MOTION OF A ONE-DIMENSIONAL NONLINEAR OSCILLATOR UNDER ADIABATIC CONDITIONS

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The motion of a one-dimensional oscillator whose Hamiltonian depends on a slowly varying parameter is analyzed by the methods of classical mechanics. It is shown that throughout the process the change of the adiabatic invariant is exponentially small. A method is proposed which permits the derivation in many important cases of a closed expression for the pre-exponential factor.

1. INTRODUCTION

A mechanical system is characterized by its integrals of motion, that is, quantities which remain constant as the system moves. If the fields acting on the system vary slowly in time and in space, then the quantities that are conserved in constant and homogeneous fields are no longer integrals of the motion and vary slowly with time; over sufficiently long times, the change may turn out to be quite appreciable. However, some of these quantities (adiabatic invariants) have the property that they are approximately conserved during the entire time of motion in a quasistatic and quasihomogeneous field. The question of which quantities are adiabatic invariants and to what accuracy they are conserved has been the subject of many recently published papers [1-6] [1)].

In this article we consider systems with one degree of freedom, having the following properties: the Hamiltonian of the system depends on an external parameter λ, which is a specified function of the time; when the parameter is constant the motion in the system is periodic; the function $\lambda = \lambda(\gamma t)$ changes little during a time of the order of T_0, the characteristic period of the system. We assume that the latter condition is satisfied if $\gamma \ll 1$, that is, γ has the meaning of a parameter that characterizes the slowness of the process.

As is well known, an adiabatic invariant for such a system is the action integral (I), calculated over a closed curve in phase space, the curve determining the periodic motion for a constant value of λ. We shall henceforth be interested in the quantity ΔI, the change in the adiabatic invariant over the entire time of the process, during which $\lambda(\gamma t)$ changes from a value λ_- as $t \to -\infty$ to a value λ_+ as $t \to +\infty$. For an arbitrary nonlinear oscillator ΔI is determined by

$$\Delta I = A \exp(B/\gamma), \qquad \operatorname{Re} B < 0 \tag{1.1}$$

(B is independent of γ), that is, ΔI is exponentially small in the parameter γ. This statement can best be proved by investigating the question of the adiabatic invariant from the point of view of quantum mechanics (see [1,2]) and then allowing the constant $\hbar$ to go to zero. Such an approach makes it possible to determine the form of the constant B for an arbitrary system of the type under consideration. However, the calculation of the factor before the exponent is apparently best carried out within the framework of classical mechanics. In the present article we consider a general method which permits us to present in many important cases a simple classification of the Hamiltonians of systems by their analytic properties and to determine the pre-exponential factor in closed form; the exponential smallness of ΔI will be proved during the course of the exposition by the methods of classical mechanics.

2. GENERAL SOLUTION OF THE PROBLEM

For a one-dimensional periodic system with Hamiltonian $H(q, p, \lambda)$ (q and p are the canonically conjugate coordinate and the momentum), the action variable I is defined by the integral $I(E, \lambda) = (2\pi)^{-1} \oint p\,dq'$, where the integration is over the trajectory $H(q, p, \lambda) = E$ (E — energy of the system).

The analysis that follows is best carried out using canonically-conjugate variables: the action

[1)]We note that the method used by Vandervoort [4] is in error.

I and the angle w, which are related to q and p with the aid of the generating function S_0:

$$S_0 = S_0(q, I, \lambda) = \int_{\bar{q}}^{q} p\,dq', \quad p = p(q, I, \lambda), \quad \bar{q} = \bar{q}(I, \lambda). \tag{2.1}$$

The function $p(q, I, \lambda)$ is obtained here from the equation

$$H(q, p, \lambda) = E(I, \lambda).$$

The Hamiltonian that determines the variation of I and w with time is of the form (see, for example, [7])

$$\mathcal{H}'(I, w, \lambda) = E(I, \lambda) + \gamma\lambda'\Lambda(w, I, \lambda), \tag{2.2}$$

where $\Lambda = (\partial S_0/\partial\lambda)_{q,I}$ is a periodic function in w with period 2π, and the prime denotes differentiation with respect to γt.

For further analysis we need the explicit form of the function Λ, written in terms of the variables q, I, and λ. From the definition of Λ it follows that

$$\Lambda = \int_{\bar{q}}^{q} \frac{H_\lambda - \bar{H}_\lambda}{H_p}\,dq', \tag{2.3}$$

where

$$H_\lambda = (\partial H/\partial\lambda)_{q,p}, \quad H_p = (\partial H/\partial p)_{q,\lambda}, \quad p = p(q, I, \lambda);$$

$\bar{H}$ —is the average of $H_\lambda(q, I, \lambda)$ taken over a period. We choose $\bar{q}(I, \lambda)$ in such a way as to make

$$\int_0^{2\pi} \Lambda(w, I, \lambda)\,dw = 0.$$

The functions $E(I, \lambda)$, $\Lambda(\omega, I, \lambda)$, and $\lambda(\gamma t)$ will be assumed analytic in all the arguments.

