ON THE THEORY OF THE PULSATIONS IN THE OUTPUT OF THE RUBY LASER

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A theory connected with nonlinear effects in the interaction of the field with the medium is developed for the pulsations in the output power of lasers. Formulas are derived for the amplitudes and frequencies of the spikes as functions of the parameters characterizing the laser. The mechanism of the spiking is the transfer of particles from the upper level to the lower one with the radiation of energy and in the opposite direction with absorption of energy. The theory is applied to the ruby laser and is compared with experiment.

ONE of the problems in the operation of optical masers is the nature of the pulsations (or spiking) in their power output. At present a large number of materials (around thirty) have been used in lasers. Of these only the gas lasers (for example, He-Ne) and several solid state lasers (for example, CaF_2 doped with Sm^{2+}) do not produce pulsations; the majority of solid-state lasers, however, have a pulsating output.

There are many theoretical papers [1,2] 1) attempting to explain this behavior, but progress has been very modest so far.

The equations which are written down and solved in these papers do not contain stable limit cycles; they contain only singular points of the stable focus type. This means that all pulsations occurring in the system die out sooner or later. From this it has been concluded that the pulsations are related exclusively to the build-up processes of laser action and hence that the pulsations should disappear in lasers operating continuously. However, undamped spiking is observed even in c.w. lasers [4]. Singer and Wong [5] were evidently the first to point out the importance of nonlinear effects in the interaction of the field with the medium for the explanation of spiking. Following their ideas, one of the present authors [3] found spiking behavior in a system described by a single relaxation time τ_1.

In the present paper the pulsations are treated on the basis of more general equations applicable to solid state lasers, and the results are compared with experiments made on a ruby laser.

We consider a two-level system ($E_2 > E_1$). There is a constant flux of active particles into the upper level. The steady state process is described by equations of the form

$$\ddot{\mathscr{E}} + \omega_0\dot{\mathscr{E}}/2Q + \omega_0^2\mathscr{E} = -4\pi\ddot{P},$$
$$\ddot{P} + 2\gamma_2\dot{P} + (\omega_0^2 + \gamma_2^2 + \Delta)P = -2d^2\omega_0 N\mathscr{E}/h,$$
$$\dot{N} + \gamma_1(N - N_0) = 2\mathscr{E}(\dot{P} + \gamma_2 P)/h\omega_0. \qquad (1)$$

Here $\mathscr{E}$ is the electric field strength2) in the cavity, P is the average value of the polarization of the active particles, N is the number of particles in the upper level, N_0 is the initial number of these particles, ω_0 is the transition frequency—assumed for simplicity equal to the cavity frequency, Q is the quality factor of the cavity, and $\gamma_2 = 1/\tau_2$, $\gamma_1 = 1/\tau_1$ where τ_2 is the transverse relaxation time and τ_1 is the interaction time of the particles with the high frequency field. (For ruby $1/\tau_1 = 1/\tau_{sp} + 1/\tau_p$, where "sp" in the subscript indicates spontaneous and "p" indicates pump); for a four-level system such as U^{3+} doped CaF_2, the magnitude of τ_1 is the time of the nonradiative transition from the lower of the two active levels to the "pumping level" (ground state); d is the modulus of the dipole moment matrix element, and Δ is the frequency shift between the polarization of the active particles and the electric field. The meaning of this last quantity for ruby will be discussed later. Equations of this type involving τ_1 have been widely used in discussing the operation of microwave masers [6,7]. Equations involving τ_1 and τ_2 are also known in the literature.

We seek a solution of the form

$$\mathscr{E} = E(t)\cos[\omega_0 t + \varphi(t)], \quad P = P_1(t)\cos[\omega_0 t + \psi(t)],$$

where E, P_1, φ, and ψ are slowly varying func-

1)Cf.[3] for an earlier bibliography.

