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RADIATION UNITS AND CRITICAL ENERGIES FOR VARIOUS SUBSTANCES

O. I. DOVZHENKO and A. A. POMANSKIĬ

P. N. Lebedev Physics Institute, Academy of Sciences, U.S.S.R.

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Values of radiation units and critical energies are derived in accordance with current theoretical and experimental notions. An analysis of the origin of the discrepancies between the radiation unit and critical energy values presented in various papers is discussed.

1. In the investigation of processes connected with the passage of electrons and photons of high energy through matter, when the energy losses are essentially due to bremsstrahlung and pair production, respectively, the range is best expressed in so-called radiation or cascade units of length (t-units). The introduction of the t-unit makes it possible to obtain the solution of the cascade equations in the form that does not depend on the substance in which the electron-photon cascade develops[1].

The advantage in introducing the t-unit follows from an analysis of the cross section of the bremsstrahlung and electron-positron pair production processes. These cross sections were first obtained in the field of the atomic nuclei by Bethe and Heitler[2,3] in the Born approximation. Neither bremsstrahlung nor pair production in the field of the atomic electrons was considered here.

Taking into account the corrections for the inaccuracy of the Born approximation and for the processes occurring in the field of the atomic electrons, the radiation unit of length should be represented in the form

$$t^{-1} = 4(N/A)\alpha r_0^2 Z(Z+\zeta)[L^{rad} - f(Z)]. \qquad (1)$$

Here t is in g/cm^2, N is the Avogadro number, Z and A are the mass and atomic numbers of the element, $\alpha = e^2/\hbar c \approx 1/137$, $r_0 = e^2/mc^2$ is the classical radius of the electron, L^{rad} is the so-called radiation logarithm, which determines the effect of the total screening of the field of the nucleus by the field of the atomic electrons, f(Z) is a correction for the inaccuracy of the Born approximation, and ζ is a quantity connected with the processes in the field of the atomic electrons, defined as

$$\zeta = l^{rad}/(L^{rad} - f(Z)) \gtrsim 1, \qquad (2)$$

where l^{rad} is a quantity that takes into account the screening in processes on atomic electrons.

In the calculation of the radiation length units, different authors[4-8] took different account of the influence of the total screening, of the processes in the field of the atomic electrons, and of the inaccuracy of the Born approximation; this has caused the values of the radiation length units to differ by 10—20%, which in turn has led to a discrepancy in the values of the critical energies. Such discrepancies can greatly distort the interpretation of the experimental results obtained in the investigation of the interactions of high-energy particles ($E \gg mc^2$) with matter.

In the present paper we analyze the causes of these discrepancies and give the values of the radiation units and of the critical energies that correspond most accurately to modern theoretical and experimental notions.

2. As was shown above, the expression for the cascade length unit contains the quantity

$$L^{rad} = \int_0^{mc/\hbar} [1 - F(q)]^2 \frac{dq}{q} + 1,$$

which is called the radiation logarithm and takes into account the total screening. The radiation logarithm depends on F(q) —the atomic form factor (atomic scattering function) —which is a measure of the amplitude of the scattered wave produced as a result of radiation of given amplitude being incident on the atom ($q = 4\pi\lambda^{-1} \sin\theta$ is the change in the momentum in elastic scattering, where θ is half the scattering angle).

The first to estimate L^{rad} was Bethe[2], who calculated F(q) using the Thomas-Fermi statistical model, and obtained $L_{TF}^{rad} = \ln(183Z^{-1/3})$. Subsequent calculations of L^{rad}[9,10], based on the Thomas-Fermi model, yielded a value L_{TF}^{rad} closer to $\ln(190Z^{-1/3})$. The calculations carried out in the present work yielded values $L_{TF}^{rad} = \ln(191Z^{-1/3})$.