As $t \to \pm\infty$ we have $\lambda' \to 0$ and consequently $I \to I_\pm = \text{const}$; the asymptotic behavior of the function w(t) as $t \to \pm\infty$ can be represented in the form

$$w(t) \to \int_{\bar{t}}^{t} \Omega(I_\pm, \lambda(\gamma t'))\,dt' + w_\pm, \quad t \to \pm\infty, \tag{2.4}$$

where the lower limit $\bar{t}$ is chosen to be an arbitrary point on the real axis; w_- and w_+ are, respectively, the initial and final phases of the motion; $\Omega(I, \lambda) = (\partial E/\partial I)_\lambda$ —the frequency of the periodic motion for constant I and λ. The final value of the adiabatic invariant (I_+) depends not only on I_-, but also on the initial phase of the motion, so that our purpose is to determine the function $I_+(w_-, I_-)$, which is periodic in w_- with period 2π.

To determine $I_+(w_-, I_-)$ we introduce the functions $S_-(w, t, I_-)$ and $S_+(w, t, I_+)$, which are the generating functions of the canonical transformations from I_-, w_- and I_+, w_+, respectively, to the "running" values I and w; the $S_\pm(w, t, I_\pm)$ satisfy the Hamilton-Jacobi equation

$$E\left(\frac{\partial S_\pm}{\partial w}, t\right) + \gamma\lambda'\Lambda\left(w, \frac{\partial S_\pm}{\partial w}, t\right) + \frac{\partial S_\pm}{\partial t} = 0 \tag{2.5}$$

and as $t \to \pm\infty$ they are determined by the following asymptotic formulas

$$S_+ = I_+ w - \int_{\bar{t}}^{t} E(I_+, t')\,dt', \quad t \to +\infty,$$

$$S_- = I_- w - \int_{\bar{t}}^{t} E(I_-, t')\,dt', \quad t \to -\infty. \tag{2.6}$$

According to the well-known properties of canonical transformations, the $S_\pm$ are interrelated by

$$S_+(w, t, I_+) = S_-(w, t, I_-(w, t, I_+)) - F(I_-(w, t, I_+), I_+),$$
$$\partial S_-/\partial I_- - \partial F/\partial I_- = 0. \tag{2.7}$$

Here $F(I_+, I_-)$ is the generating function of the canonical transformation from $I_- w_-$ to $I_+ w_+$; $I_-(w, t, I_+)$ is determined in implicit form by the second equation of (2.7). Knowledge of the functions $S_\pm$ makes it possible to investigate the motion of the system completely, and in particular to determine the value of $I_+(w_-, I_-)$.

The method presented below for finding $I_+(w_-, I_-)$ is based on an investigation of the behavior of the functions $S_\pm(w, t, I_\pm)$, which are analytically continued in the complex t plane. Before we proceed to the exposition of the method, we indicate some essential properties[2)] of the Fourier coefficients (B_n) of the functions $I_+(w_-, I_-)$. The coefficients B_n are determined by the formula

$$B_0 = I_- \; B_n = B^*_{-n} = A_n(I_-)\exp\left\{in\int_{\bar{t}}^{t_0} \Omega(I_-, t')\,dt'\right\}, n \geqslant 1. \tag{2.8}$$

Here $t_0(I_-)$ is that singular point of the functions $\lambda(\gamma t)$, $T(I_-, t) = 2\pi/\Omega(I_-, t)$ and $\Lambda^{(n)}(I_-, t)$ ($\Lambda^{(n)}$ are the Fourier coefficients of the function $\Lambda(w, I_-, t)$), which lies closest to the real axis in the half plane

$$\mathrm{Im}\left\{\int_{\bar{t}}^{t_0} \Omega(I_-, t')\,dt'\right\} > 0;$$

the A_n depend on the parameter γ in such a way that the point $\gamma = 0$ is not an essential singularity. In this case $|\mathrm{Im}\, t_0| \sim T_0/\gamma$ and consequently the Fourier coefficients B_n are exponentially small quantities of the order of $\exp(-n\kappa/\gamma)$ ($\kappa \sim 1$).

2)The proof of these properties is given in the appendix.

The Hamilton-Jacobi equation contains a small parameter γ, and therefore for finite $|t| \lesssim T_0/\gamma$ and $\operatorname{Im} w \lesssim 1$ the functions $S_\pm$ can be represented in the form of an asymptotic series in powers of γ. In the zeroth approximation when $I = I_-$ we have

$$w(t) = \int_{\bar{t}}^{t} \Omega(I_-, t')\, dt' + w_-,$$

that is, the motion of the system is quasiperiodic (the coefficients of the next higher powers can be readily obtained with the aid of formulas A_1 and A_3 of the appendix). However, if we consider the motion of the system for real t and w_-, then the asymptotic methods will not make it possible to determine ΔI, which, in the case when $\operatorname{Im} w_- = 0$, is exponentially small in the parameter γ.

