j} \int d^3\mathbf{k}\, \{\lambda^j_{\alpha,s}\, [(\lambda^j_{\alpha,s} f^\alpha)\, (2\pi)^3\, \varepsilon^j_\mathbf{k} \delta\, (\omega - \omega^j_{s,\alpha})]\},$$

$$\frac{\partial \varepsilon^j_\mathbf{k}}{\partial t} = \left\{\sum_{s,\alpha} \int d^3\mathbf{p}\, (2\pi)^3\, \pi^j_{s,\alpha}\, \delta\, (\omega - \omega^j_{s,\alpha})\, \lambda^j_{\alpha,s} f^\alpha\right\} \varepsilon^j_\mathbf{k}, \tag{5}$$

where we have the operator

$$\boldsymbol{\lambda}^j_{\alpha,s} = \frac{m_0^\alpha s \omega_H^\alpha}{p_\perp \omega} \frac{\partial}{\partial p_\perp} + \frac{k^j_z}{\omega} \frac{\partial}{\partial p_z},$$

m_0^α is the rest mass, $\omega_H = e^\alpha H_0 / m_0^\alpha c$, and $k^j_z = \mathbf{k}^j \mathbf{H}_0 / H_0$.

We shall investigate the case in which we can consider the interaction of electromagnetic waves with particles of only one kind for S = 1 (the normal Doppler effect), and in the general case we shall present only an intuitive derivation of the condition for plateau instability. Dipole radiation will be assumed and relativistic effects will be neglected. Then $\omega^j_{S=1} = \omega^j(p_z, \theta)$ where θ is the angle between $\mathbf{H}_0$ and $\mathbf{k}^j$, and

$$\pi^j_{s=1} = p_\perp^2 P_j\, (\mathbf{k}, p_z, \theta). \tag{6}$$

Subject to the foregoing limitations, following integration with respect to $d\omega$ the quasi-linear equations become (omitting the index α)

$$\frac{\partial f}{\partial t} = \frac{m_0}{p_\perp} \frac{\partial}{\partial p_\perp} \left\{p_\perp^2 \left[A \frac{m_0}{p_\perp} \frac{\partial f}{\partial p_\perp} + B \frac{\partial f}{\partial p_z}\right]\right\} + \frac{\partial}{\partial p_z} \left\{p_\perp^2 \left[B \frac{m_0}{p_\perp} \frac{\partial f}{\partial p_\perp} + D \frac{\partial f}{\partial p_z}\right]\right\},$$

$$\frac{\partial \varepsilon^j_\mathbf{k}}{\partial t} = \Gamma^j \varepsilon^j_\mathbf{k} \int_0^\infty p_\perp^3 \left\{\frac{\omega_H}{\omega_j} \frac{m_0}{p_\perp} \frac{\partial f}{\partial p_\perp} + \frac{k^j_z}{\omega_j} \frac{\partial f}{\partial p_z}\right\} dp_\perp, \tag{7}$$

where

$$A(p_z) = \sum_j \int d\Omega P_j\, (2\pi)^3 \frac{(k^j)^2\, (\partial k/\partial\omega)\, \varepsilon^j_\mathbf{k}}{|1 - (\partial k^j_z/\partial\omega)\, (P_z/m_0)|} \left(\frac{\omega_H}{\omega_j}\right)^2,$$

$$B(p_z) = \sum_j \int d\Omega P_j\, (2\pi)^3 \frac{(k^j)^2\, (\partial k^j/\partial\omega)\, \varepsilon^j_\mathbf{k}}{|1 - (\partial k^j_z/\partial\omega)\, (p_z/m_0)|} \frac{\omega_H}{\omega_j} \frac{k^j_z}{\omega_j},$$

$$D(p_z) = \sum_j \int d\Omega P_j\, (2\pi)^3 \frac{(k^j)^2\, (\partial k^j/\partial\omega)\, \varepsilon^j_\mathbf{k}}{|1 - (\partial k^j_z/\partial\omega)\, (p_z/m_0)|} \cdot \left(\frac{k^j_z}{\omega_j}\right)^2,$$

$\Gamma^j = (2\pi)^4 P_j m_0 / |k^j_z|$ and Ω is the solid angle.