In the calculation of the cascade length unit, Rossi and Greisen[4] used $L_{TF}^{rad} = \ln(183Z^{-1/3})$. Later on Kirpichev and Pomeranchuk[10] have shown that the use of the Thomas-Fermi method for light elements is incorrect, and proposed to calculate L^{rad} by using the experimental values of the atomic form factors, which are known from x-ray scattering.

The difference in the radiation logarithms calculated on the basis of the Thomas-Fermi model and data on atomic form factors, as used by Kirpichev and Pomeranchuk, is approximately 10%. Starting from this, Belen'kiĭ increased the theoretical (Thomas-Fermi) values of the radiation logarithms by 10% for all the elements, and on this basis he compiled a table of radiation units[5]. Thus, the radiation length units as given by Belenkiĭ were approximately 10% smaller than the units given by Rossi and Greisen.

This difference was assumed by Belen'kiĭ to remain constant on going from light elements to heavy ones, although the greater the number of electrons in the atom, the better the approximation that the statistical Thomas-Fermi model should give. However, even the data used by Kirpichev and Pomeranchuk did not approach the theoretical values, at least with increasing Z (see Fig. 1). An analysis of these data has shown that they actually represent not the results of experiments aimed at a direct determination of the values of the atomic form factors, but a set of values of F(q), which are of practical utility for the analysis of the experiments on the investigation of crystal structure. To obtain the true values of F(q) from the crystallographic data it is necessary to correct the latter for the thermal motion of the lattice atoms, for the zero-point energy of the lattice, for the overlap of the atoms in the crystals, and for extinction. An account of these corrections leads to values of F(q) which coincide with those calculated with the Hartree-Fock model of the atom[11-16].

Starting from this fact, we have used in the calculations of L^{rad} the theoretically calculated values of $f(q) = ZF(q)$, given in Vol. 3 of the new edition of the International Tables[7]. The values of L^{rad} were calculated for all the elements of practical interest. In the calculations for hydrogen, we used the f(q) obtained from the exact quantum-mechanical wave functions. For the other elements, L^{rad} was calculated using the latest variant of the self-consistent field method with account of the electron exchange. For atoms with $Z > 20$, in addition, we used the electron-density distributions as given by the Thomas-Fermi-Dirac model[18].

However, the calculated values of f(q) given in the table are in some cases insufficient to yield the exact value of L^{rad}, since the f(q) have been calculated up to values $10^8 q/4\pi$ usually not exceeding 2, whereas for the exact estimate of L^{rad} it is necessary to know f(q) for somewhat larger values of $10^8 q/4\pi$. In such cases f(q) was additionally calculated for large values of q; account was taken here of the fact that scattering at very large angles is determined only by the K electrons, which can be represented by hydrogen-like wave functions in the 1s state[16]. This extrapolation was applied to the f(q) calculated by the self-consistent field method. Where the L^{rad} were calculated by the Thomas-Fermi-Dirac model, the f(q) for large scattering angle (large q), that is, near the nucleus, were extrapolated by the Thomas-Fermi model without account of the electron exchange. The error introduced by such an extrapolation is negligibly small.

Figure 1 shows a plot of radiation logarithm against log Z, for different elements and under different assumptions. It is seen that the values of L^{rad} obtained in the present work are very close to $L^{rad} = \ln(191Z^{-1/3})$ and radically contradict the values obtained by Kirpichev and Pomeranchuk. Since the values obtained in the present paper for the radiation logarithms were calculated using f(q) which are in very good agreement with the experimental data, it is necessary to use these very values of L^{rad} in the calculation of the radiation unit length (1).