The situation changes radically for complex t and w_-, defined by the conditions

$$\operatorname{Im} w_- \approx \operatorname{Im} \int_{\bar{t}}^{t_0} \Omega(I_-, t')\, dt', \quad \operatorname{Im} w(t, w_-, I_-) \lesssim 1. \quad (2.9)$$

The formulas (2.8) show that in the case when $\operatorname{Im} w_-$ satisfy the first condition in (29), $I_+(w_-, I_-)$ is no longer exponentially small and can be determined with the aid of asymptotic methods. The criterion for the applicability of these methods is the inequality $|\gamma\lambda'\Lambda(w, I_-, t)| \ll I_-/T_0 \sim I_0/T_0$ (I_0 – characteristic value of the action variable), which is satisfied if the second relation of (2.9) holds.

From the zeroth-approximation equations it follows that the values of t satisfying conditions (2.9) lie on a line R_0 which is parallel to the real axis as $\operatorname{Re} t \to -\infty$ and pass at distances $\lesssim T_0$ from the singular point $t_0(I_1)$. In accordance with the definition of $t_0(I_-)$, the functions $\lambda(\gamma t)$, $T(I_-, t)$, which govern the quasiperiodic mode of motion, have a singularity at $t = t_0$. This means that in the section of the counter R_0, where $|t - t_0| \lesssim T_0$, the adiabaticity condition $|\partial T/\partial t| \ll 1$ is violated, and it is necessary to determine $I(t, w_-, I_-)$ and $w(t, w_-, I_-)$ exactly by expanding the function $\lambda = \lambda(\gamma t)$ in the Hamiltonian of the system near t_0. At the points of the contour R_0 where $|t - t_0| \gg T_0$, it is possible to use the asymptotic series for $S_\pm(w, t, I_\pm)$ to determine $I(t)$ and $w(t)$; the functions $S_-(w, t, I_-)$ and $S_+(w, t, I_+)$ can be represented in the form of an asymptotic series in powers of γ, for values of t lying on the left and right branches of R_0, respectively. By determining the motion in the vicinity of t_0, we can "join" the asymptotic solutions to the left and to the right of the point t_0 and find the function

$$I_+(w_-, I_-) \quad \text{for} \quad \operatorname{Im} w_- \approx \int_{\bar{t}}^{t} \Omega(I_-, t')\, dt'.$$

ΔI can be determined by continuing the function $I_+(w_-, I_-)$ analytically in the region $\operatorname{Im} w_- \lesssim 1$ (there are no singular points of $I_+(w_-, I_-)$ in this region) and to calculate its first Fourier coefficient.

3. CLASSIFICATION OF THE SINGULAR POINTS AND VIRTUAL SCATTERING

In this section we shall investigate a case when the singular points of the functions $T(I_-, \lambda)$ and $\Lambda^{(n)}(I_-, \lambda)$ ($\lambda = \lambda(\gamma t)$) do not coincide with the singularities of $\lambda'(\gamma t)$ and are closest to the real axis. We shall assume throughout the following arguments that the analytic function $H(q, p, \lambda)$ has no singularities for finite values of q, p, and λ. These assumptions are sufficient to indicate the classification for the singularities of T and $\Lambda^{(n)}$ and to find the form of the pre-exponential factor.

1. For arbitrary complex values of E and λ, the functions $T(E, \lambda)$ and $\Lambda^{(n)}(E, \lambda)$ are determined by the formulas

$$T(E, \lambda) = \oint \frac{dq}{H_p}, \quad \Lambda^{(n)}(E, \lambda) = \frac{1}{in} \oint \frac{H_\lambda}{H_p} \exp\left\{ in\Omega \int^{q} \frac{dq'}{H_p} \right\} dq;$$

$$H_\lambda \equiv H_\lambda(p(q, E, \lambda), q, \lambda), \quad H_p \equiv H(q, p(q, E, \lambda), \lambda). \quad (3.1)$$

The function $p(q, E, \lambda)$ is determined here from the equation $H(q, p, \lambda) = E$, and the integration is carried out in the q plane over the closed contour L enclosing the two turning points $q_{1,2}(E, \lambda)$, in which $H_p(q, E, \lambda) = 0$. (For real E and λ we have $\operatorname{Im} q_{1,2} = 0$.) The singularities of the integrand in (3.1) are the zeroes of the function $H_p(q, \lambda, E)$. In the general case considered here, that of a nonlinear oscillator, we can have, in addition to $q_1(E, \lambda)$ and $q_2(E, \lambda)$, other values $q_i' = q_i'(E, \lambda)$ (outside the contour L), for which $H_p(q, E, \lambda) = 0$. The points q_i' can be interpreted as virtual turning points. If L has finite dimensions, then the singularities of $T(E, \lambda)$ and $\Lambda^{(n)}(E, \lambda)$ can occur only in the case when some of the points $q_i'(E, \lambda)$ are at an infinitesimally short distance away from the integration curve[3]. On the other hand, when finding the singularities of T and $\Lambda^{(n)}$ it is necessary to take account of the fact that the values of T and $\Lambda^{(n)}$ do not change if the curve L is deformed in such a way that it does not cross the

[3]If q_1 and q_2 are the only zeroes of H_p, then the singularity of T and $\Lambda^{(n)}$ is possible only if the length of the contour L becomes infinite, that is, as $|q_1 - q_2| \to \infty$. This occurs, for example, with a linear oscillator.

