AN INVESTIGATION OF THE S MATRIX IN THE COMPLEX ANGULAR MOMENTUM PLANE IN THE QUASI-CLASSICAL CASE

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An effective method is developed for the study of the scattering matrix $S(\nu)$ in the complex angular momentum plane $l = \nu - 1/2$ in nonrelativistic quantum mechanics. The method is based on a study of the properties of the quasiclassical solutions of the radial Schrödinger equation in the complex radius-vector plane.

1. INTRODUCTION

RECENTLY the analytic properties of the scattering matrix $S(\nu)$ in the complex angular momentum plane have been intensively studied. These studies were aimed mainly at a clarification of the high energy asymptotic behavior of the scattering amplitude in the relativistic region. However in the absence of a consistent quantum field theory it is necessary to make definite assumptions about the location of the poles of $S(\nu)$ in the complex plane. These assumptions are based on the analogy with nonrelativistic quantum mechanics. For this reason the study of the analytic properties of $S(\nu)$ in nonrelativistic quantum mechanics constitutes an important problem.

On the other hand such a study is of interest for its own sake since it makes possible a full solution of the problem of quasiclassical scattering of particles by potential fields.

In what follows we consider the motion of a particle in a spherically symmetric potential $U(r)$. It is assumed that the particle energy $E \gg U(r)$ on the real axis r. In addition we take $U(r)$ to be an even analytic function of r having no singularities on the real axis. These conditions are not essential limitations on the method considered and are adapted for the sake of simplification only.[1)]

In this note we outline the method for investigation of $S(\nu)$ in the complex ν plane in the quasiclassical case. Applications of the method for a complete clarification of the asymptotic behavior and of the poles of $S(\nu)$ and for calculations of the scattering amplitude will be given in a following communication of the authors.

2. DEFINITION AND BASIC PROPERTIES OF $S(\nu)$

Let us suppose that the potential $U(r)$ falls off faster than $1/r$ as $r \to \infty$ and that $r^2U(r) \to 0$ as $r \to 0$.

For physically meaningfull positive half-integer values of $\nu = l + 1/2$ the function $S(\nu)$ is defined as follows. There exists a solution $j_\nu(r)$ of the radial Schrödinger equation

$$\frac{d^2 j_\nu}{dr^2} + \left(k^2 - u - \frac{\nu^2 - 1/4}{r^2}\right) j_\nu = 0$$
$$(k^2 = 2mE, \quad u = 2m\,U(r)), \tag{2.1}$$

which behaves like $r^{\nu+1/2}$ as $r \to 0$. As $r \to \infty$ the function $j_\nu(r)$ takes the asymptotic form

$$j_\nu(r) = T_\nu \cos\left(kr - \nu\frac{\pi}{2} - \frac{\pi}{4} + \delta_{\nu-1/2}\right) = A_\nu e^{ikr} + B_\nu e^{-ikr}. \tag{2.2}$$

By definition

$$S(\nu) = e^{2i\delta_{\nu-1/2}} = ie^{i\nu\pi} A_\nu / B_\nu. \tag{2.3}$$

These definitions can be extended without any changes to arbitrary complex values of ν provided that $\operatorname{Re}\nu > 0$. It is only necessary to remark that $j_\nu(r)$ has a branch point at $r = 0$ for other than half-integer values of ν. We agree to draw the cut along the negative real r semiaxis.

For $\operatorname{Re}\nu < 0$ the definition of $j_\nu(r)$ needs to be made more precise. At first glance the definition of $j_\nu(r)$ by its behavior as $r \to 0$ is in this case ambiguous. Indeed, an addition of the second

[1)]The case when U (r) is not regular on the real axis has been investigated by the same method [1] on the example of a rectangular spherical well.

independent solution $j_{-\nu}(r)$ of Eq. (2.1) with an arbitrary constant coefficient will leave unaffected the behavior of j_ν as $r \to 0$. We therefore make out definition more precise by assuming that $j_\nu(r)$ can be represented by the series:

$$j_\nu(r) = \frac{\sqrt{2\pi}\, r^{\nu+1/2}}{2^\nu \Gamma(\nu+1)} \sum_{n=0}^{\infty} a_n \left(\frac{r}{2}\right)^{2n} \qquad (a_0 = 1). \quad (2.4)$$

For noninteger ν the series for $j_\nu(r)$ and $j_{-\nu}(r)$ have no common powers of r. In this way the definition (2.4) fixes $j_\nu(r)$ uniquely for all ν other than negative integers. In that last case we shall understand by $j_\nu(r)$ the limit to which this function tends as ν tends to the corresponding value.

The factor $1/\Gamma(1+\nu)$ in Eq. (2.4) insures the analyticity of $j_\nu(r)$ for negative integer ν. According to the general theory of differential equations of the Fuchs class (see, e.g., [2]), for negative integer ν the function $j_\nu(r)$ coincides with $j_{-\nu}(r)$ apart from a constant factor. Had we defined $j_\nu(r)$ without the factor $1/\Gamma(1+\nu)$, then $j_\nu(r)$ would develop poles for negative ν. Let us recall that our considerations apply only to even potentials $U(r)$. For other potentials an analogous situation develops also for halfinteger values of ν.