3. All the considerations in the preceding section applied only to processes occurring in the field of the atomic nuclei. In this case it is pos-

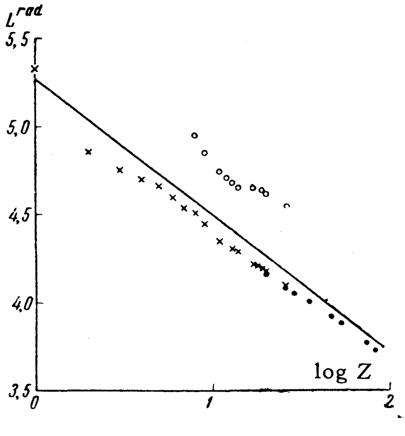

FIG. 1. Dependence of L^{rad} on log Z as given by various authors: ○ – Kirpichev and Pomeranchuk, × – Hartree-Fock, ● – Thomas-Fermi-Dirac; solid line – after Thomas-Fermi.

sible to neglect the momentum transferred to the nucleus, assuming the wavelength of the scattered radiation to be equal to the wavelength of the incident radiation. However, as can be seen from the exact expression for the t-unit (1), it is necessary to take into account processes in the field of the atomic electrons.

The first to indicate the need for taking into account the radiation and pair production in the field of atomic electrons were Landau and Rumer [19], in a paper on the theory of cascade showers. Assuming the cross section of the radiation processes in the field of the electron to be equal to the cross section of these processes on the proton, and assuming, accordingly, that the contribution of these electrons is determined by the formulas of Bethe and Heitler in which Z^2 is replaced by Z, the number of the electrons in the atom, they have reached the conclusion that the total cross section is proportional to $Z(Z+\zeta)$, with ζ taken equal to unity.

However, the replacement of Z^2 by $Z(Z+1)$ is valid only for heavy nuclei. For small Z (light nuclei) an account of the influence of the radiation processes in the field of the atomic electrons should be more accurate; for example, one of the principal electromagnetic processes which contribute to the absorption of high-energy photons in light elements is the production of electron-positron pairs by the photon in the Coulomb field of the electron (usually called triplet formation).

The theory of radiation processes occurring in the field of atomic electrons was developed by Wheeler and Lamb[20]. The corresponding cross sections were calculated in the Born approximation with account of screening. On the basis of the statistical Thomas-Fermi model, they obtained for ζ the following expression:

$$\zeta = \ln (1440\, Z^{-2/3}) / \ln (183\, Z^{-1/3}). \qquad (2')$$

In addition, they obtained values of ζ using atomic wave functions for hydrogen $\zeta^H = 1.16$ and nitrogen $\zeta^N = 1.20$ in place of 1.39 and 1.31 which follow from (2′).

In the present investigation we used the results of Wheeler and Lamb in the calculation of the t-units. For the heavy elements, ζ was determined from (2), where $l^{rad} = \ln(1440Z^{-2/3})$ and $L^{rad} = \ln(191Z^{-1/3})$, while for the light and medium elements the values of ζ were obtained by interpolating between $\zeta^H = 1.16$, $\zeta^N = 1.20$, and the ζ calculated by Eq. (2) for the heavy elements, where the statistical model of the atom is valid. As a result we obtained values which increase weakly with increasing Z, from 1.16 for hydrogen to 1.27 for lead.

4. Bethe and Heitler[2,3] calculated the bremsstrahlung and pair production cross sections, using the Born approximation of collision theory, valid when

$$2\pi Z/137\beta_0 \ll 1, \qquad 2\pi Z/137\beta \ll 1, \qquad (3)$$

where Z is the atomic number of the scattering force center, and β_0 and β are the velocities (as fractions of the velocity of light) of the incident particle at the start and at the end of the collision process, respectively.

Owing to the inexact value of the Born cross sections in the field of the nuclei of the heavy elements, many authors have undertaken to calculate these cross sections without using the Born approximation[21-23]. As followed from several experimental data[24-26], the correction for the deviation from the Born approximation for the cross sections of the bremsstrahlung and pair production turned out to be almost proportional to Z^2 and was independent of the degree of screening1). An analytic expression for the correction for the deviation of f(Z) in (1) from the Born approximation was obtained in the form

$$f(Z) = \left(\frac{Z}{137}\right)^2 \sum_{\nu=1}^{\infty} \frac{1}{\nu\,[\nu^2 + (Z/137)^2]}\,. \qquad (4)$$

This expression was used in the present paper in the calculation of the values of the t-units by Eq. (1). It must be noted that in the calculation of the t-unit for lead, contained in the paper by Belen'kiĭ and Ivanenko[8], the correction for the deviation from the Born approximation was assumed to be zero for bremsstrahlung, owing to the erroneous conclusions of the paper of Davis et al[22], while for the other elements the correction for deviation from the Born approximation was not taken into account at all, as was the case also in the work of Bethe and Ashkin[6].