Let us consider the one-dimensional (in θ) packet of the j_0-th normal wave propagating at the angle θ_0. We introduce the new variables

$$q = p_\perp^2 \frac{1}{2m_0} - \int_{p_z^0}^{p_z} \frac{B}{D} dp_z, \qquad p = p_z - p_z^0, \tag{8}$$

where p_z^0 is the lower boundary of the packet-plasma interaction region. In this coordinate system for a one-dimensional packet (A/B = B/D) we obtain

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial p} \left\{p_\perp^2 D \frac{\partial f}{\partial p}\right\}, \tag{9}$$

from which it follows that a one-dimensional packet endeavors to level out the distribution function along the line q = const (to establish a plateau), i.e., $f \to f^0 = f^0(q)$.

Equation (9) also permits an evaluation of the time t_0 required for a change of the distribution function. If L is the width of the interaction region with respect to p_z, we have

$$t_0^{-1} \approx L^{-2} \langle p_\perp^2 D \rangle \approx m_0 \varkappa T_0 L^{-2} D, \tag{10}$$

where T_0 is the plasma temperature and κ is Boltzmann's constant. Then the inequality (1), subject to the condition

$$(\partial\omega/\partial k)^2 \sim \omega^2/k^2 \gg \varkappa T_0/m_0,$$

acquires the form

$$\Delta\omega \gg \sqrt{eE_0 k/m_0}\, (\varkappa T_0/m_0 c^2)^{1/4}, \tag{11}$$

where E_0 is the effective amplitude of the electric field in the packet (the electromagnetic energy of the packet in unit volume is $E_0^2/8\pi$).

We now obtain the electromagnetic wave increments in the state with a plateau when $f = f_0(q)$:

$$\frac{\partial \varepsilon^j_\mathbf{k}}{\partial t} \frac{1}{\varepsilon^j_\mathbf{k}} = \Gamma^j \int_0^\infty p_\perp^3 \frac{\partial f^0}{\partial q} dp_\perp \left[\frac{\omega_H}{\omega_j} \frac{m_0}{p_\perp} \frac{\partial q}{\partial p_\perp} + \frac{n^j_z}{c} \frac{\partial q}{\partial p_z}\right]$$

$$= \Gamma^j \frac{\omega_H}{\omega_j} \left[1 - \frac{k^j_z}{k^{j_0}_z}\right] \int_0^\infty p_\perp^3 \frac{\partial f^0}{\partial q} dp_\perp$$

$$= -2m_0^2 \Gamma^j \frac{\omega_H}{\omega_j} \left[1 - \frac{k^j_z}{k^{j_0}_z}\right] \int_{q_0}^\infty f^0(q)\, dq;$$

$$n^j_z = \frac{k^j_z c}{\omega_j}, \qquad q_0 = -\int_{p_z^0}^{p_z} \frac{B}{D} dp_z. \tag{12}$$

When $f^0(q) > 0$ the waves for which

$$k^j_z(p_z, \theta) > k^{j_0}_z(p_z, \theta_0) \tag{13}$$

are unstable on the plateau formed by the j-th wave.

The steps of calculation in (12) are valid if the particles of the plateau formed by the j-th wave are "resonant" for the j-th wave. This has been taken into account in (13) by the function $k_z^j = k_z^j(p_z)$, which is obtained from the Doppler equation (3) and the function $k_z^j = k_z^j(\omega_j)$.

3. The instability criterion (13) has a simple quantum-mechanical interpretation. When a photon having the frequency ω_j and wave vector k^j is emitted a change of electron (or ion) state occurs:[11,13]

$$\Delta p_\perp = -\frac{\hbar s m_0 \omega_H}{p_\perp}, \qquad \Delta p_z = -\hbar k_z^j \tag{14}$$

where s is the index of the harmonic. Therefore a strong one-dimensional packet levels out the distribution function (in the interaction region) along the directions

$$(dp_\perp/dp_z)_j = \Delta p_\perp/\Delta p_z = s m_0 \omega_H / p_\perp k_z^j. \tag{15}$$

The particle distribution perpendicular to the lines represented by (15) falls off with increasing distance from the coordinate origin (if the original distribution function was Maxwellian).

From the quantum-mechanical point of view instability occurs when the upper energy levels involved in photon emission have greater populations than the lower levels. This means that for instability of the j-th wave on the plateau formed by the j-th packet a certain relation must be satisfied between the derivatives in (15). Thus, if k_z^{j0}, $k_z^j > 0$, a radiative transition to a state with smaller population (i.e., instability) occurs when

$$(dp_\perp/dp_z)_j < (dp_\perp/dp_z)_{j_0} \quad \text{or} \quad s_j/k_z^j < s_{j_0}/k_z^{j_0}. \tag{16}$$

This condition must be satisfied for the same particles, i.e., (neglecting the transverse Doppler effect) for the same p_z.