The function $j_\nu(r)$ defined in the indicated manner is an entire function of ν. The same applies to the coefficients A_ν and B_ν. It therefore follows that $S(\nu)$ can have only poles as a function of ν,

It is useful to give one more definition of $j_\nu(r)$, equivalent to Eq. (2.4). Let us consider in the limit as $|r| \to 0$ a line in the complex r plane on which $|r^\nu| = |r^{-\nu}| = 1$. It is not hard to see that this line is the logarithmic spiral

$$\operatorname{Re} \nu \ln |r| - \operatorname{Im} \nu \arg r = 0, \quad (2.5)$$

which passes through all the sheets of the Riemann surface of $j_\nu(r)$. Let us require that $j_\nu(r)$ behave like $r^{\nu+1/2}$ as we approach zero along this spiral. Such a definition is unambiguous for complex ν since the other independent solution $j_{-\nu}(r)$ behaves like $r^{-\nu+1/2}$ and has the same magnitude (in modulus) as the first and hence is easily distinguishable against its background. For negative ν again additional comments are needed, since in that case the spiral becomes a circle which does not go through zero. In that case $j_\nu(r)$ may again be defined as a limit.

Since the potential is even, we can establish a relation between $j_\nu(r)$ and $j_\nu(-r)$. The function $j_\nu(-r)$ is, just like $j_\nu(r)$, a solution of Eq. (2.1) that goes like const $r^{\nu+1/2}$ as $r \to 0$. It then follows from the uniqueness of $j_\nu(r)$ that

$$j_\nu(e^{\pm i\pi} r) = e^{\pm i(\nu+1/2)\pi} j_\nu(r). \quad (2.6)$$

Let us define now the functions $h_\nu^{(1)}(r)$ and $h_\nu^{(2)}(r)$ as those solutions of Eq. (2.1) which go like $e^{\pm ikr}$ as $r \to +\infty$. As $r \to -\infty$, with the point $r = 0$ circuited from above, the $h_\nu^{(1)}$ and $h_\nu^{(2)}$ go over into superpositions of incoming and outgoing waves:

$$h_\nu^{(1)} = a_\nu e^{ikr} + b_\nu e^{-ikr}, \qquad h_\nu^{(2)} = c_\nu e^{ikr} + d_\nu e^{-ikr}. \quad (2.7)$$

The functions $h_\nu^{(1)}$ and $h_\nu^{(2)}$ are entire even functions of ν. Consequently the same is true of the coefficients a_ν, b_ν, c_ν, and d_ν. From Eq. (2.2) and the definitions of $h_\nu^{(1),(2)}$ it follows that

$$j_\nu(r) = A_\nu h_\nu^{(1)} + B_\nu h_\nu^{(2)}. \quad (2.8)$$

Let us establish the relation between $S(\nu)$ and the coefficients a_ν, b_ν, c_ν, d_ν. To that end we use Eq. (2.6), setting in it $r \to -\infty$ along the upper edge of the cut. Making use of the asymptotic behavior (2.2) and (2.7) we get

$$e^{i(\nu+1/2)\pi}(A_\nu e^{-ikr} + B_\nu e^{ikr})$$
$$= A_\nu (a_\nu e^{ikr} + b_\nu e^{-ikr}) + B_\nu (c_\nu e^{ikr} + d_\nu e^{-ikr}). \quad (2.9)$$

On equating the coefficients of e^{ikr} and e^{-ikr} we get a homogeneous system of linear equations for A_ν and B_ν:

$$(e^{i(\nu+1/2)\pi} - b_\nu) A_\nu - d_\nu B_\nu = 0,$$
$$-a_\nu A_\nu + (e^{i(\nu+1/2)\pi} - c_\nu) B_\nu = 0. \quad (2.10)$$

Setting the determinant of the system (2.10) equal to zero we obtain

$$e^{2i(\nu+1/2)\pi} - e^{i(\nu+1/2)\pi}(b_\nu + c_\nu) + b_\nu c_\nu - a_\nu d_\nu = 0. \quad (2.11)$$

On the other hand it follows from the constancy of the Wronskian of $h_\nu^{(1)}$ and $h_\nu^{(2)}$ that

$$a_\nu d_\nu - b_\nu c_\nu = 1. \quad (2.12)$$

On substituting of Eq. (2.12) into Eq. (2.11) we find

$$b_\nu + c_\nu = 2i \cos \nu\pi. \quad (2.13)$$

From the second of the Eqs. (2.10) we get with the help of Eq. (2.13)

$$A_\nu / B_\nu = (e^{-i(\nu+1/2)\pi} + b_\nu) / a_\nu. \quad (2.14)$$

Finally, from the definition (2.3) and from (2.14) we get

$$S(\nu) = 1/a_\nu + e^{i(\nu+1/2)\pi} b_\nu / a_\nu. \quad (2.15)$$

Equation (2.15) makes it possible to explicitly resolve $S(\nu)$ into even and odd parts. In particular the odd part $S_a(\nu)$ is given by

$$S_a(\nu) = \frac{1}{2}[S(\nu) - S(-\nu)] = -\sin\nu\pi\,(b_\nu/a_\nu). \quad (2.16)$$

For $E > 0$, which is the only case of interest to us, $S(\nu)$ satisfies the unitarity condition derived by Regge:

$$S^*(\nu^*)\,S(\nu) = 1. \quad (2.17)$$

It follows from Eq. (2.17) that the poles and zeros of $S(\nu)$ lie at complex conjugate points.