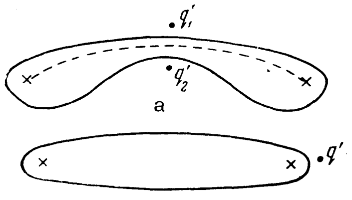

FIG. 1

points q_i'. From the considered properties of the contour integral, which determines T and $\Lambda^{(n)}$, it follows that for finite dimensions of L the investigated functions can have singularities in the following cases:

a) For some values of E and λ the points $q_1'(E,\lambda)$ and $q_2'(E,\lambda)$ coalesce, and their approach occurs in such a way that q_1' and q_2' are outside the contour L, on opposite sides of the curve L' joining q_1 and q_2 and located inside L (this case corresponds to the figure of 1a).

b) The external point $q'(E,\lambda)$ coalesces with one of the points $q_1(E,\lambda)$ or $q_2(E,\lambda)$ which are situated inside the contour L (Fig. 1b).

Thus, a singularity of one or the other branch of the multiply-valued functions T and $\Lambda^{(n)}$ always occurs when the two turning points coalesce. For each given value of E there exist, generally speaking, several values $q_0^{\nu}(E)$ at which the different pairs of turning points $q_i(\lambda, E)$ coalesce. If the points $q_i(\lambda E)$ and $q_k(\lambda E)$ coalesce at the point $q_0^{\nu}(E)$, the latter is determined from the expression

$$q_0^{\nu}(E) = q_i(\lambda_0(E), E) = q_k(\lambda_0(E), E),$$

where $\lambda_0(E)$ is specified in implicit form by the equation[4)] $q_i(\lambda, E) = q_k(\lambda, E)$. Starting from the definition of $q_i(\lambda, E)$, we can readily show that $q_0(E)$ is the point of total stoppage of the system, that is, when $q = q_0(E)$ and $p = p_0(E) = p(q_0(E), \lambda_0(E), E)$ the following equations are satisfied

$$\partial H(q, p, \lambda)/\partial p = \partial H(q, p, \lambda)/\partial q = 0, \quad H(q, p, \lambda) = E.$$

This property of the singular points $q_0(E)$ defines the character of the dependence of T on E and λ when q_1' and q_2' (case a) or q' and q_1 (case b) are at a short distance from $q_0(E)$. The difference $\Delta = \lambda - \lambda_0(E) \ll 1$ in this case, and at the points of the contour L for which the inequality $|q - q_0(E)| \gg |q_1' - q_2'|$ (or $|q' - q_1|$) is satisfied we have $H_p(q,\lambda,E) \approx H_p(q, \lambda_0(E), E)$. In the vicinity of $q_0(E)$ the values of $H_p(q, \lambda, E)$ are determined by expanding the Hamiltonian $H(q,p,\lambda)$:

$$H(q, p, \lambda) = E + H_\lambda \Delta + \tfrac{1}{2}(H_{pp}(\Delta p)^2 + 2H_{qp}\Delta q \Delta p + H_{qq}(\Delta q)^2), \tag{3.2}$$

where $\Delta p = p - p_0(E)$, $\Delta q = q - q_0(E)$, and the derivatives are taken at $p = p_0(E)$, $q = q_0(E)$, $\lambda = \lambda_0(E)$. Taking into consideration the location of the contours relative to the points q_1', q_2' and q', q_1 (Figs. 1a and b), we obtain from this expansion and from (3.1)

$$T(E, \lambda) = T_0(E) \ln [f_0(E)/(\lambda - \lambda_0(E))], \quad \lambda - \lambda_0(E) \ll 1. \tag{3.3}$$

Here $f_0(E) \sim 1$ and $T_0(E) \sim T_0$ are expressed in terms of the expansion coefficients of the Hamiltonian: in case a) we have $T_0(E) = 2D^{-1/2}$, and in case b) we obtain $T_0(E) = D^{-1/2}$, where $D(E)$ is the determinant of the quadratic form in the expansion (3.2). Formula (3.3) shows that $T(E,\lambda)$ diverges logarithmically when $\lambda \to \lambda_0(E)$.