The t-unit values obtained in the present paper for several elements, and also the values of the t-units given in many other papers[6-8], are listed in Table I. For complex elements the t-units were calculated from the formula[4]

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1) The experimental data on the absorption of photons with energy from 40 to 1000 MeV in different substances are gathered in the paper by Malamud,[25] while data on the bremsstrahlung cross sections (from 17 to 500 MeV) are given by Brown.[26] Both sets of data are in full agreement with modern theoretical notions used in the present paper to calculate the t-units.

Table I. Values of t-units and critical energies ϵ for different elements

Element	Z	Bethe and Ashkin[5]		Belen'kii and Ivanenko[5,8]		Rossi [7]			Present work		
		$t, \frac{g}{cm^2}$	ε, MeV ($\delta = 0$)	$t, \frac{g}{cm^2}$	ε, MeV ($\delta = 0$)	$t, \frac{g}{cm^2}$	ε, MeV $\delta = 0$	ε, MeV $\delta > 0$	$t, \frac{g}{cm^2}$	ε, MeV $\delta = 0$	ε, MeV $\delta > 0$
H	1	58	340	62					62.8	350	
He	2	85	220						93.1	250	
Li	3								83.3	180	138
Be	4								66.0	141	110
B	5								53.6	115	91
C	6	42.5	103	40		44.6	102	76	43.3	96	79
N	7	38	87	35.4		39.4	88.7		38.6	85	
O	8	34.2	77	32.1		35.3	77.7		34.6	75	
F	9								33.4	67.5	
Na	11								28.2	55.5	46.5
Al	13	23.9	47	22.7	37.2	24.5	48.8		24.3	47	40
Si	14								22.2	44	37.5
Cl	17								19.5	36	
Ar	18	19.4	34.5	18		19.8	35.2		19.7	34	
K	19								17.4	31.7	27.8
Ca	20								16.3	30.4	26.7
Fe	26	13.8	24	12.6	18.4	14.1	24.3	21	13.9	23.3	20.7
Cu	29	12.8	21.5	11.6		13.1	21.8		13.0	20.9	18.8
Br	35								11.5	17.2	15.7
Ag	47								9.0	12.7	11.9
I	53								8.5	11.3	10.7
Xe	54								8.5	11.0	
W	74								6.8	8.3	8.08
Pb	82	5.8	6.9	5.55	6.4	6.5	7.8	7.6	6.4	7.5	7.4

Remark: The difference between the values of the t-units determined from Eq. (1) in the present paper and in other papers is due to the different values of L^{rad}, ζ, and $f(Z)$ assumed in the calculations.

1. Bethe and Ashkin[6]:

$$L^{rad} = \ln (183\, Z^{-1/3}), \quad \zeta = \ln (1440\, Z^{-2/3})/\ln (183\, Z^{-1/3}), \quad f(Z) = 0$$

(ζ is introduced in accordance with the Thomas-Fermi model for all the elements: no account is taken in $f(Z)$ of the inaccuracy in the Born approximation).

2. Belen'kii and Ivanenko[5,8]:

$$L^{rad} \approx 1.1 \quad \ln (191 Z^{-1/3}), \quad \zeta = 1, \; f(Z) = 0.06\,(Z/82)^2 \ln (191\, Z^{-1/3})/[1 + 0.06\,(Z/82)^2],$$

with $f(Z)$ introduced only for Pb.