Since a_ν and b_ν are entire functions of ν it follows that the poles of $S(\nu)$ are at the same time zeros of a_ν. The converse of this statement is, however, not true since we could have simultaneously the equality $b_\nu = -e^{-i(\nu+1/2)\pi}$. That this equality should hold is by no means an accident. Suppose that at the point ν_0 there is indeed a pole, i.e., that $a_{\nu_0} = 0$ and $b_{\nu_0} \neq -e^{-i(\nu_0+1/2)\pi}$. Then a zero of a_ν occurs also at $-\nu_0$. But at that point, as will be shown, $b_{-\nu_0} = -e^{-i(-\nu_0+1/2)\pi}$. Indeed, suppose this to be false. Then at the points $\pm\nu_0$ there are poles and at the points $\pm\nu_0^*$ there are zeros of $S(\nu)$. But that is in contradiction with Eq. (2.15) when the parity of a_ν and b_ν is taken into account. Consequently at the pole point ν_0 we have

$$b_{\nu_0} = -e^{-i(-\nu_0+1/2)\pi}. \quad (2.18)$$

In conclusion we mention one more important property of $S(\nu)$: for positive values of E the poles of $S(\nu)$ cannot lie on the real ν axis. Indeed, for real ν the function $j_\nu(r)$ is real (since the boundary condition at $r \to 0$ and the coefficients in the Schrödinger equation are real). On the other hand it follows from Eqs. (2.2) and (2.3) that the pole of $S(\nu)$ cannot be real.

3. QUASICLASSICAL SOLUTION OF THE SCHRÖDINGER EQUATION AND LEVEL LINES

In the quasiclassical case ($ka \gg 1$) considered here the functions $S(\nu)$, b_ν/a_ν, and others can be found explicitly. To that end we make use of the method developed in [4,5]. Our method is based on a study of the behavior of the functions j_ν and h_ν in the complex planes of r and ν.

It is well known that in the quasiclassical approximation the Schrödinger equation (2.1) has solutions in the form (see, for example, [6]):

$$Z_\pm(r, \bar{r}) = \frac{1}{\sqrt{p_\nu}}\exp\left(\pm i\int_{\bar{r}}^{r} p_\nu dr\right), \quad (3.1)$$

$$p_\nu = \sqrt{k^2 - u - \nu^2/r^2}. \quad (3.2)$$

The general solution may be expressed in the form

$$Z = \lambda Z_+ + \mu Z_-. \quad (3.3)$$

Equation (3.3) does not define λ and μ unambiguously. It is necessary to impose upon these coefficients an additional condition which may be chosen in the following form:

$$dZ/dr = ip_\nu(\lambda Z_+ - \mu Z_-). \quad (3.4)$$

When the condition (3.4) is satisfied the quantities λ and μ are slowly varying functions of r.

There exist lines in the complex r plane on which the functions $\exp(\pm i\int_{\bar{r}}^{r} p_\nu dr)$ oscillate without changing in modulus. These are, obviously, the level lines

$$\mathrm{Im}\int_{\bar{r}}^{r} p_\nu dr = \mathrm{const}. \quad (3.5)$$

On these lines the quantities λ and μ remain constant with a relative accuracy dp_ν^{-1}/dr. However, this assertion makes sense only if the two terms on the right side of Eq. (3.3) do not differ appreciably in magnitude. If, however, the ratio of the magnitudes is less than or of the order of dp_ν^{-1}/dr, then the extraction of the smaller term from the background of the larger exceeds the accuracy.

The quasiclassical approximation becomes inapplicable near the "turning points" at which p_ν vanishes. Therefore, after passing through the neighborhood of a turning point along a level line the coefficients λ and μ may have changed substantially. The same may happen in the neighborhood of any singularity of the potential $U(r)$.

On moving along a path that crosses the level lines we find that one of the exponentials grows and the other diminishes. Therefore if the solution Z was correctly given on some level line in the form of the sum (3.3), in which the two terms are of comparable magnitude, then as we move across the level lines the solution will be given by just one of the exponentials (the growing one). If the coefficient of the growing exponential was zero to start out with, then the accuracy of the method is not sufficient to determine the behavior of the solution in the corresponding region.

In the following we will make constant use of various pictures showing the distribution of the level lines. We therefore study now the distribution of the level lines for various values of ν. Let U_0 be a quantity characteristic of the potential on the real r axis. First we note that for $U_0/E \ll 1$ the level lines corresponding to motion in the potential field are nearly the same in the entire r plane as the level lines for free motion [1] depicted in Fig. 1.