2)Spatial effects are not taken into account in the present treatment and the electric field is considered to be nearly harmonic.

tions of time. Application of the Van der Pol method to the system (1) leads to the following set of equations:

$$\begin{aligned} dx/dt &= -\omega_0 x/2Q + (\omega_0/2Q)\, y \sin_1\Phi, \\ dy/dt &= -\gamma_2 y + \gamma_2\, xz \sin_1\Phi, \\ dz/dt &= -\gamma_1 (z - z_0) - \gamma_1 (z_0 - 1)\, xy \sin_1\Phi, \\ d\Phi/dt &= -B + (\gamma_2\, xz/y + \omega_0 y/2Qx) \cos_1\Phi. \end{aligned} \quad (2)$$

The following dimensionless quantities have been introduced

$$x = E/E_0;\ y = P_1/P_0;\ z = N/N_0';\ \sin_1\Phi = \sin\Phi/\sin\Phi_0;\ \cos_1\Phi = \cos\Phi/\sin\Phi_0$$

along with the notations $B = (\Delta + \gamma_2^2)/2\omega_0$, $\Phi = \psi - \varphi$. E_0, P_0, N_0' and Φ_0 are the solutions of (1) for the case of stationary harmonic behavior.

For simplicity further investigation will be made for the case of zero frequency mismatch ($B = 0$).

Equations (1) admit the case of stationary harmonic behavior:

$$E_0 = \sqrt{4\pi Q h N_0 \gamma_1},\quad P_0 = \sqrt{N_0 h \gamma_1/4\pi Q},$$
$$N_0' = h\gamma_2/4\pi Q d^2,\quad \Phi_0 = \pi/2. \quad (3)$$

When the condition

$$z_0 > [(\omega_0/2Q)^2 + \omega_0\gamma_1/2Q + 3\omega_0\gamma_2/2Q]\,/\gamma_2(\omega_0/2Q - \gamma_1 - \gamma_2) \quad (4)$$

is fulfilled[3)] the harmonic mode will be unstable and pulsations will arise. We now investigate further the nature of the pulsating mode.

It is convenient to introduce a "dimensionless time"

$$\tau = \omega_1 t \quad (4a)$$

(ω_1 will be defined below). Equations (2) can then be written in the form

$$\begin{aligned} \mu\dot{x} &= -x + y \sin_1\Phi, \\ \mu\,[\dot{\Phi} - (\mu_2 xz/\mu y)\cos_1\Phi] &= yx^{-1}\cos_1\Phi, \\ \dot{y} &= -\gamma_2 y/\omega_1 + (\gamma_2/\omega_1)\, xz \sin_1\Phi, \\ \dot{z} &= -\mu_1 z + \mu_1 z_0 - \mu_1 (z_0 - 1)\, xy \sin_1\Phi. \end{aligned} \quad (5)$$

We have introduced here the small parameters

$$\mu = \omega_1 2Q/\omega_0 \ll 1,\quad \mu_1 = \gamma_1/\omega_1 \ll 1,\quad \mu_2 = 2\gamma_2 Q/\omega_0 \ll 1. \quad (6)$$

The last of these conditions is connected with the fact that when $\gamma_2 > \omega_0/2Q$ condition (4) will never be satisfied, i.e., the harmonic mode will always be stable and there will be no spiking.

The system (5) can be transformed to a nonlinear equation of the third order:

$$\ddot{x} - \frac{\dot{x}^2}{x} + \frac{\gamma_1\gamma_2(z_0 - 1)}{\omega_1^2(1 + \mu_2)}\, x(x^2 - 1)$$
$$= -\mu\Big[(1 + \mu_2)^{-1}\Big(\dddot{x} - \frac{\dot{x}\ddot{x}}{x}\Big) + \mu_1(1 + \mu_2)\,\ddot{x} + \frac{\mu_1}{\mu}\dot{x} + \frac{\gamma_1\gamma_2(z_0 - 1)}{\omega_1^2(1 + \mu_2)}\, x^2\dot{x}\Big]; \quad (7)$$

it is convenient to take

$$\omega_1^2 = \frac{\gamma_1\gamma_2(z_0 - 1)}{1 + \mu_2} = \frac{d^2E_0^2}{h^2(1 + \mu_2)}.$$

Here we have used Eq. (3). Then, accurate to first order in μ, (7) takes the form

$$\ddot{x} - \dot{x}^2/x + x(x^2 - 1) = -\mu\,[\dddot{x} - \dot{x}\ddot{x}/x + \mu_1\dot{x}/\mu + \dot{x}x^2]. \quad (8)$$

This is a nonlinear equation with a small nonlinearity in the right-hand member. It can be solved by standard methods [8]. For $\mu = 0$ the equation can be integrated:

$$\dot{x} = \pm x\sqrt{C + 2\ln x - x^2}. \quad (9)$$

Equation (9) defines a family of cycles in the Van der Pol plane, depending on the quantity C (cf. [3]). The motion in each cycle is stable.