2. After clarifying the character of the singularity $T(E,\lambda)$, we can proceed to find the function $I_+(w_-, L)$. For this purpose it is necessary to investigate the motion of the system when t changes in the vicinity of the singular point $t_0(L)$, moving along the line R_0 (see Sec. 2). By definition, and in accordance with the foregoing analysis, $t_0(I_-)$ is one of the solutions of the equation

$$\lambda(\gamma t) = \lambda_0(E(I_-, t)).$$

The motion in the vicinity of t_0 is best investigated by considering the time variation of one of the canonically conjugate variables $q(t)$ [or $p(t)$] and the "moving" difference

$$\Delta(t) = \lambda(\gamma t) - \lambda_0(E(I(t), t)).$$

(When $|t - t_0| \ll T_0/\gamma$ we have $\Delta(t) \ll 1$.) The variables Δ and q are connected by a simple relation, which can be obtained from the equations

$$\dot{q} = H_p, \quad \dot{\Delta} = \gamma\lambda'(1 - H_\lambda\, \partial\lambda_0/dE),$$

$$d\lambda_0/dE = 1/H_\lambda(q_0(E), p_0(E), \lambda_0(E)). \tag{3.4}$$

Dividing the second equation by the first and recognizing that

$$\lim_{q \to q_0(E)} \frac{1 - H_\lambda(q, \lambda_0(E), E)\, d\lambda_0/dE}{H_p(q, \lambda_0(E), E)}$$

exists, we obtain accurate to terms $\sim \gamma^2 \ln \gamma$

$$\frac{d\Delta(q, I_-)}{dq} = \frac{\gamma\lambda'}{H_\lambda^0}\chi(q, I_-),$$

$$\chi = \frac{H_\lambda^0 - H_\lambda(q, \lambda_0, E_0)}{H_p(q, \lambda_0, E_0)}, \quad \lambda_0 = \lambda(\gamma t_0), \ E_0 = E(I_-, t_0). \tag{3.5}$$

[4)] A situation is possible in which the number of coalescing turning points exceeds 2. It is then necessary to satisfy the following system of equations: $q_1'(\lambda,E) = q_2(\lambda,E) = \ldots = q_n(\lambda,E)$. When $n > 3$, the system is indeterminate, and when $n = 3$ it has a unique solution. It follows therefore that this situation is accidental.

The variable $q(t)$ should move over a contour L (in the complex plane q) such that as t is varied along R_0 the quantity $\operatorname{Im} w(t, I_-, w_-)$ remains finite. Since $w(q, E, \lambda)$ is given by the formula

$$w(q, E, \lambda) = 2\pi \int_{\bar{q}}^{q} \frac{dq'}{H_p(q', E, \lambda)} \Bigg/ \oint \frac{dq'}{H_p(q', E, \lambda)},$$

$\operatorname{Im} w(t)$ will be finite only when the relative placements of the contour L_0 and of the "external" turning points $q_i'(t) = q_i'(\lambda(\gamma t), E(t))$ are topologically similar to the arrangement of q_i' and the contour of integration $L(t)$ in the formula for $T(E(t), \lambda(\gamma t))$. When $|t - t_0| \ll T_0/\gamma$, the external points $q_i'(t)$, as they move in the complex plane, approach the contour $L(t)$ and "pinch" it in the way shown in Figs. 1a and b. Then[5] $q_{1,2}'(t)$ is determined by the formula

$$q_{1,2}'(t) = q_0 \pm \left[\frac{-2H_\lambda H_{pp}}{D} \Delta(t)\right]^{1/2}, \quad q_0 = q_0(E_0), \tag{3.6}$$

which is obtained by expanding the Hamiltonian in the vicinity of q_0. It follows from (3.5) and (3.6) that $q(t)$ "rotates" over the closed contour L_0, which encloses the points $q_1(t)$ and $q_2(t)$ and p passes at a distance $\lesssim \gamma^{1/2}|q_1 - q_2|$ from the point q_0.

When t varies from some value t_- on the left branch R_0 to a value t_+ on the right branch ($T_0 \ll |t_\pm - t_0| \ll T_0/\gamma$), the motion of $q(t)$ over L_0 can be broken up into the following three stages: 1) the variable q moves first from the point $q(t_-)$ to the point $\bar{q}_1$ which does not lie in the vicinity of $q_0(t_0)$ (the choice of $\bar{q}_1$ is arbitrary); 2) starting with this instant, $q(t)$ executes N revolutions ($\gamma^{-1} \gg N \gg 1$); 3) it moves from the point $\bar{q}_1$ to $q(t_+)$. Accordingly, we obtain from the equation $q = H_p$, accurate to terms of order γ, the equality

$$t_+ - t_- = \int_{q(t_-)}^{\bar{q}_1} \frac{dq'}{H_p(q', \lambda(\gamma t_-), E(t_-))} + \sum_{k=1}^{N} T_k + \int_{\bar{q}_1}^{q(t_+)} \frac{dq'}{H_p(q', \lambda(\gamma t_+), E(t_+))}, \tag{3.7}$$