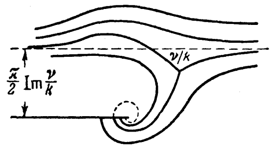

FIG. 1

As $r \to 0$ $p_\nu \approx i\nu/r$ and the level lines are given by the logarithmic spirals Re $(\nu \ln r)$ = const [see Eq. (2.5)], passing through all the sheets of the Riemann surface $Z_\nu(r, \bar{r})$. Figure 1 shows a turn of each spiral, the turn lying on the "physical" r sheet. The point $r_1 = \nu/k$ represents a turning point (complex turning point in a centrifugal potential): $p_\nu(r_1) = 0$. Near r_1 we have approximately $p_\nu(r) \approx \text{const}\sqrt{r - r_1}$. Therefore three branches of level lines depart from the turning point at an angle of $2\pi/3$ to each other.

As $r \to \pm\infty$ we have $p_\nu \to k$ and the level lines approach asymptotically straight lines parallel to the real r axis. A more detailed study shows that the level lines starting from r_1 approach at infinity the straight line Im $r = (\pi/2)$ Im r_1. In the case Re $\nu > 0$, Im $\nu > 0$, depicted in Fig. 1, the right branch of this line approaches its asymptote from below and the left branch from above.

Let us assume that the singularities of $u(r)$ that lie nearest to the real r axis are simple poles. Since $u(r)$ is real on the real axis and is even, it follows that there must be four such poles: at $\pm r_0$ and $\pm r_0^*$. Let us denote the residue of $u(r)$ at the point r_0 $(0 < \arg r_0 < \pi/2)$ by R. Then the residues at the points $-r_0$ and $\pm r_0^*$ are respectively $-R$ and $\pm R^*$. Near the poles lie zeros of p_ν^2 which we will denote by $\pm r_2$ and $\pm r_3$:

$$r_2 = r_0 + \frac{R}{k^2 - \nu^2/r_0^2}, \qquad r_3 = -r_0^* - \frac{R^*}{k^2 - \nu^2/r_0^{*2}}. \quad (3.6)$$

For the case of motion in a "weak" potential field $(U_0/E \ll 1)$ the level lines can be appreciably distorted only in a relatively small neighborhood of the poles of the potential (Fig. 2). The structure of the level lines near poles of $u(r)$ can be understood as follows (4). The poles $\pm r_0$ and $\pm r_0^*$ represent branch points of infinite order of solutions of the Schrödinger equation. Let us agree to draw cuts from them parallel to the imaginary r axis. The level lines go off to one side only of the points $\pm r_0$ and $\pm r_0^*$, because in the neighborhood of r_0

$$p_\nu \approx \text{const}/\sqrt{r - r_0}, \qquad \int_{r_0}^{r} p_\nu\, dr \approx \text{const}\,(r - r_0)^{1/2}.$$

The points $\pm r_2$ and $\pm r_3$ represent turning points

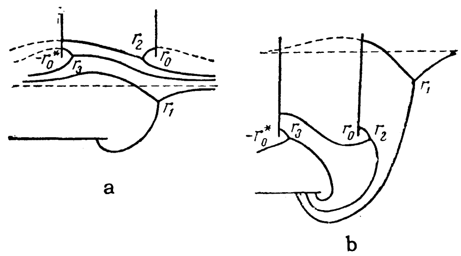

FIG. 2

and three branches of level lines depart from them, just like from r_1. Two of them go to $\pm\infty$ (Fig. 2a), or to zero and infinity (Fig. 2b). On the same sheet of the r Riemann surface as $r \to \pm\infty$ and $r \to 0$ there cannot exist two lines of the same level. It therefore follows that the third branch of the level lines that comes out of the turning points $\pm r_2$ and $\pm r_3$ necessarily goes under the cut onto another sheet of the Riemann surface (as shown on Fig. 2 by dashed lines).

We do not present a detailed picture in the lower half-plane in view of the fact that the level lines are symmetric with respect to zero. The remaining changes, introduced into the picture of the level lines by the potential, reduce to insignificant shifts and deformations of these lines by amounts of relative order U_0/E.

Real values of ν present a special case. In that case the free motion level lines have the characteristic form depicted in Fig. 3 ("eye"). For motion in a potential field for $E \gg U_0$ one has the picture shown in Fig. 4. The poles may turn out to lie outside (Fig. 4a) as well as inside (Fig. 4b) the "eye."

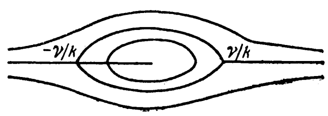

FIG. 3

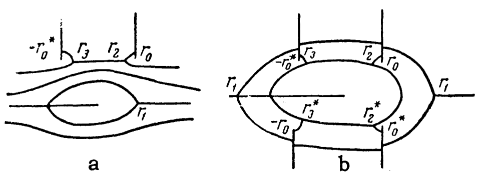

FIG. 4

For values of ν equal in magnitude but opposite in sign we obtain the same picture of level lines.