3. Rossi[7]:

$$L^{rad} = \ln (183 Z^{-1/3}), \quad \zeta = 1, \quad f(Z) = 0.12\,(Z/82)^2 \ln (183\, Z^{-1/3})/[1 + 0.12\,(Z/82)^2].$$

4. Present work: The values of L^{rad} are shown in Fig. 1; $\zeta \geq 1$ (smoothly varying with increasing Z, from 1.6 to H to 1.27 for Pb), $f(Z)$ is determined from (4).

The critical energies ϵ without account of the density effect ($\delta = 0$) and with account of this effect ($\delta > 0$) are determined in [6,7] by the method of Rossi and Greisen[4], while in [5] and in the present work they are determined by averaging over the equilibrium spectrum. The difference between the critical energies obtained by different authors is due essentially, as already indicated, to the difference in the values of the t-units.[4]

$$1/t = \sum_i p_i/t_i, \tag{5}$$

where t_i —radiation unit of the i-th element contained in the complex matter with a weight concentration t_i. The corresponding values of the t-units are represented in Table II, while Fig. 2 shows the dependence of the t-unit (in meters) for air on the atmospheric pressure for three different temperatures.

Since the available experiments cover only a limited interval of electron and photon energies, it is advantageous to discuss the applicability of the t-units obtained in the present paper for different primary electron and photon energies. As regards the lower limit, the existing experimental data confirmed the correctness of the formulas of Bethe and Heitler (with suitable correction for the inaccuracy of the Born approximation and for processes occurring in the field of the atomic electrons) up to energies of 10 MeV[24]. At lower energies, the photon energy losses due to the Compton and photoelectric effects and due to electron ionization become appreciable, so that the very concept of radiation unit of length begins to become meaningless.

On the other hand, as was first pointed out by

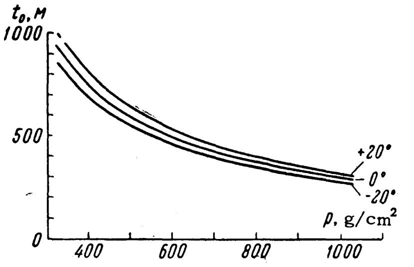

FIG. 2

Table II. Values of t-units and critical energies ϵ for complex substances

Substance	t, $\frac{g}{cm^2}$	ε, MeV	
		$\delta = 0$	$\delta > 0$
Air	37.1	81	
	34.2[5,8]	72[5]	
	36.5[6]	83[6]	
	37.7[7]	84.2[7]	
Water	36.4	90.0	73.0
	33.9[5,8]	—	—
	35.9[6]	93[6]	—
	37.1[7]	83.8[7]	65[7]
LiH	80.0	207	157
C_3H_8	45.9	129	—
C_8H_8 (polystyrene)	44.4	110	88.0
$Al_2SO_3 \cdot 2SiO_2 \cdot 2H_2O$ (clay)	28.8	60.9	50.6
SiO_2 (quartz)	27.4	56.2	47.3
$CaCO_3$ limestone	24.2	51.2	43.4
NaCl	22.2	42.1	36.2
NIKFI-R nuclear emulsion	11.4	17.8	16.4

Landau and Pomeranchuk[27], the bremsstrahlung and pair production theory developed by Bethe and Heitler breaks down in condensed media at sufficiently high energies, owing to the influence of multiple scattering, which leads to a reduction in the probability of γ-quantum emission with increasing energy of the primary electron, if the latter amounts to

$$E_0/mc^2 \gtrsim {}^1/_{60}\sqrt{E_0 k t_0/(E_0 - ck)\hbar},$$

where k —momentum of the radiated gamma quantum and t_0 —radiation length unit (in centimeters) in the given medium.