By expanding ln (x) around 1 it is simple to obtain the amplitude of the field as a function of time (the argument of the sine varies from $-\pi/2$ to $+\pi/2$):

$$E = E_0(2 - C_1) \times [2 \mp \sqrt{2C_1}\sin\{dE_0 h^{-1}\sqrt{2 - C_1}(t + h_\pm)\}]^{-1} \quad (10)$$

($h_\pm$ is an arbitrary constant) and the period

$$T = 2\pi h/dE_0\sqrt{2 - C_1}, \quad (11)$$

where $C_1 = C - 1$, and where E_0 is determined by (3). We have reverted to the time t in (10) and (11) by using (4a); ω_1 is taken correct to first order in μ.

Evaluation of the constant C_1 is elementary but somewhat time-consuming. One obtains the expression

$$C_1 = 2\Big[1 - \gamma_1\frac{\omega_0}{2Q}\Big(\frac{h}{dE_0}\Big)^2\Big]\Big[3 - \frac{1}{4}\gamma_1\frac{\omega_0}{2Q}\Big(\frac{h}{dE_0}\Big)^2\Big]^{-1}. \quad (12)$$

The calculation of C_1 is also carried out to first order in μ. A zero value for C_1 indicates that the cycle degenerates into a point and the behavior becomes stationary and harmonic. In this case the pulses disappear and hence condition (4) need not be satisfied. In fact if the inequality in condition (4) is replaced by an equality, and Eqs. (3)

3)Investigation of the stable system (1) with zero frequency detuning was first carried out by A. N. Oraevskii. It was he who obtained this condition.

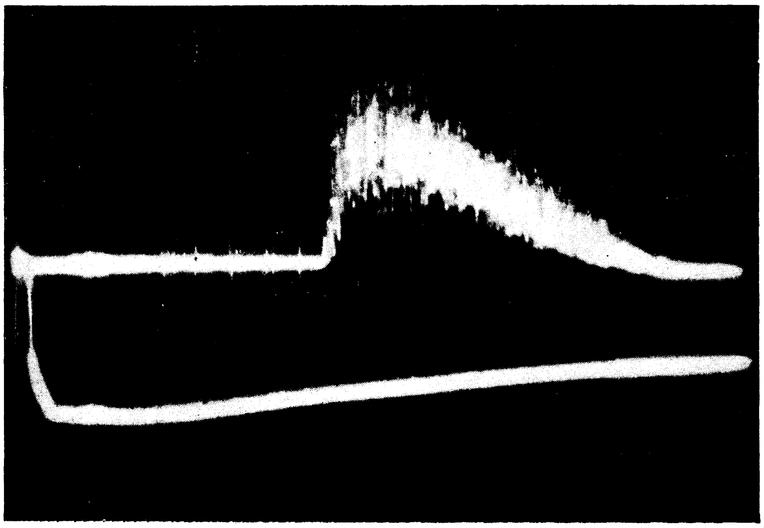

FIG. 1. An oscillogram of the output of the crystal (top trace). The modulation, which decreases during the pulse, is clearly visible. The bottom trace shows the pulse of the flash lamp.