The Schrödinger equation solutions that are of interest to us are prescribed by boundary conditions at zero and at infinity. As r tends to zero or infinity the potential $U(r)$ plays a negligibly small role and therefore the coefficients λ and μ

in Eq. (3.3) tend to constant values. The decomposition (3.3) acquires in these regions an asymptotically exact meaning, independent of the magnitude of the ratio λ/μ. Along the level lines the coefficients λ and μ are conserved with quasi-classical accuracy. They may however change if the level line passes through a sufficiently small neighborhood of a turning point or a singular point of the potential. Beside, there exist level lines that do not go through the regions $\operatorname{Re} r \to \pm\infty$ or $r \to 0$ (see Fig. 2). In order to obtain the coefficients λ and μ in the entire plane one must state the "connecting formulas" across the singular points and the turning points, i.e. one must find out how λ and μ vary in the neighborhood of these points.

4. CONNECTING FORMULAS

The connecting formulas are established as follows. The asymptotic solution Z is known on one side of the singular point. It is necessary to connect with it the solution that is exact in the neighborhood of the singular point, and then find the asymptotic behavior of the exact solution on the other side of the singular point. The asymptotic solution so obtained determines the changed coefficients λ and μ.

For the one-dimensional case the connecting formulas have been obtained by the method outlined in a number of papers.[4,7,8] We give here just the results that will be useful to us later.

a) Simple turning point.[7] Near the simple turning point r_1 one has approximately $p_\nu = \text{const}\sqrt{r - r_1}$. The level lines have the form shown in Fig. 5. Let us take the lower limit of

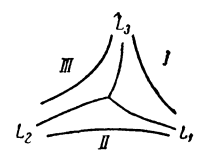

FIG. 5

integration $\overline{r}$ in Eq. (3.1) equal to r_1. Let the coefficient λ of the exponential Z_+, which decreases on motion inward of region I, be equal to unity on the branch L_1, and let the coefficient μ of Z_- be equal to zero.[2)] Then, as before, $\lambda = 1$ and $\mu = 0$ on L_3 and $\lambda = 1$ and $\mu = -i$ on L_2.

Thus, the solution expressible in terms of a single exponential on the level line $\operatorname{Im}\int_{r_1}^{r} p_\nu dr = 0$ remains a single exponential on another line of the same level if the transition between these lines can be achieved by going through a region within which the solution is small. If instead the solution is large within that region, then in the transition there is added to the original exponential another one with the coefficient $\mp i$ depending on whether the passage was clockwise or counter-clockwise.

b) Pole and turning point near each other.[4] The level lines are shown schematically in Fig. 6.

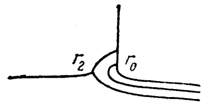

FIG. 6

We set in this case $\overline{r} = r_2$. If $\lambda = 1$ and $\mu = 0$ on L_1 (it is assumed that Z_+ grows inward region II) then we have on L_2

$$\lambda = 1, \qquad \mu = F(\xi_2), \tag{4.1}$$

$$\xi_2 = \frac{1}{\pi}\int_{r_2}^{r_0} p_\nu dr \approx -i\sqrt{k^2 - \frac{\nu^2}{r_0^2}}\,\frac{r_2 - r_0}{2} \approx -i\frac{R}{\sqrt{k^2 - \nu^2/r_0^2}}, \tag{4.2}$$

$$F(\xi) = 2\pi i e^{-2\xi \ln(-\xi/e)}/\Gamma(-\xi)\,\Gamma(1-\xi) \tag{4.3}$$

For $|\xi| \gg 1$ the function $F(\xi) = -i$ and we return again to the case a). Let us also write out the asymptotic behavior of $F(\xi)$ for small ξ:

$$F(\xi) \approx -2\pi i \xi. \tag{4.4}$$

On going around counterclockwise as before $\lambda = 1$, but $\mu = -F(\xi_2)$.

c) Two zeros near each other[8] (Fig. 7). Let

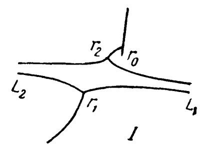

FIG. 7

us choose $\overline{r} = r_1$ (r_1 is that turning point from which departs the level-line that goes to zero). Let $\lambda = 1$ and $\mu = 0$ on L_1 (Z_+ decreases into the region I). Then on L_2:

$$\lambda = 1, \qquad \mu = G(\eta), \tag{4.5}$$

$$\eta = \frac{1}{\pi}\int_{r_1}^{r_2} p_\nu\, dr, \tag{4.6}$$

$$G(\eta) = \sqrt{2\pi}\, i e^{\eta - \eta\ln(-\eta)}/\Gamma(1/2 - \eta). \tag{4.7}$$

2)The symbolz Z_+ and Z_- [see Eq. (3.1)] do not determine the solution uniquely, since p_ν changes sign upon crossing a turning point. To make the solutions unique it is necessary to specify the direction in which they decrease or increase.

As $\eta \to \infty$, $G(\eta) \to +i$ and we come back to case a), going around counterclockwise.

In some cases there arises the problem of connecting formulas when two turning points and a pole lie close to each other. Unfortunately this problem leads to a differential equation with unknown explicit solution and consequently the connecting formulas for this case have not been found.

5. LOCATION OF SINGULAR POINTS

In what follows we shall call singular points not only the poles of the potential but also the turning points.

