RELATIVISTICALLY INVARIANT TRANSFORMATIONS IN WAVE FUNCTION SPACE

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Transformations in undor space are considered. The corresponding conservation laws are derived. Larmor transformations and the reversal of boson masses are examined. Extended Foldy-Tani and Cini-Touschek transformation groups are set up in boson theory (one of them is the analog of the usual FT transformation and has been discussed previously[30,31] without being included in the general scheme of the Kemmer theory).

In this paper we shall consider certain conservation laws which arise as a consequence of the invariance of the Lagrangian under transformations in wave-function space. These transformations are not in general imitated by any coordinate transformations of the conformal group. The results obtained in the investigation of these transformations allow one, in accordance with Schwinger's dynamical principle, to establish quantization laws which depend on the order of the Lagrangian. We shall also consider concrete examples: the group of Larmor transformations for a boson field and the canonical transformations of the Hamiltonian related to it.

1. WAVE FUNCTION TRANSFORMATIONS AND CONSERVATION LAWS

The analysis of the invariant properties of Lagrangians and the attendant wave equations is usually restricted to coordinate transformations and the conservation laws following from Noether's theorem.[1-4] There exists a possibility, which has been relatively little investigated, of extending the group of transformations of wave functions to those which are not directly imitated by transformations in four-dimensional space. As examples we might mention the γ_5 transformations for fermion fields[5-7] and the Larmor transformation for boson fields.[8,9] As was discovered recently, such transformations can be incorporated in the Lorentz group of spaces with more than four dimensions,[10] but we shall not, in the present paper, develop this idea, which is interesting in its own right.

We shall investigate wave equations of first and second order[11-13] with a constant proper mass m ($\hbar = c = 1$):

$$(\alpha_\lambda \partial_\lambda + m)\psi = 0, \tag{1.1}$$

$$(-\alpha_\lambda \alpha_\sigma \partial^2_{\lambda\sigma} + m^2)\psi = 0, \tag{1.2}$$

derived from the Lagrangians

$$L_0^{(1)} = -im\left(\psi^+\, {}^1/_2\,(I + m^{-1}\alpha_\lambda\partial_\lambda)\,\psi + {}^1/_2\,(I - m^{-1}\alpha_\lambda\partial_\lambda)^T\psi^+\cdot\psi\right), \tag{1.3}$$

$$L_0^{(2)} = -2im\,{}^1/_2\,(I - m^{-1}\alpha_\lambda\partial_\lambda)^T\psi^+\cdot{}^1/_2\,(I + m^{-1}\alpha_\lambda\partial_\lambda)\,\psi, \tag{1.4}$$

where ψ^+ is the charge conjugate and transpose of the function ψ, and α_μ is one of the representations of the Dirac or Kemmer algebras.

The variational principle for $L_0^{(1)}$ or $L_0^{(2)}$ must be invariant under transformations of the conformal group with the restrictions introduced by Hill.[3] Let us consider the class of transformations in the space of ψ defined by the relations

$$\delta\psi = \delta u(\alpha)\,\psi, \qquad \delta\psi^+ = \psi^+\delta u^+(\alpha) \tag{1.5}$$

and commuting with the Lorentz transformation ($\delta u_l = \xi[\alpha_\lambda\alpha_\sigma]\delta l_{\lambda\sigma}$). The latter condition is satisfied in at least two cases:

$$\text{a)}\quad \delta u^+(\alpha) = -\delta u(\alpha) = i\delta\eta\alpha, \qquad [\alpha\alpha_\mu] = \alpha^2 - I = 0, \tag{1.6}$$

where η is a continuous real parameter (the infinitesimal phase transformation of the wave function reduces to such a transformation with $\alpha = I$). The invariance of L_0 generates in this case the conservation law

$$\partial_\lambda\left(\psi^+\alpha\,(\delta_l L_0/\delta(\partial_\lambda\psi^+)) - (\delta_r L_0/\delta(\partial_\lambda\psi))\,\alpha\psi\right) = 0. \tag{1.7}$$