In addition, Ter-Mikaelyan has shown[28] that the polarization of the medium also leads to a decrease in the probability of emission of low-energy γ quanta[2]. A more rigorous analysis, which takes into account both multiple scattering and polarization of the medium, was made by Migdal[29]. The expressions he obtained for the bremsstrahlung and pair-production cross sections are valid when $E_0 \gg mc^2$. The calculations carried out by Varfolomeev[30] using Migdal's formulas for photonuclear emulsion show that the radiation losses of an electron with energy 10^{13} eV are smaller than the corresponding value as given by Bethe and Heitler by approximately 15%; for $E_0 = 10^{12}$ eV the difference is less than 1%. These estimates impose a limitation on the applicability of the given values of the t units at high incident-particle energy. For lead, a noticeable deviation from the Bethe and Heitler cross section begins at energies $\sim 5 \times 10^{12}$ eV; for light elements it begins at much higher energies.

5. The ionization losses of the high-energy electrons passing through matter are best expressed in losses per unit radiation length. The average ionization loss of an electron per t-unit is usually called the critical energy. It can be shown[4] that the energy lost by the electron over one radiation length unit to radiation is also approximately equal to the critical energy.

The values of the critical energies for different substances have been obtained in several investigations[4-7]. In cosmic-ray research one usually uses the values of the critical energies obtained either by Belen'kiĭ[5] or by Rossi[7], which differ from each other by 10—20%.

Rossi[7] assumes for the critical energy ϵ a value equal to the energy lost by an electron with the same energy ϵ to ionization and excitation of the atoms along a path equal to one radiation length. Considering separately the short-range and long-range collisions, in which the electron loses respectively an energy larger or smaller than a certain value $\eta = 5 \times 10^6$ eV, he obtains the values of ϵ directly from the formula for the ionization losses, using the values obtained by him for the radiation length units.

On the other hand, Belen'kiĭ[5] defines the critical energy as the mean value of the energy losses, averaged over the equilibrium energy spectrum[31], per radiation length unit, and uses his own t-unit calculations (see Tables I and II). However, as will be shown below, the different approach to the definition of the critical energy hardly influences the value of ϵ, and the discrepancy in the values of the critical energy were due to the differences in the t-unit lengths used by Belen'kiĭ[5] and Rossi[7]. Nonetheless, the method proposed by Belen'kiĭ for the determination of the critical energy[5] is more rigorous, for in most experiments connected with high-energy radiation it becomes necessary to deal with the energy spectra of the

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[2] All this pertains to the pair-production process.

particles, and not with particles of definite energies, so that in the calculation of the values of the critical energy for different substances we have used the method proposed by Belen'kiĭ.

Let us consider in greater detail the method of determining the critical energy with averaging over the equilibrium spectrum. For the average ionization losses per t-unit we use an expression that holds true for energies $E \gg mc^2$ [5]:

$$-dE/dt = 2\pi r_0^2 N (Z/A) mc^2 t \, [L^{ion} - \delta], \tag{6}$$

where N, Z, A, r_0, and t have the same meaning as in (1);

$$L^{ion} = \ln \frac{E^3}{mc^2 I^2(Z)} - 1, \tag{7}$$

I(Z) — average ionization excitation potential of the atoms of the medium, and δ — correction for the density effect.

Using (6), Belen'kiĭ[5] obtained, averaging over the equilibrium spectrum, the critical energies for certain substances, neglecting the density effect ($\delta = 0$). It is clear, however, that it is wrong to neglect the density effect in dense substances when $\epsilon \gg mc^2$, as follows also from the calculations of Rossi[7].

In the relativistic region, δ can be represented in the form[32]

$$\delta = -\ln(1 - \beta^2) - \ln \frac{I^2(Z)\,\pi m}{h^2 e^2 n} - 1, \tag{8}$$

where n — number of electrons per cm³ of the medium. In the calculation of ϵ we shall take into account the density effect in accordance with (8); thus, we neglect the density effect for the region of energies $E < E_c$, where E_c is determined from the condition $\delta = 0$:

$$E_c \approx 2.93 \cdot 10^4 \, I(Z) \sqrt{A/\rho Z}, \tag{9}$$

where ρ is the density of the substance in g/cm³. The error introduced by this approximation does not exceed 1%.