If the turning point $r_1 \approx \nu/k$ is far from the pole r_0 then, as was already shown, the second turning point r_2 is close to r_0 [see Eq. (3.6)]. Let us introduce the notation

$$\sigma = (\nu - kr_0)/kr_0, \qquad x = (r - r_0)/r_0, \qquad U_0 = R/2mr_0. \tag{5.1}$$

In terms of σ, x the function p_ν^2 takes on the form

$$p_\nu^2 \approx k^2 x^{-1} (2x^2 - 2\sigma x - U_0/E). \tag{5.2}$$

The roots of the equation $p_\nu^2 = 0$ are

$$x_{1,2} = \sigma/2 \pm \sqrt{\sigma^2/4 + U_0/2E}. \tag{5.3}$$

A measure of the distance between the points r_i and r_k is given by the quasiclassical "phase"

$$(r_i, r_k) = \int_{r_k}^{r_i} p_\nu dr \approx kr_0 \sqrt{2} \int_{x_k}^{x_i} \sqrt{\frac{(x - x_1)(x - x_2)}{x}}\, dx$$
$$(i, k = 0, 1, 2;\ x_0 = 0). \tag{5.4}$$

The points r_i and r_k are to be considered as distant from each other if $|(r_i, r_k)| \gg 1$. As we have already mentioned, our method yields effective results if even just two of the three points r_0, r_1, and r_2 are distant from each other. Let us study first the "distances" (r_i, r_k) as functions of σ for fixed values of U_0/E, and kr_0.

1. $|\sigma| \gg |U_0/E|^{1/2}$.

In this case $|x_1| \approx |\sigma| \gg |x_2| \approx |U_0/E\sigma|$. Neglecting x_2 compared with x_1 we find

$$(r_0, r_1) \approx kr_0 \sqrt{2} \int_{x_1}^{0} \sqrt{x - x_1}\, dx = i\,\frac{2^{3/2}}{3} kr_0 \sigma^{3/2},$$

$$(r_0, r_2) \approx ikr_0 \sqrt{2\sigma} \int_{x_2}^{0} \sqrt{\frac{x - x_2}{x}}\, dx = -\frac{\pi}{\sqrt{2}} kr_0 \frac{U_0}{E\sqrt{\sigma}}. \tag{5.5}$$

Since $(r_0, r_1) \gg (r_0, r_2)$, it follows that $(r_1, r_2) \approx (r_0, r_2)$.

2. $\sigma \ll (U_0/E)^{1/2}$.

Then $x_1 \approx -x_2 \approx \sqrt{U_0/2E}$,

$$(r_0, r_1) \approx kr_0 \int_{x_1}^{0} \sqrt{\frac{x^2 - x_1^2}{x}}\, dx = iCkr_0 \left(\frac{U_0}{2E}\right)^{3/4},$$
$$(r_0, r_2) \approx i\,(r_0, r_1), \qquad (r_1, r_2) = (i - 1)(r_0, r_1). \tag{5.6}$$

Here C is a constant of order of magnitude of unity.

3. $\sigma \sim (U_0/E)^{1/2}$.

Here one must consider separately the case when σ is near the branch points of the roots $\pm i\sqrt{2U_0/E}$. Let

$$\sigma = i\sqrt{2U_0/E} + \varepsilon \qquad (\varepsilon \ll \sqrt{U_0/E}). \tag{5.7}$$

Then

$$x_{1,2} \approx i\sqrt{U_0/2E} \pm (2U_0/E)^{1/4} \sqrt{2i\varepsilon}. \tag{5.8}$$

Hence

$$(r_0, r_1) \approx (r_0, r_2) \approx -\sqrt{2}\, e^{3\pi i/4} kr_0 (U_0/2E)^{3/4}, \tag{5.9}$$

$$(r_1, r_2) \approx \text{const}\ kr_0 (U_0/E)^{1/4} \varepsilon. \tag{5.10}$$

If instead $\varepsilon \sim \sqrt{U_0/E}$ then

$$(r_0, r_1) \sim (r_0, r_2) \sim (r_1, r_2) \sim kr_0 (U_0/E)^{3/4}.$$

Let us consider now the various possible relations between the parameters U_0/E and kr_0.

A. $kr_0 (U_0/E)^{3/4} \gg 1$.

A.1. $\sigma \gg (U_0/E)^{1/2}$.

In accordance with Eq. (5.5) one has in this case $(r_0, r_1) \gg 1$, $(r_1, r_2) \gg 1$. For (r_0, r_2) one has according to Eq. (5.5) $(r_0, r_2) \gg 1$ if $\sigma \ll \sigma \ll (kr_0U_0/E)^2$, and $(r_0, r_2) \lesssim 1$ if $\sigma \gtrsim (kr_0U_0/E)^2$. Regardless of the value of (r_0, r_2) the case A.1. is "soluble" (a pair of "distant" points exists).

A.2. $\sigma \ll \sqrt{U_0/E}$.

In accordance with Eq. (5.6) (r_0, r_1), (r_0, r_2), and $(r_1, r_2) \gg 1$. Case A.2. is also soluble.

A.3. $\sigma \sim (U_0/E)^{1/2}$.