Since $\alpha = I$ exists in all representations, we can, by defining it as degenerate in terms of the corresponding group of isospin space, $\alpha \in IG(I_S)$, interpret (1.6) as conservation of the current $j_\mu^{(S)}$. In many-dimensional representations of the alge-

algebras $\alpha \neq I$ can also commute with the α_μ;

b) $\delta u^+(\alpha) = i\delta\eta^+\alpha, \qquad \delta u(\alpha) = i\delta\eta\alpha,$

$$\{\alpha\alpha_\mu\} = \alpha^2 - I = 0. \tag{1.8}$$

The corresponding conservation law contains sources:

$$\partial_\lambda\,(\psi^+\alpha\,(\delta_l L_0/\delta(\partial_\lambda\psi^+)) + (\delta_r L_0/\delta(\partial_\lambda\psi))\,\alpha\psi) + \psi^+\alpha\,(\delta_l L_0/\delta\psi^+) + (\delta_r L_0/\delta\psi)\,\alpha\psi = 0. \tag{1.9}$$

It is further necessary to distinguish two variants: b_1) $\alpha = \alpha_c$ and $[\alpha_c R_\mu] = 0$, and b_2) $\alpha = \alpha_{ac}$ and $\{\alpha_{ac} R_\mu\} = 0$ (R_μ are the matrices of coordinate reflections).

In the variant b_1) the conserved quantity is an ordinary vector and has, if α_c is unique, the meaning of the current vector of the field action, s_μ, introduced by the author[14] and independently by Freistadt.[15] The vector s_μ can be introduced on the basis of classical considerations by representing the field action as a functional,

$$S[\sigma] = \int\limits_{\sigma(x)} s_\mu(x')\,d\sigma_\mu(x'), \tag{1.10}$$

According to (1.9), it has the following form for the Lagrangians (1.3) and (1.4):

$$s_\mu^{(1)} = \psi^+\alpha_c\alpha_\mu\psi, \qquad s_\mu^{(2)} = (2m)^{-1}\,(\psi^+\alpha_c\alpha_\mu\alpha_\lambda\partial_\lambda\psi + \partial_\lambda\psi^+\alpha_\lambda\alpha_\mu\alpha_c\psi). \tag{1.11}$$

Diagonalization of α_c leads to the projection operators $\frac{1}{2}(I \pm \alpha_c)$:

$$\psi_{\rm I} = {}^1\!/_2\,(I + \alpha_c)\,\psi, \qquad \psi_{\rm II} = {}^1\!/_2\,(I - \alpha_c)\,\psi; \tag{1.12}$$

since $\{\alpha_c\alpha_\mu\} = 0$, the canonical momenta $\pi_{\rm I}$ conjugate to the canonical coordinates $\psi_{\rm II}$ can be defined by the relations

$$\pi_{\rm I}^+ = \delta_r s_0/\delta\psi = \delta_r L_0/\delta\,(\partial_t\psi_{\rm II})\,\alpha_c,$$
$$\pi_{\rm I} = \delta_l s_0/\delta\psi^+ = \alpha_c\delta_l L_0/\delta\,(\partial_t\psi_{\rm II}^+). \tag{1.13}$$

Using the action operator we can obtain the commutation relations for ψ with the help of Schwinger's dynamical principle. In the theory with $L_0^{(1)}$ the commutation relations are different for fermions and bosons, while in the theory with $L_0^{(2)}$ they are identical:[16]

$$[\psi_\alpha^+(x)\,\psi_\beta(x')] = i\delta_{\alpha\beta}\,\Delta\,(x - x'), \tag{1.14}$$

and the Pauli principle for fermions must be incorporated directly in the scattering matrix.

In the second variant, b_2), where $\alpha = \alpha_{ac}$ and $\{\alpha_{ac} R_\mu\} = 0$, the conserved quantity is the pseudovector of the spin, and the conservation law (1.9) represents the known Uhlenbeck-Laporte equation for fermions[17] and bosons.