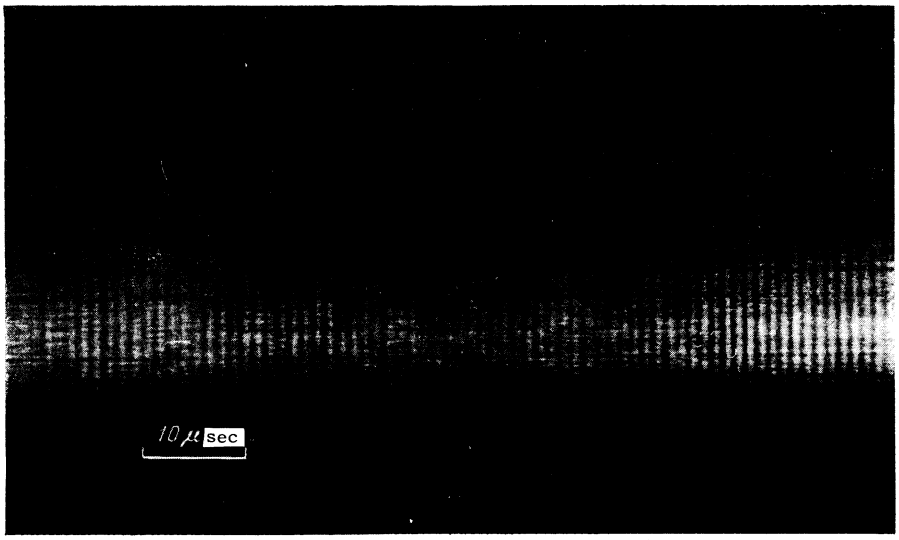

FIG. 2. Time sweep of the output of the ruby laser. Each bright band is a spike.

and (6) are taken into account, then, to first order,

$$1 = \gamma_1 (dE_0/h)^{-2}\, \omega_0/2Q,$$

which clearly coincides with the condition that C_1 be zero.

A few remarks may be made about the solution we have found. As long as condition (4) is not satisfied there will be no cycles and the solution will be $E = E_0$ (3). When condition (4) is satisfied pulsations will occur on top of a constant average field of amplitude E_0. As long as the amplitude of these pulsations is small they will be approximately sinusoidal, although harmonics will be present. The frequency of the pulsations (11) is determined by the average amplitude of the field and increases with increasing amplitude. The mechanism of this process consists in the transfer of particles from the upper level to the lower with release of energy (the rising portion of a spike) and the transfer of particles under the influence of the field from the lower level to the upper with the absorption of energy (the falling portion of a spike).

The frequency of the spikes is of order dE_0/h. This is precisely the frequency at which particles are transferred from level to level. The pulses occur under conditions of strong saturation, wherein each active particle goes several times from the upper to the lower level and back during the time of interaction with the radiation field. Because of the population inversion, transitions from the upper level to the lower always predominate on the average. This of course is true for the pulsating mode as well (the amplitude of the field never falls to zero).

The appearance of the pulsations means that the Q of the cavity has become "too high" and the cavity can not get rid of part of the energy in the field. This part of the field is absorbed by the particles and then re-radiated. This "repumping" is the basis of the spiking behavior.

COMPARISON WITH EXPERIMENT

Application of this general theory to a particular laser requires certain clarifications. We are interested in a laser operating on the $^2E \rightarrow {}^4A_2$ transition in ruby (the R_1 transition). Although ruby is a three-level system (the green band being the third level), for the existing pumping levels it may be adequately described by two levels with a constant influx of particles. This is due to the fact that the third level is practically empty (the lifetime of the nonradiative transition from it is of the order of 10^{-8} seconds), and (for constant pumping) the modulation of the constant flow of particles to the ^2E level due to the pulsations does not exceed several percent. A second question involves the width of the ^2E level. Evidently it is clear [9] that this width is not due to relaxation but rather to an adiabatic shift of the position of the ^2E level as a result of the lattice vibrations. This causes a decrease in the interaction of the ion with the laser field and can be taken into account by introducing in (1) a quantity Δ, which is the effective frequency shift between the electric field and the polarization. This problem still requires an exact solution, but we consider a very simple model of the line: the broadening of the upper level is taken into account by introducing the Hamiltonian

$$\hat{H} = \hat{V} \cos \omega_2 t, \tag{13}$$

where the only non-zero matrix element of $\hat{V}$ is $\langle {}^2E \mid \hat{V} \mid {}^2E \rangle$ (cf. [9]), and ω_2 satisfies the relation: ω of the pulsations $\ll \omega_2 \ll \omega_0$. Calculation shows that Eqs. (1) with the term Δ and the equation with the Hamiltonian (13) give the same simplified equations.

An attempt to verify Eqs. (10), (11), and (12) via Q, τ_1, τ_2 and the other parameters is apparently doomed to failure. The question of the magnitude of τ_2 remains open in general. The other parameters, for example τ_1 (determined by the pump), although clear theoretically are difficult to obtain experimentally. Hence we have attempted to write down verifiable formulas in terms of observable quantities rather than in terms of the above parameters.