According to Eqs. (5.9) and (5.10), (r_0, r_1) and $(r_0, r_2) \gg 1$. In addition $(r_1, r_2) \overset{\gg}{\lesssim} 1$ if $\varepsilon \overset{\gg}{\lesssim} (kr_0)^{-1}(U_0/E)^{-1/4}$. Thus case A is fully soluble.

B. $kr_0 (U_0/E)^{3/4} \lesssim 1$.

B.1. $\sigma \gg (kr_0)^{-2/3} \gtrsim (U_0/E)^{1/2}$.

According to Eq. (5.5) $(r_0, r_1) \sim (r_1, r_2) \gg 1$, $(r_0, r_2) \lesssim 1$. Case B.1. is soluble.

B.2. $\sigma \lesssim (kr_0)^{-2/3}$.

According to Eqs. (5.5), (5.6), (5.9) and (5.10) we have

$$(r_0, r_1),\ (r_0, r_2),\ (r_1, r_2) \lesssim 1.$$

Case B.2. is not soluble.

In case B our method permits the study of the asymptotic behavior not in the entire ν plane but only outside a neighborhood $\sim(kr_0)^{1/3}$ of the point $\nu_0 = kr_0$.

6. BEHAVIOR OF j_ν AND h_ν IN THE COMPLEX r PLANE

The knowledge of level lines and connecting formulas makes it possible to obtain quasiclassical asymptotic solutions of the Schrödinger equation for arbitrary complex values of ν and r. Let us show how this is done on the example of the simplest picture of level lines, Fig. 2a.

Let us start with $h_\nu^{(1)}$. By definition this function has for Re $r \to +\infty$ just one exponential which increases in the direction of decreasing Im r (for brevity we shall say in what follows that it increases on moving down). First let us find its asymptotic behavior on level lines going through singular points.

The sections of the level lines between the points r and r′ will be denoted by [r, r′]. On the lines $[+\infty, r_1]$, $[+\infty, r_2]$, and $[+\infty, r_3]$ we have everywhere

$$h_\nu^{(1)} \approx \sqrt{k}\, e^{-i(+\infty,\bar{r})} Z_+(r, \bar{r}), \tag{6.1}$$

where we have introduced the notation

$$(\pm\infty, \bar{r}) = \int_{\bar{r}}^{\pm\infty} (p_\nu - k)\, dr - k\bar{r}. \tag{6.2}$$

The normalization factor $e^{-i(+\infty, r_1)}$ was introduced into Eq. (6.1) in order that $h_\nu^{(1)}$, in agreement with its definition, have the asymptotic behavior e^{ikr} as $r \to +\infty$.

The asymptotic behavior, Eq. (6.1), is also true on the line $[-\infty, r_1]$. On the line $[r_1, 0]$ we have according to the connecting formula a)

$$h_\nu^{(1)} = \sqrt{k}\, e^{-i(+\infty, r_1)} (Z_+(r, r_1) - iZ_-(r, r_1)). \tag{6.3}$$

On the line $[-\infty, r_3]$ we have according to the connecting formula b)

$$h_\nu^{(1)} = \sqrt{k}\, e^{-i(+\infty, r_3)} (Z_+(r, r_3) + F(\xi_3) Z_-(r, r_3)), \tag{6.4}$$

where $F(\xi)$ is defined by Eq. (4.3) and

$$\xi_3 = \frac{1}{\pi} \int_{r_3}^{-r_0^*} p_\nu\, dr \approx i \frac{R^*}{\sqrt{k^2 - \nu^2 / r_0^{*2}}}. \tag{6.5}$$

Let us agree to denote by the symbol $\sim$ points that lie on unphysical sheets of r. On the line $[-\tilde{\infty}, r_2]$ the asymptotic behavior of $h_\nu^{(1)}$ again coincides with Eq. (6.1). On the line $[-\tilde{\infty}, r_2]$ we have according to the connecting formula b)

$$h_\nu^{(1)} = \sqrt{k}\, e^{-i(+\infty, r_2)} (Z_+(r, r_2) + F(\xi_2) Z_-(r, r_2)). \tag{6.6}$$

The level lines that go through singular points correspond to borders of regions within each of which the asymptotic behavior is determined by the exponential that increases as one moves from the border into the given region.

In the case considered here the asymptotic behavior of $h_\nu^{(1)}$ is below the lines $[-\tilde{\infty}, r_2]$ and $[-\tilde{\infty}, r_3]$ close to the asymptotic behavior of the corresponding function for free motion $h_\nu^{0(1)}$. Above these lines in $h^{(1)}$ differs substantially from $h_\nu^{0(1)}$. Formula (4) makes it possible to obtain the coefficients a_ν, b_ν [see definition (2.7)]

$$a_\nu = \exp\{-i(+\infty, r_3) + i(-\infty, r_3)\}, \tag{6.7}$$

$$b_\nu = F(\xi_3) \exp\{-i(+\infty, r_3) - i(-\infty, r_3)\}. \tag{6.8}$$

The asymptotic behavior of $h_\nu^{(1)}$ in the lower half plane, we well as the asymptotic behavior of $h_\nu^{(2)}$, are found in an analogous manner.