Analogous conditions can be given for an arbitrary direction of packet motion. It appears in the general case that only a plateau produced by the Cerenkov effect is always stable with respect to the normal Doppler effect. All other combinations of radiation mechanisms can result in instability.

4. We shall now consider the development of instability on the plateau for the case represented by (7). If a plasma contains packets with different values of k_z the quasi-linear equation for f in the coordinate system (8) is

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial t} + \frac{\partial}{\partial q}\left\{p_\perp^2 F \frac{\partial f}{\partial q}\right\} + \frac{\partial}{\partial p}\left\{p_\perp^2 D \frac{\partial f}{\partial p}\right\},$$
$$F = (AD - B^2)/D. \tag{17}$$

We shall clarify the pattern of instability development by studying (17) in an idealized case where:

1) The coefficients A, B, and D (and therefore the wave energy ϵ_k^j) are independent of p_z. In order to have zero particle flux through the boundaries of the interaction region we must require fulfillment of the boundary condition $\partial f/\partial p = 0$.

2) The term $(\partial f/\partial q)(\partial q/\partial t)$ can be neglected in (17). This is justified if

$$\partial q/\partial t \ll m_0 F. \tag{18}$$

This condition is clearly satisfied for small increments, small L, and large packet energies.

3) The condition of packet "narrowness" is fulfilled:

$$\varkappa T_0 \gg L\omega_H/k_z^j, \tag{19}$$

which permits the assumption $p_\perp^2 = 2m_0 q$ with zero as the lower limit of q.

Subject to the foregoing limitations, Eq. (17) becomes

$$\frac{1}{2m_0}\frac{\partial f}{\partial t} = \frac{\partial}{\partial q}\left\{qF\frac{\partial f}{\partial q}\right\} + qD\frac{\partial^2 f}{\partial p^2}, \tag{20}$$

from which it follows that if in a moving system of coordinates (q, p) f is independent of p at the initial time, then subsequently we will have $f = f^0(q, t)$. This function satisfies the equation

$$\frac{1}{2m_0}\frac{\partial f}{\partial t} = \frac{\partial}{\partial q}\left\{qF\frac{\partial f^0}{\partial q}\right\}. \tag{21}$$

On the other hand, it can be shown that for arbitrary initial conditions the solution of (20) approaches f^0. The time constant is $\tau_0 = L^2/m_0\kappa T_0 D$.

We shall assume that for t = 0 we have $f = f^0(q)$, i.e., we shall consider times $t > \tau_0$. Then $f = f^0(q, t)$, obeying Eq. (21), while the energies of the original and growing packets (ϵ_k^0 and ϵ_k^r, respectively) are described [see Eq. (12)] by

$$\frac{\partial \varepsilon_k^0}{\partial t} = 2m_0\Gamma^0 \int_0^\infty q\frac{\partial f^0}{\partial q}dq\left(\frac{\omega_H}{\omega_0} - \frac{k_z^0}{\omega_0}\frac{B}{D}\right)\varepsilon_k^0,$$
$$\frac{\partial \varepsilon_k^r}{\partial t} k = 2m_0\Gamma^r \int_0^\infty q\frac{\partial f^0}{\partial q}dq\left(\frac{\omega_H}{\omega_r} - \frac{k_z^r}{\omega_r}\frac{B}{D}\right)\varepsilon_k^r. \tag{22}$$

Since f^0 satisfies (21), we have

$$\int_0^\infty q\frac{\partial f^0}{\partial q}dq = -\int_0^\infty f^0 dq = \text{const}$$
$$= -N_0(\varkappa T_0)^{-1/2}\exp\{-\langle p_z\rangle^2/2m_0\varkappa T_0\} = -N$$

where N_0 is the number of particles per cm^3 and $\langle p_z\rangle$ is the mean longitudinal momentum in the interaction region.

We shall assume that growing waves also have a one-dimensional spectrum (varying in the angle θ) in accordance with the appropriate initial conditions, for example. When instability develops on the plateau due to the thermal noise level the assumption of a one-dimensional ϵ_k^r packet is sometimes justified. On the other hand, if for growing waves k_z^r depends weakly on θ, which occurs, for example, in the Alfvén branch within the region of ionic cyclotron resonance, [14,15] then assuming that the dependence of ϵ_k^r on θ during the development of instability varies very little, we can write an equation for the mean (with respect to θ) energy of growing waves. Proceeding now to the total packet energies E_0 and E_r and the plasma energy E_p per cm^3, Eq. (22) and the energy conservation law following directly from (5) yield