Let us determine the average value of $\overline{L^{ion} - \delta}$, integrated over the equilibrium spectrum

$$\overline{L^{ion} - \delta} = \int\limits_{E_{min}}^{\infty} [L^{ion} - \delta] \times \frac{\partial N(E_0, E)}{\partial E} dE \Bigg/ \int\limits_{E_{min}}^{\infty} \frac{\partial N(E_0, E)}{\partial E} dE, \tag{10}$$

where $N(E_0, E) = E_0 \chi(\omega)/\epsilon$ — equilibrium spectrum of the electrons for $E_0 \gg E$ [5],

$$\chi(\omega) = 1 - \omega e^{\omega} \int\limits_{\omega}^{\infty} \frac{e^{-x}}{x} dx, \qquad \omega = \frac{2.29E}{\varepsilon}.$$

Using (7) and (8) we obtain, integrating (10),

$$\overline{L^{ion} - \delta} = 3 \ln \frac{E_{min}}{mc^2} + 3 \frac{1 - \chi(\omega_{min})}{\omega_{min} \chi(\omega_{min})} + 2 \ln \frac{mc^2}{I(Z)} - 1 - 2 \frac{1 - \chi(E_{\kappa} \omega_{min}/E_{min})}{\omega_{min} \chi(\omega_{min})} \frac{E_{min}}{E_c}. \tag{11}$$

By way of the lower limit of integration it is sensible to take $E_{min} = 2mc^2$, according to Belen'kiĭ[5], and then we obtain from (6) and (11) the following transcendental equation for the determination of $\epsilon = 2.29 E_{min}/\omega_{min}$

$$1 = D\omega_{min} + C \left\{ \frac{1 - \chi(\omega_{min})}{\chi(\omega_{min})} - \frac{2}{3\varkappa} \frac{1 - \chi(\varkappa \omega_{min})}{\chi(\omega_{min})} \right\}, \tag{12}$$

where

$$D = \frac{\pi N r_0^2}{2.29} \frac{Zt}{A} \left[2 \ln \frac{mc^2}{I(Z)} - 1 + 3 \ln 2 \right], \tag{13}$$

$$C = \frac{3\pi N r_0^2}{2.29} \frac{Zt}{A}, \tag{14}$$

$$\varkappa = \frac{E_c}{2mc^2} \approx 2.87 \cdot 10^{-2} \, I(Z) \sqrt{\frac{A}{\rho Z}}. \tag{15}$$

Expression (12) differs from the corresponding equation (16.2) of Belen'kiĭ[5] in the presence of a second term in the curly bracket, by means of which the density effect is taken into account.

To determine ω_{min} (and consequently also ϵ) from (12) it is necessary to know the average ionization potential I(Z), which enters into the expression for the parameter D [see (13)]. According to Bloch's calculations[33], based on the statistical Thomas-Fermi model, I(Z) = kZ, where k is a quantity which is constant for all the elements. However, theoretically k cannot be calculated with high accuracy[34,35], so that it is advantageous to use for I(Z) the experimental data. Since the experimental data on I(Z) for different elements also have a very large scatter, we have analyzed the results of the available experiments on the determination of I(Z),[3)] and have reached the conclusion that the average proportionality coefficient k can be taken equal to 12.3 eV. It must be pointed out that the discrepancy in the values of I(Z) does not introduce an essential error in the value of the critical energy, since the average ionization potential enters into the expression for ϵ under the logarithm sign.

Starting with k = 12.3 eV, and using (12)—(14), we have obtained the values of the critical energies ϵ both without account of the density effect ($\delta = 0$) and with account for this effect ($\delta > 0$). These results are given in Tables I and II, which list also

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[3)] The experimental papers on the determination of I(Z) are discussed in the review of Sternheimer.[32]

the values of ϵ obtained by Belen'kiĭ[5] and Rossi [7]. The correct values of ϵ are those obtained with account of the density effect (the correction for the density effect is negligibly small for gases).