Let us turn to the functions j_ν. According to its definition j_ν is represented by one exponential on the line $[0, r_1]$. Taking into account the normalization (2.4) the asymptotic behavior of j_ν on $[0, r_1]$ has the form

$$j_\nu \approx N(\nu) Z_+(r, r_1), \quad N(\nu) = (2\pi)^{1/2} \nu^{\nu+1/2} e^{-\nu} / \Gamma(\nu+1). \tag{6.9}$$

The normalization factor $N(\nu) \approx 1$ for $|\nu| \gg 1$ (an exception is provided by the neighborhood of the negative semiaxis ν). In the case under consideration (Re $\nu > 0$) j_ν decreases as $r \to 0$. It therefore follows that on $[-\infty, r_1]$ the asymptotic behavior (6.9) holds. On $[+\infty, r_1]$ we have according to the connecting formula a)

$$j_\nu \approx N(\nu) (Z_+(r, r_1) + iZ_-(r, r_1)). \tag{6.10}$$

Above the lines $[r_1, \pm\infty]$ there remains just the one exponential that grows on moving upward, say Z_+.[3] It therefore follows that on the lines $[-\infty, r_3]$, $[+\infty, r_3]$, $[-\tilde{\infty}, r_2]$, and $[+\infty, r_2]$ the asymptotic behavior (6.9) holds.

On the line $[-\tilde{\infty}, r_3]$ we have according to b)

$$j_\nu \approx N(\nu)\, e^{i(r_3, r_1)} (Z_+(r, r_3) - F(\xi_3) Z_-(r, r_3)). \tag{6.11}$$

Analogously we find on the line $[+\tilde{\infty}, r_2]$

$$j_\nu \approx N(\nu)\, e^{i(r_2, r_1)} (Z_+(r, r_2) + F(\xi_2) Z_-(r, r_2)). \tag{6.12}$$

From Eq. (6.10) we easily obtain the coefficients A_ν, B_ν [see definition (2.2)]:

$$A_\nu = N(\nu)\, e^{i(+\infty, r_1)}, \qquad B_\nu = iN(\nu)\, e^{-i(+\infty, r_1)}, \tag{6.13}$$

and $S(\nu)$, making use of definition (2.3):

$$S(\nu) = ie^{i\nu\pi} A_\nu / B_\nu \approx e^{2i(+\infty, r_1) + i\nu\pi}. \tag{6.14}$$

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[3]On going round point r_1 the solution Z_+ goes formally over into iZ_-. Therefore the continuation of j_ν from the lines $[-\infty, r_1]$ and $[+\infty, r_1]$ upwards gives the same result. This question is considered in detail in [7] and in [1].

Equation (6.14) agrees with the usual quasi-classical expression for $e^{2i\delta l}$ (see, e.g., [6], p. 471). The only difference is that the angular momentum l and the turning point r_1 are complex quantities.

Let us compare now Eq. (6.14) with the other formula (2.15), which expresses $S(\nu)$ in terms of a_ν and b_ν. We will show that within the limits of accuracy the first term in Eq. (2.15) coincides with the right side of Eq. (6.14). Indeed, according to Eq. (6.7)

$$a_\nu^{-1} \approx \exp\{i(+\infty, r_3) + i(r_3, -\infty)\}. \quad (6.15)$$

The integral in the exponent in Eq. (6.15) is taken from $-\infty$ to $+\infty$ along a complex contour that passes above the branch point r_1 of p_ν. Let us draw a cut from r_1 to $+\infty$ and displace the contour of integration to the real axis. At that there arises a second contour of integration encompassing the cut. The integral along the contour encompassing the cut is equal to $2(+\infty, r_1)$. The integral along the real axis, with the point $r = 0$ passed from above, is as a consequence of the evenness properties equal to $\nu\pi$ (half of the residue at the point $r = 0$). Thus

$$(+\infty, r_3) + (r_3, -\infty) = 2(+\infty, r_1) + \nu\pi. \quad (6.16)$$

Equation (2.15) also makes possible the evaluation of the odd part $S_a(\nu)$ of the scattering matrix $S(\nu)$. From Eqs. (6.7) and (6.8) we find

$$S_a(\nu) = -\sin\nu\pi\,(b_\nu/a_\nu) \approx -\sin\nu\pi\, F(\xi_3)\, e^{2i(r_3, -\infty)}. \quad (6.17)$$

On comparison of Eq. (6.17) with (6.14) we discover that $S_a(\nu)$ is exponentially small in comparison with $S(\nu)$:

$$|S_a(\nu)/S(\nu)| = \tfrac{1}{2}|F(\xi_3)|\,|e^{2i(r_3, r_1)}|. \quad (6.18)$$

Although the error in the determination of the quantity $1/a_\nu$ in Eq. (6.15) is substantially larger than $S_a(\nu)$ it is legitimate to keep the second term in Eq. (2.15) because the decomposition into an odd and even part is an exact decomposition.

We remind the reader that only one of the possible level line pictures was considered here. Any other distribution of the lines can be investigated by the same methods.

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