$$E_0 + E_r + E_p = \text{const},$$

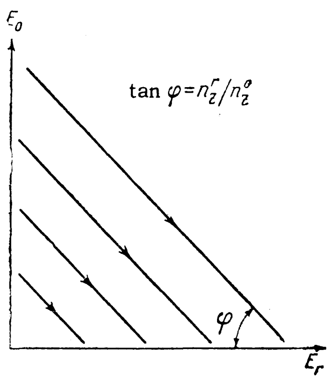
$$\frac{dE_0}{dt} = -(2\pi)^4 P_0 P_r 2m_0^3 \omega_H N / \frac{1}{|k_z^0| \omega_r^2 \omega_0} \frac{(k_z^r - k_z^0) E_0 E_r}{\omega_0^{-2} P_0 E_0 k_z^0 + \omega_r^{-2} P_r E_r k_z^r}$$

$$\frac{dE_r}{dt} = (2\pi)^4 \frac{P_0 P_r 2m_0^3 \omega_H N}{|k_z^r| \omega_0^2 \omega_r} \frac{(k_z^r - k_z^0) E_0 E_r}{P_0 E_0 k_z^0 \omega_0^{-2} + P_r E_r k_z^r \omega_r^{-2}}. \tag{23}$$

In the cited case of non-one-dimensional growing waves, P_r must be understood as the mean value with respect to θ.

The trajectories on the (E_0, E_r) "phase" plane are represented in the figure. The energy transferred to growing waves is

$$E_r(t = \infty) = E_0(t = 0)\, n_z^0/n_z^r. \tag{24}$$



The energy transferred to the plasma is

$$\Delta E_p = (1 - n_z^0/n_z^r)\, E_0(t = 0). \tag{25}$$

The solutions of (23) are easily obtained. Thus the time of energy transfer t_{trans} for which $E_0(t_{\text{trans}}) = E_r(t = 0)$ subject to the condition $E_0(t = 0)/E_r(t = 0) \gg n_z^r/n_z^0$ is

$$t_{\text{trans}} = \frac{1}{\gamma_0}\left(1 + \frac{P_0 \omega_r^2}{P_r \omega_0^2}\right) \ln \frac{E_0(t = 0)}{E_r(t = 0)}, \tag{26}$$

where γ_0, the damping increment of the j_0-th wave on the plateau formed by the j-th waves, equals in order of magnitude the linear damping increment of the j-th wave within a plasma in equilibrium.

Equation (23) can be derived from simple considerations. For saturation we have the following relationship of the increments:

$$\gamma_j = \gamma_{0j}(1 + \tau_{0j}/\tau_j)^{-1}, \tag{27}$$

where τ_j, the time required to equate the level populations determining the increment, is inversely proportional to the wave energy, and τ_{0j} is the relaxation time determining γ_{0j}. In our case the relaxation processes are waves of other types; therefore both τ_j and τ_{0j} are determined by (9), and γ_{0j} is the increment corresponding to the plateau established by the j_0-th wave (for γ_{0j_r}) or by the j_r-th wave (for γ_{0j_0}). For narrow packets we obtain (23) using (9) and (27).

5. We shall now estimate the nonlinear damping decrement of a stable non-one-dimensional packet. According to (9) and (12) a stable one-dimensional packet in a collisionless plasma following the establishment of a plateau is not damped, [6,9] while a non-one-dimensional packet is attenuated, as a general rule. We shall assume fulfillment of the conditions (18) and (19) and that the distribution function in the interaction region is represented by (21). Then the time dependence of wave energy (for $t > \tau_0$) is given by (25), where the parameters B and D are determined by a wave of only one type, the summation over j being omitted in (7).

In order to evaluate the attenuation of a slightly non-one-dimensional electromagnetic wave packet we shall assume that only k_z depends on θ, and that the width $\Delta\mu$ of the packet with respect to $\mu = \cos\theta$ is considerably smaller than the region in which k_z changes significantly. With a series expansion of k_z in powers of μ and assuming a rectangular packet shape (with respect to μ), we obtain for the energy $E = \int \epsilon\, d\Omega$

$$\frac{dE}{dt} = -(2\pi)^4 \frac{2Pm_0^3 \omega_H N}{\omega |k_z|^3} \left(\frac{\partial k_z}{\partial \mu}\right)^2 \frac{(\Delta\mu)^2}{12} E, \tag{28}$$

which becomes invalid when, as a result of energy transfer, the packet is considerably narrowed with respect to θ and its shape changes appreciably.

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