2. LARMOR TRANSFORMATION AND REVERSAL OF BOSON MASS

The discussion of these transformations is most convenient within the framework of the complete reducible four-dimensional representations of the Kemmer algebra spanned by the basis of the irreducible eight-dimensional representation of the Dirac algebra,[18]

$$\beta_\mu^{(\pm)} = {}^1\!/_2\,(\gamma_\mu \pm \bar{\gamma}_\mu), \tag{2.1}$$

where $G_{16}(\gamma) \times G_{16}(\bar{\gamma}) = G_{256}(\gamma, \bar{\gamma})$ is the group of the irreducible eight-dimensional representations of the anticommutative algebra.

The additivity of the representations γ_μ and $\bar{\gamma}_\mu$ implies the additivity of $L(\gamma)$ and $L(\bar{\gamma})$ and the multiplicativity of the transformation operators, $U(\beta) = U(\gamma)\,U(\bar{\gamma})$. For the Lagrangian $L_0^{(1)}(\gamma)$ (such fields will be called Larmor fields) we have invariance with respect to the following important group of transformations (in the matrix for the 16 component undor $\psi(x)$ we must take $|m|$):

$$\psi' = U_L^+\psi U_L = \eta_L\gamma_5\psi, \qquad \psi^{+\prime} = U_L^+\psi^+U_L = -\eta_L\psi^+\gamma_5,$$
$$m' = -m, \tag{2.2}$$

$$\psi' = \bar{U}_L^+\psi\bar{U}_L = \bar{\eta}_L\bar{\gamma}_5\psi, \qquad \psi^{+\prime} = \bar{U}_L^+\psi^+\bar{U}_L = +\bar{\eta}_L\psi^+\bar{\gamma}_5,$$
$$m' = +m, \tag{2.3}$$

$$\psi' = U_R^+\psi U_R = \eta_R R_5\psi, \qquad \psi^{+\prime} = U_R^+\psi^+U_R = -\eta_R\psi^+R_5,$$
$$m' = -m, \tag{2.4}$$

where $R_5 = R_1R_2R_3R_4$ is a matrix of class α_c of Sec. 1 introduced earlier[19] and corresponding to the complete space inversion for bosons: $u_J^+ = u_J = R_5$, $\eta_L^2 = \bar{\eta}_L^2 = \eta_R^2 = 1$. The transformations (2.2) to (2.4) form the commutative group of the Larmor transformations, since $u_L\bar{u}_L = u_R$, etc.

The Larmor bosons with $L_0^{(1)}(\bar{\gamma})$ exhibit similar properties. In Kemmer bosons u_L and $\bar{u}_L$ reverse the behavior of ψ upon reflection[8], while the transformation u_R leaves the field equations invariant. If the sign of m is also fixed in the field equations, then u_R reverses the masses of all bosons. The application of u_L and $\bar{u}_L$ to the particular case of the electromagnetic field has already been considered by Larmor[20] and, recently, Takabayasi.[21] The author[8,9] has treated the transformations u_L and $\bar{u}_L$ in boson theory. The analog of u_L is also known in the theory of spinor fields.[5]

Phase transformations with $\delta\varphi = \delta\eta\gamma_5$ leave the equations of Larmor bosons with $L_0^{(1)}(\bar{\gamma})$ ininvariant, whereas phase transformations with

$\delta\varphi = \delta\eta\bar{\gamma}_5$ leave Larmor bosons with $L_0^{(1)}(\gamma)$ invariant. In the first case [for $\eta \to \delta\eta$ both cases are analogous to variant a) of Sec. 1], the pseudovector current $\psi^+\gamma_5\bar{\gamma}_\mu\psi$ of a field with $\bar{\gamma}_\mu$ is conserved, and in the second case, the pseudovector current $\psi^+\bar{\gamma}_5\gamma_\mu\psi$ of a field with γ_μ is conserved. Neither current vanishes for electrically neutral fields.