where T_k is the change in t occurring during the k-th revolution. In completing the k-th revolution, the system passes twice in the vicinity of $q_0(t_0)$ with different values of the moving difference Δ_k' and Δ_k''. (Henceforth in the text the prime over Δ_k and q_0 will indicate the sequence with which the vicinity of $q_0(t_0)$ is passed for specified $\bar{q}_1$.) Taking this circumstance into account, we obtain with the aid of (3.4) the following expression for T_k:

$$T_k = \frac{1}{2} T_0(E_0) \left(\ln \frac{f_0(E_0)}{\Delta_k'} + \ln \frac{f_0(E_0)}{\Delta_k''}\right). \tag{3.8}$$

The dependence of Δ_k' and Δ_k'' on I_- and w_- will be determined with the aid of the asymptotic series where $I = \partial S_-(w, t, I_-)/\partial w$ [formula (A.3) of the appendix], confining ourselves to first order terms in γ. Taking (2.3) into account, we obtain after simple calculations

$$\Delta_k' - \Delta_k'' = \Delta_T \oint_{L_0''} \chi\, dq \Bigg/ \oint_{L_0} \chi\, dq = \Delta_2, \quad \Delta_T = \gamma\lambda' \oint_{L_0} \chi\, dq / H_\lambda^0,$$

$$\Delta_k' = \left(k - N_- - \frac{w_-'}{2\pi}\right)\Delta_T + \Delta_T \int_{\bar{q}_0}^{q_0'} \chi\, dq \Bigg/ \oint_{L_0} \chi\, dq,$$

$$\bar{q}_0 = \bar{q}(I_-, \lambda_0). \tag{3.9}$$

Here L_0' and L_0'' are the parts of the contour L_0 separated by the point q_0, with L_0' being the part containing the point $\bar{q}_1$;

$$w_-' = w_- + \int_{\bar{t}}^{t_0} \Omega(I_-, t')\, dt';$$

N_- is a positive integer determined by the choice of the point t_- ($\gamma^{-1} \gg N_- \gg 1$).

Let us calculate the derivative $\partial w(t_+, w_-, I_-)/\partial w_-$. To this end we substitute expressions (3.8) and (3.9) in (3.7) and differentiate the latter with respect to w_-. Using the partial-fraction expansion of the logarithm derivative of the Γ function [8], and also the well-known properties of the Γ function:

$$z\Gamma(z) = \Gamma(z+1), \; \Gamma(z)\Gamma(1-z) = \pi/\sin \pi z,$$

we obtain as a result of transformations carried out accurate to $\gamma \ln \gamma$, the formula

$$\frac{\partial w(t_+, w_-, I_-)}{\partial w_-} = 1 - \frac{2\pi^2 T_0(E_0)}{T(t_+)} \left\{\frac{\sin(w_-' + \alpha(I_-))}{\cos(w_-' + \alpha(I_-)) - \sin(\rho_1 - \rho_2)\pi/2} - i\right\};$$

$$\rho_{1,2} = \frac{\Delta_{1,2}}{\Delta_T} \quad (\rho_1 + \rho_2 = 1), \alpha(I_-) = 2\pi \int_{\bar{q}_0}^{q_0'} \chi\, dq \Bigg/ \oint \chi\, dq + \pi\rho_2. \tag{3.10}$$

Expression (3.10) does not depend on the choice of $\bar{q}_1$.

On the other hand, when $t = t_+$ the following equality holds

$$w = \int_{\bar{t}}^{t_+} \Omega(I_+, t')\, dt' + w_+(w_-, I_-) + 0(\gamma).$$

[5]In case b) $q_{1,2}'$ should be replaced by q' and q_1. In view of the fact that the course of the succeeding argumentation is perfectly analogous for cases a) and b), we discuss only case a) in detail.

Differentiating this equation with respect to w_- and using the formula (3.3) for $\Omega(E,\lambda)$ ($E = E(I_+, \lambda)$), and also the relation $\partial\Delta/\partial I_+ = (2\pi/\Delta_T H_\lambda^0)\,\partial\Delta/\partial t$, we obtain

$$\left(\frac{\partial w}{\partial w^-}\right)_{t_+, I_-} = 2\pi \frac{\partial I_+}{\partial w_-} \frac{1}{\Delta_T H_\lambda^0} (\Omega(I_+, t_+) - \Omega(I_+, t_0)) + \left(\frac{\partial w_+}{\partial w_-}\right)_{I_-} + \frac{\partial}{\partial w_-}\int_{t}^{t_0} \Omega(I_+, t')dt'.$$