If we take account of Δ in (1) [which is equivalent to (2) with $B \neq 0$] then we easily obtain in place of (10) and (11)

$$E = E_0 \frac{2 - C_1}{2 \mp \sqrt{2C_1} \sin \{dE_0 h^{-1} \sqrt{2 - C_1}(t + h_\pm) \sin \Phi_0\}}, \tag{10a}$$

$$T = 2\pi h / dE_0 \sqrt{2 - C_1} \sin \Phi_0, \tag{11a}$$

where $\sin \Phi_0 = [1 + (2BQ/\omega_0)^2]^{-1/2}$ is a function only of temperature.

The quantities E^2 and T are observed in experiment. It is clear from (10a) that

$$E_{max}^2 = E_0^2 \left(\frac{2 - C_1}{2 - \sqrt{2C_1}} \right)^2; \quad E_{min}^2 = E_0^2 \left(\frac{2 - C_1}{2 + \sqrt{2C_1}} \right)^2. \tag{14}$$

Knowing E_{max}^2 and E_{min}^2 we can determine E_0 and C_1 and verify (11a). For a given temperature (11a) has the form

$$T \sim 1/E_0 \sqrt{2 - C_1} \tag{15}$$

(the coefficient of proportionality depends also on the number of modes excited).

The test of (15) was carried out under pulsed conditions. It is clear that if the pulse of pumping light is sufficiently long and all quantities vary slowly, one may make use of (10a) and (11a) (the amplitudes of approximately 10^3 spikes vary by a factor of three).

The experimental set-up was similar to the one used in [10]. The laser output was recorded with an FÉU photomultiplier and a high speed SFR camera. The angular distribution of the emission from the crystal was obtained in the focal plane of the SFR objective. Using a small slit 0.1 mm wide, a narrow band was isolated from the angular

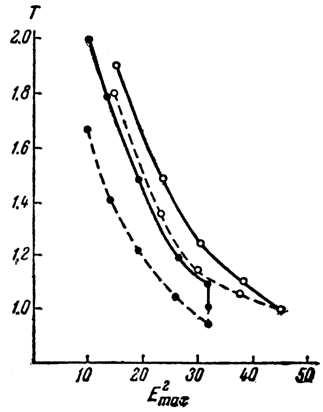

FIG. 3. The dependence of the period T on E_{max}^2 for two different crystals with the laser operated at low temperature (108°K) and close to threshold. (The voltage on the flash lamp was 2.6 kV, the threshold voltage 2.5 kV.) The experimental dependence is shown by the solid lines, and the theoretical dependence by the dashed lines (the units are arbitrary).

distribution and its image projected by a second lens and a rotating mirror onto the film. A typical oscillogram of the radiation from the crystal—obtained on an OK-17M oscilloscope—is shown in Fig. 1 (top trace). The pumping lamp pulse as recorded by a photocell is shown below in Fig. 1. Figure 2 shows the time scan of the output of the ruby laser; a marked periodicity is seen. The mirror rotated at 7500 rpm; the period of the spikes is about 1 μsec. The vertical spread characterizes the angular distribution.

The values of E^2_{max} and E^2_{min} during the pulse were determined from the oscillograms. The "theoretical" relationship (15) (as a function of the inverted population) was then constructed. The experimental value of T was obtained by counting the number of spikes on the film in the SFR.

Measurements were made for various temperatures (103—125°K) and for various voltages on the flash lamp. When the laser is operated near threshold, (15) gives good agreement with experiment (see Fig. 3 for typical behavior). For operation far above threshold, (15) gives a more rapid variation than experiment. This is connected with the fact that well above threshold the spectral distribution changes markedly during the pulse, and this is not taken into account in our equations. (The angular dependence varies weakly, about 10—25%, under the conditions of our experiment.) If one takes into account the fact that when n modes are excited one determines from the oscillograms quantities proportional to nE^2_{max} and nE^2_{min}, Eq. (15) will have the form

$$T \sim \sqrt{n/n_0}\,/\,E_0\sqrt{2-C_1},$$

where n is the number of modes at a given point in the output pulse (with maximum value n_0). The value of n decreases with decreasing E^2, which produces a more gradual dependence for T.

These observations are supported by experiments in which simultaneous measurements were made of the variation in the angular and spectral composition of the beam (using an interferometer).

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170