No similar invariances exist in the case of Kemmer bosons with $\beta_\mu^{(\pm)}$. They do not even emerge in the limit $m \to 0$. As for spinor fields, the reduction of the representations has in this case the effect that the transformation with the operator $\exp(i\eta R_5)$ goes over into a trivial phase transformation, for already in the 8×8 representation the quantity $R_5 \to \pm I$ and the transformations u_L and $\bar{u}_L$ coincide.

Transformations of the type of a general rotation in isospace,

$$\psi' = a\psi + b\gamma_5\psi^c, \qquad \psi^{c\prime} = a\psi^c - b\gamma_5\psi, \qquad |a|^2 + |b|^2 = 1 \tag{2.5}$$

leave $L_0(\gamma)$ invariant, whereas $L_0(\bar{\gamma})$ remains invariant under (2.5) with $\bar{\gamma}_5$. All boson equations are invariant under (2.5) with the matrix R_5.

Lagrangians of second order, $L_0^{(2)}$, are invariant under all transformations of the form

$$\psi^{+\prime} = \eta\psi^+\alpha_{ac}, \qquad \psi' = \eta\alpha_{ac}\psi, \qquad \eta^2 = 1, \tag{2.6}$$

which enlarges the number of conservation laws:

$$\partial_\lambda\,(\psi^+\alpha_{ac}^{(s)}\alpha_\lambda\alpha_\sigma\partial_\sigma\psi + \partial_\sigma\psi^+\alpha_\sigma\alpha_\lambda\alpha_{ac}^{(s)}\psi) = 0. \tag{2.7}$$

Moreover, the new group $\exp(\eta R_5)$ is added to the transformation groups with the operators $\exp(i\eta\gamma_5)$ and $\exp(i\eta\bar{\gamma}_5)$. Supernumerary degrees of freedom appearing in the equations of second order can, if necessary, be dealt with by extracting part of the components of ψ with the help of the invariant projection operators

$$P^{(\pm)}\,(\alpha_{ac}^{(s)}) = {}^1\!/_2\,(I \pm \alpha_{ac}^{(s)}). \tag{2.8}$$

The same operators taken in various combinations may also include the interaction with other fields.

3. CANONICAL TRANSFORMATIONS OF THE HAMILTONIAN

The construction of a complete group of canonical transformations of the Hamiltonian of a free spinor field has been considered by Pac[22] with the help of methods developed by Watanabe.[23] In the theory of bosons, the factorization of the group of basic reflections,

$$G_R \in I,\ R_4,\ R_5,\ R_0 = R_4R_5 \tag{3.1}$$

yields two groups, including the Larmor transformations:

$$G_0 \in I,\ \gamma_4,\ \gamma_5,\ \gamma_0 = i\gamma_4\gamma_5,$$
$$\bar{G}_0 \in I,\ \bar{\gamma}_4,\ \bar{\gamma}_5,\ \bar{\gamma}_0 = i\bar{\gamma}_4\bar{\gamma}_5, \tag{3.2}$$

These have the property

$$[\bar{G}_0 H_0\,(\gamma)] = [G_0 H_0\,(\bar{\gamma})] = 0; \tag{3.3}$$

therefore, the transformations studied in [22] for an anticommuting algebra have additional degeneracies in the case of Larmor bosons.

Let us now consider the generalization of the two basis classes of canonical transformations of the free field Hamiltonian, the Foldy-Tani[24,25] and the Cini-Touschek[26] transformations, to the theory of boson fields. It is sufficient to consider the case of a Larmor field with γ_μ, since the generalization of the results to the case of a Kemmer field presents no difficulties if one makes use of the multiplicativity of the free field transformation operators mentioned in Sec. 2.