Comparing this expression with (3.10) we find, that the derivative $(\partial I_+/\partial w_-)_{I_-}$ is determined, accurate to $\gamma^2 \ln \gamma$, by the formula

$$\left(\frac{\partial I_+}{\partial w_-}\right)_{I_-} = -\frac{1}{2}\Delta_T H_\lambda^0 T_0(E_0)\left\{\frac{\sin(w_-' + \alpha)}{\cos(w_-' + \alpha) - \sin(\rho_1 - \rho_2)\,\pi/2} - i\right\}. \tag{3.11}$$

When $\operatorname{Im} w_- \sim 1$, the main contribution to ΔI is made by the Fourier coefficients of the function $I_+(w_-, I_-)$ with $n = \pm 1$. Therefore in case a) we obtain from (3.11)

$$\Delta I = 2\operatorname{Re}\left\{\Delta_T H_\lambda^0 T_0(E_0)\, e^{i\alpha} \sin\frac{\pi(\rho_1 - \rho_2)}{2} \exp i\left(w_- + \int_{t}^{t_0} \Omega(I_-, t')\,dt'\right)\right\}. \tag{3.12}$$

In case b) analogous calculations lead to the result

$$\Delta I = 4\operatorname{Re}\left\{\Delta_T H_\lambda^0 T_0(E_0) \exp i\left(\alpha + w_- + \int_{t}^{t_0} \Omega(I_-, t')\,dt'\right\},\right.$$

$$\alpha = \int_{q_0}^{q_0(I_-)} \chi\,dq \Big/ \oint \chi\,dq, \tag{3.13}$$

where $q_0(I_-)$ is the point at which q_1 and q' coalesce. The quantities α, Δ_T, H_λ^0, $\rho_{1,2}$, and $T_0(E_0)$, which enter into the expression for the pre-exponential factor, are the parameters of the problem, characterizing the motion of the system in the vicinity of $t_0(I_-)$.

3. The results obtained become particularly lucid for a charged particle moving in a slowly time-varying potential well $U(q, \lambda)$, the form of which is shown in Fig. 2. The system of trajectories in the phase plane q, p, corresponding to a definite λ and different E, is shown in Fig. 3. The self-intersecting "figure-8" trajectory is determined by an energy $E_0(\lambda)$ equal to the value of the potential $U(q_0(\lambda), \lambda)$ at the maximum of the barrier separating regions 1 and 2 (Fig. 2). The intersection point of the "figure-8" $B_0(\lambda) = \{q_0(\lambda), P_0(\lambda)\}$ is the point of complete stop-

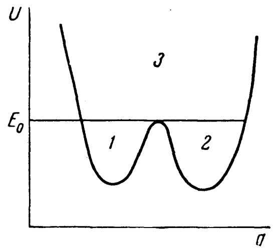

FIG. 2

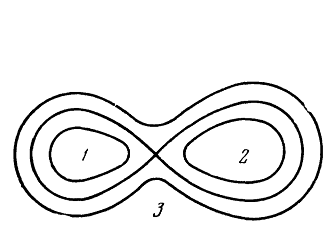

FIG. 3

page of the system (the turning points $q_1'(\lambda, E)(q')$ and $q_2'(\lambda, E)(q_1)$ coalesce at $q_0(\lambda)$), and consequently, in accordance with the results obtained above, the period $T(E, \lambda)$ has a logarithmic singularity in λ when $\Delta = \lambda - \lambda_0(E) \ll 1$ ($\lambda_0(E)$ is the inverse of $E_0(\lambda)$).

The character of the motion of the system under consideration depends essentially on the distance between t_0, which is a zero of $\Delta(t)$, and the real axis. If t_0 lies on the real axis, then the particle moving in the field $U(q, \lambda)$ experiences a unique "scattering" on the singular point[6)] $B(\lambda(t_0))$. This phenomenon consists in the fact that when the sign of $\Delta(t)$ is reversed there occurs an abrupt transition (within a time $\sim T_0$) from one mode of the quasiperiodic motion to the other. For example, if the particle was situated in region 3 at the initial instant, then upon scattering by $B(\lambda(\gamma t_0))$, depending on the exact initial conditions, it falls into either region one or two (see Figs. 2 and 3). In the papers by I. Lifshitz, the author, and Nabutovskiĭ [5-6], where this phenomenon was considered in detail[7)], it was shown that scattering by a singularity can be characterized by quantities W_1 and W_2, which determine the probability that the particle will fall into one region or the other. Thus, for the transition from the region 3 into 1 or 2, the scattering probabilities $W_{1,2}$ are determined by the formulas

$$W_1 + W_2 = 1, \quad W_1 = \rho_1 \sigma(\rho_1), \quad \sigma(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases},$$

where $\rho_{1,2}$ have already been determined above [formulas (3.10) and (3.9); in this case the counter integrals in the expression for ρ_{1-2} are taken along the loops of the "figure-8" separating the regions 1 and 2, so that $\operatorname{Im} \rho_{1,2} = 0$].

In the problem which we have considered, ΔI

[6)]In this case ΔI is no longer exponentially small; we can show that ΔI is then nearly equal to γ_0'.