If we plot the values obtained by us against Zt/A, we find that the critical energy for any substance can be obtained with a sufficient degree of accuracy from the relation $\epsilon = B(Zt/A)^{\mu}$, where $B \approx 2.45$ and $\mu \approx 1.2$ if we disregard the density effect, and $B \approx 2.66$ and $\mu \approx 1.1$ if account is taken of the density. The same relation is satisfied by the values of ϵ obtained by Rossi[7], if his t-unit values are employed.

For complicated substances, the values of ϵ must be defined not as by Rossi[7] but in the following manner[5]

$$\varepsilon = t \sum_i p_i \varepsilon_i / t_i, \tag{16}$$

where t — radiation unit of the complex substance, determined from (5), while ϵ_i and t_i are the critical energy and the t-unit of the i-th element contained in the complex substance with weight concentration p_i.

The critical energies obtained in this manner for complex substances also satisfy the relation $\epsilon = B(Zt/A)^{\mu}$ given above, where the effective values of Z and A of the complex substance should be defined as

$$(Z/A)_{\text{eff}} = \sum_i p_i Z_i / A_i$$

(Z_i and A_i — mass number and atomic weight of the i-th element with weight concentration p_i). Yet Rossi[7] determined the critical energy for water by using the following value

$$(Z/A)_{\text{eff}} = \sum_i p_i Z_i \Big/ \sum_i p_i A_i,$$

which holds true only for the determination of the t-unit but not for the critical energy. Indeed, since the energy losses of the electron to radiation depend on the number of atoms per cm^3 of matter, and the energy losses to ionization and excitation depend on the number of electrons in 1 cm^3 of the medium, $(Z/A)_{\text{eff}}$ will differ, depending on the method of determination. For water the value of $(Z/A)_{\text{eff}}$ calculated on the basis of the number of atoms per cm^3 (Rossi) is ~ 0.505, whereas $(Z/A)_{\text{eff}}$ calculated on the basis of the number of electrons per cm^3 is ~ 0.555, so that Rossi[7] obtained for water too low a value of ϵ (see Table II).

The method considered above for the determination of the critical energy averaged over the equilibrium spectrum presupposes that the energy of the incident electron is $E_0 \gg \epsilon$, in which connection the upper limit of integration in (10) is assumed to be infinite. We can show, however, that this does not introduce an appreciable error in the value of ϵ. Indeed, if we neglect the density effect for simplicity, then the expression for ϵ for an electron of energy E_0 will, in analogy with (12), be

$$1 = D\omega_{min} + C\,[1 - \chi(\omega_{min}) - G(\omega_0, \omega_{min})]/\chi(\omega_{min}), \tag{17}$$

where D and C are determined by (13) and (14), $\omega_0 = 2.29\, E_0/\epsilon$, and

$$G(\omega_0, \omega_{min}) = \frac{\omega_{min}}{\omega_0} \frac{1 - \exp(\omega_{min} - \omega_0)\, \chi(\omega_0)/\chi(\omega_{min})}{1 - (\omega_{min}/\omega_0)\exp(\omega_{min} - \omega_0)\, \chi(\omega_0)/\chi(\omega_{min})}. \tag{18}$$

The quantity $G(\omega_0, \omega_{\text{min}})$ gives a small correction to the value of ω_{min} determined from D and C. Thus, for aluminum $\epsilon = 47$ MeV if $E_0/\epsilon \to \infty$ and $\epsilon = 45$ MeV for $E_0/\epsilon \approx 1.3$. It follows therefore, as indicated above, that the method of determining the critical energy when averaging over the equilibrium spectrum[5], and the method proposed by Rossi[7], lead to practically identical results if we use the same values of the radiation length units.

Recently Mazyukevich et al[36] reported critical energy values obtained by averaging over the equilibrium spectrum for the majority of the elements of the periodic system. However, the critical energies given in [36] are in error, owing to incorrect values of the radiation unit lengths employed.

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