Let us introduce the unitary canonical transformations with the Hermitian matrices γ_I and γ_{II} (generalized Foldy-Tani transformations)

$$U_{FT}^+ = N^{-1/2}\,(\gamma_I + \gamma_{II}\gamma_4 H_0\,(\gamma)/|E|),$$
$$U_{FT} = N^{-1/2}\,(\gamma_I + H_0\,(\gamma)\gamma_4\gamma_{II}/|E|), \tag{3.4}$$
$$U_{FT}^+U_{FT} = \gamma_I^2 = \gamma_{II}^2 = I,$$
$$N = (2\,|E| + (1+s)\,m)/\,|E| \tag{3.5}$$

and the generalized Cini-Touschek transformations

$$U_{CT}^+ = N^{-1/2}\,(\gamma_I + \gamma_{II}\alpha_p\,(\gamma)\,H_0\,(\gamma)/\,|E|),$$
$$U_{CT} = N^{-1/2}\,(\gamma_I + H_0\,(\gamma)\,\alpha_p\,(\gamma)\,\gamma_{II}/\,|E|), \tag{3.6}$$
$$U_{CT}^+U_{CT} = \gamma_I^2 = \gamma_{II}^2 = I, \quad N = (2\,|E| + (1+s)\,|\mathbf{p}|)/|E|, \tag{3.7}$$

where $\alpha_p(\gamma) = i\gamma_4(\boldsymbol{\gamma}\mathbf{p})/|\mathbf{p}|$, $\alpha_p^2(\gamma) = I$, and $s = \pm 1$ defines the commutation rules for γ_I and γ_{II}:

$$\gamma_I\gamma_{II} - s\gamma_{II}\gamma_I = 0. \tag{3.8}$$

The matrices γ_I and γ_{II} can commute differently, with $\alpha_p(\gamma)$ and γ_4, thereby changing the result of the transformation. Specifying their behavior by the indices s_I', s_{II}', s_I'', and s_{II}'':

$$\gamma_I\alpha_p\,(\gamma) - s_I'\alpha_p\,(\gamma)\,\gamma_I = \gamma_{II}\alpha_p\,(\gamma) - s_{II}'\alpha_p\,(\gamma)\,\gamma_{II}$$
$$= \gamma_I\gamma_4 - s_I''\gamma_4\gamma_I = \gamma_{II}\gamma_4 - s_{II}''\gamma_4\gamma_{II} = 0, \tag{3.9}$$

we obtain as a condition for the absence of matrices in N

$$s_I's_I'' - ss_{II}'s_{II}'' = 0, \qquad \gamma_I\gamma_{II} = I \text{ for } s = +1. \tag{3.10}$$

Specifically, the following transformations are possible:

I. $s = +1$, $s'_I = s'_{II}$, $s''_I = s''_{II}$; in this case

$$U^+_{FT} H_0(\gamma) U_{FT} = s''_I \gamma_4 |E|, \qquad U^+_{CT} H_0(\gamma) U_{CT} = s'_I \alpha_p(\gamma) |E|. \tag{3.11}$$

For $s'_I = +1$ there exist four transformations with $\gamma_I, \gamma_{II} \in \bar{G}_0$ and four transformations with $\gamma_I, \gamma_{II} \in R_5\bar{G}_0$; for $s'_I = -1$ there exist four variants with $\gamma_I, \gamma_{II} \in R_4\bar{G}_0$ and four variants with $\gamma_I, \gamma_{II} \in R_0\bar{G}_0$. The transformations FT and CT correspond to $\gamma_I = \gamma_{II} = I\bar{G}_0$.

The transformation FT diagonalizes H_0 in the theory of the electron and permits a very simple formulation of quantum electrodynamics in configuration space.[27] This is not possible for boson fields, since γ_4 and $\bar{\gamma}_5$ are not diagonalized, so that β_4 is not diagonalized either. The transformation which diagonalizes H_0 in the theory of Kemmer fields will be considered below.