[7)]In those papers we investigated the equivalent problem, that of the motion of a quasiparticle with the arbitrary dispersion law, placed in a magnetic field that varies slowly in time and in space, and also in a weak electric field parallel to the magnetic field.

is exponentially small, and during the entire process the particle is situated in the region having the same quasiperiodic mode of motion. Therefore the difference $\Delta(t)$ vanishes only when $t_0(I_-)$ is complex with $|\mathrm{Im}\, t_0| \sim T_0/\gamma$. This value of $t_0(I_-)$ determines the virtual stopping point $B(\lambda(\gamma t_0))$, and the real quantities $\rho_{1,2}$ can be interpreted as the probabilities of virtual scattering by the point $B(\lambda(\gamma t_0))$.

In conclusion we note that the results obtained do not hold true for systems which have only two turning points (for example, for a linear oscillator with slowly varying frequency). However, even in this case the method indicated in Sec. 2 frequently makes it possible to calculate ΔI in closed form [8)].

I am deeply grateful to I. M. Lifshitz for continuous interest in the work and for valuable discussions.

APPENDIX

Let us find the form of $S_\pm(w, t, I_\pm)$ with the aid of the iteration method. Choosing as the zeroth approximation

$$S_{0,\pm} = I_\pm w - \int\limits_{\tilde{t}}^{t} E(I_\pm, t')\, dt',$$

we obtain

$$S_\pm = S_{0,\pm} + \sum_{m=1}^{\infty} \gamma^m S_{m,\pm}(w, t, I_\pm, \gamma), \quad \text{(A.1)}$$

where $S_{m,\pm} \to 0$ as $t \to \pm\infty$. Substituting (A.1) in (2.5) and equating terms with equal powers of γ, we obtain the linear equations which are satisfied by $S_{m,\pm}$:

$$\begin{gathered}\Omega(I_\pm, t)\,\partial S_{1,\pm}/\partial w + \partial S_{1,\pm}/\partial t = -\lambda'\Lambda(w, I_\pm, t),\\ \Omega(I_\pm, t)\frac{\partial S_{2,\pm}}{\partial w} + \frac{\partial S_{2,\pm}}{\partial t} = -\frac{1}{2}\frac{\partial\Omega}{\partial I} I_{1,\pm}^2(w, t, I_\pm)\\ -\lambda'\Lambda_I(w, I_\pm, t)\, I_{1,\pm}(w, t, I_\pm), \ldots\\ \Omega(I_\pm, t)\,\partial S_{m,\pm}/\partial w + \partial S_{m,\pm}/\partial t = \psi_m(w, t, I_\pm). \quad \text{(A.2)}\end{gathered}$$

The functions ψ_m are polynomials of degree m in $I_{1,\pm}, I_{2,\pm}, \ldots, I_{m,\pm}$; the coefficients of the polynomials are expressed in terms of the partial derivaatives of the functions $E(I, \lambda)$ and $\Lambda(w, I, \lambda)$ with respect to I (the derivatives are taken at $I = I_\pm$). Solving (A.2) by the method of characteristics, we obtain formulas for $S_{m,\pm}$:

$$\begin{aligned} S_{1,\pm} &= -\int\limits_{\pm\infty}^{t} \lambda'\Lambda\Big(w - \int\limits_{t_1}^{t} \Omega(I_\pm, t_1')\, dt_1', I_\pm, t_1\Big)\, dt_1\\ &\approx -\lambda'\int^{w} \Lambda(w', I_\pm, t)\, dw'/\Omega(I_\pm, t), \ldots\\ S_{m,\pm} &= -\int\limits_{\pm\infty}^{t} \lambda'\psi_m\Big(w - \int\limits_{t_1}^{t} \Omega(I_\pm, t_1')\, dt_1', I_\pm, t_1\Big)\, dt_1. \end{aligned} \quad \text{(A.3)}$$

By examining the asymptotic behavior of $S_+(w, t, I_+)$ as $t \to -\infty$, we can show that I_+, I_-, and w_- are related by

$$I_- = I_+ + \int\limits_{-\infty}^{\infty} f\Big(w_- + \int\limits_{\tilde{t}}^{t} \Omega(I_+, t')\, dt', t, I_+\Big)\, dt, \quad \text{(A.4)}$$

where $f(\theta, t, I_+) = f(\theta + 2\pi, t, I_+)$, the mean value of f is zero, and the Fourier coefficients of the function f vary slowly in time, while the singular points of these coefficients coincide with the singularities of $\Lambda^{(n)}$ and $\lambda(\gamma t)$. From (A.4) there follow directly the properties indicated in Sec. 2 for the quantity B_n.

[8)]For a linear oscillator, the method of Sec. 2 coincides with the method used by Pokrovskiĭ and Khalatnikov[9] to determine the coefficient of the superbarrier reflection in the quasiclassical approximation.

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