IIa. $s = -1$, $s'_I = s'_{II} = +1$; there are two variants:

$$s''_I = -s''_{II} = +1, \qquad U^+_{FT} H_0(\gamma) U_{FT} = -\gamma_I \gamma_{II} \gamma_4 |E|,$$
$$U^+_{CT} H_0(\gamma) U_{CT} = H_0(\gamma) \tag{3.12}$$

six transformations with $\gamma_I, \gamma_{II} \in \bar{G}_0, R_5\bar{G}_0$; and

$$s''_I = -s''_{II} = -1, \qquad U^+_{FT} H_0(\gamma) U_{FT} = +\gamma_I \gamma_{II} \gamma_4 |E|,$$
$$U^+_{CT} H_0(\gamma, m) U_{CT} = H_0(\gamma, -m) \tag{3.13}$$

six transformations with $\gamma_I, \gamma_{II} \in R_5\bar{G}_0, \bar{G}_0$.

IIb. $s = -1$, $s'_I = s'_{II} = -1$, $s''_I = -s''_{II} = \pm 1$; here

$$U^+_{FT} H_0(\gamma) U_{FT} = \gamma_I \gamma_{II} \gamma_4 |E|,$$
$$U^+_{CT} H_0(\gamma) U_{CT} = H_0(\gamma) \tag{3.14}$$

sixteen transformations with $\gamma_I, \gamma_{II} \in R_4\bar{G}_0, R_0\bar{G}_0$.

IIc. $s = -1$, $s'_I = -s'_{II} = +1$; we have

$$s''_I = s''_{II} = +1, \qquad U^+_{FT} H_0(\gamma) U_{FT} = -\gamma_I \gamma_{II} \gamma_4 |E|,$$
$$U^+_{CT} H_0(\gamma) U_{CT} = +\gamma_I \gamma_{II} \alpha_p(\gamma) |E| \tag{3.15}$$

six transformations with $\gamma_I, \gamma_{II} \in \bar{G}_0, R_4\bar{G}_0$;

$$s''_I = -s''_{II} = -1, \qquad U^+_{FT} H_0(\gamma, m) U_{FT} = H_0(\gamma, -m),$$
$$U^+_{CT} H_0(\gamma) U_{CT} = -\gamma_I \gamma_{II} \alpha_p(\gamma) |E| \tag{3.16}$$

ten transformations with $\gamma_I, \gamma_{II} \in R_0G_0, R_5\bar{G}_0$.

IId. $s = -1$, $s'_I = -s'_{II} = -1$; here

$$s''_I = s''_{II} = +1, \qquad U^+_{FT} H_0(\gamma, \mathbf{p}) U_{FT} = H_0(\gamma, -\mathbf{p}),$$
$$U^+_{CT} H_0(\gamma) U_{CT} = \gamma_I \gamma_{II} \alpha_p(\gamma) |E| \tag{3.17}$$

six transformations with $\gamma_I, \gamma_{II} \in \bar{G}_0, R_4\bar{G}_0$;

$$s''_I = s''_{II} = -1, \qquad U^+_{FT} H_0(\gamma) U_{FT} = -H_0(\gamma),$$
$$U^+_{CT} H_0(\gamma) U_{CT} = \gamma_I \gamma_{II} \alpha_p(\gamma) |E| \tag{3.18}$$

ten transformations with $\gamma_I, \gamma_{II} \in R_0\bar{G}_0, R_5\bar{G}_0$.

Among the transformations (3.13) and (3.15) there is a transformation which diagonalizes $H_0(\gamma)$, where $\pm\gamma_I\gamma_{II}\gamma_4 = R_0$. For Kemmer fields, this operation does not give the needed result, since the corresponding operators $U_{FT}(\gamma)$ and $U_{FT}(\bar{\gamma})$ do not commute and $U(\beta) \neq U(\gamma)U(\bar{\gamma})$. The exclusion of the longitudinal components of the Kemmer field and use of the Hamiltonian $H_0^T(\beta) = \beta_4(m^{-1}(\beta, \mathbf{p})^2 + m)$ in U_{FT} leads to a very narrow group of transformations (four in number) giving the same result: $U^+_{FT} H_0^T(\beta) U_{FT} = R_0 |E|$.

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