REGGE POLES IN QUASICLASSICAL POTENTIAL WELL PROBLEMS

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The problem of poles of the scattering phase-shift (Regge poles) is investigated for the case of a rectangular and spherically symmetric potential well. In this case the scattering phase-shift has an explicit expression in terms of Bessel functions. In looking for poles of the scattering phase-shift, a previously developed method is used to trace the properties of the phase shift along the level lines. Two series of poles are found: "physical" and "unphysical." The character of the motion of the poles as the energy varies is then clarified. Finally, some general relations are established between the number of levels and the number of resonances. The simplest form of the potential well has been chosen in order not to complicate the calculations with inessential details. However, the results remain in essence of general validity for potentials which are singular outside the point r = 0.

THE analytic properties of the scattering matrix $S(\nu)$ in the complex angular momentum plane have been very intensively investigated recently [1-4]. We have recently proposed a method for finding the poles of $S(\nu)$ (the Regge poles) for quasiclassical potentials [5]. In the present paper this method is used for analyzing the simplest problem of Regge poles for the case of a spherically symmetric rectangular potential well. In this case $S(\nu)$ can be explicitly expressed in terms of Bessel functions. The problem for other non-analytic potentials, for which the potential energy is represented by different functions on different portions of the r axis, can be solved in quite analogous manner. The simplest form of potential well has been chosen for detailed study, in order not to encumber the analysis with inessential details. The results remain essentially valid for potentials having their singularities outside the point $r = 0$.

In what follows $\hbar = 1$ is assumed throughout.

1. THE FUNCTION $S(\nu)$. SOME GENERAL THEOREMS

We recall the usual definitions. One solves the radial Schrödinger equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left(k^2-2\mu U(r)-\frac{\nu^2-1/4}{r^2}\right)R=0, \quad (1)$$

where $k^2 = 2\mu E$. The potential $U(r)$ is specified as follows:

$$U(r) = -U_0 \text{ for } r < a, \qquad U(r) = 0 \text{ for } r > a. \quad (2)$$

A physically meaningful solution R_ν behaves as $r^{\nu-1/2}$ (for half-integral values of $\nu > 0$) as $r \to 0$. We are, however, interested in R_ν for an arbitrary complex ν. As $r \to \infty$ the function can be asymptotically represented as

$$R_\nu = \frac{1}{r}\left(a(\nu)\, e^{ikr} + b(\nu)\, e^{-ikr}\right).$$

The function $S(\nu)$ (the scattering matrix) is related to the asymptotic relation by the equation

$$S(\nu) = ie^{i\nu\pi} a(\nu)/b(\nu). \quad (3)$$

It is not difficult to derive some general properties of the function $S(\nu)$ (which are valid for any potential $U(r)$ which is real for $r > 0$). The function $S(\nu)$ possesses the following properties which are not connected with the concrete form of the potential $U(r)$; the latter is supposed to satisfy the conditions: $rU(r) \to 0$ for $r \to \infty$, $r^2U(r) \to 0$ for $r \to 0$, and $r^2U(r)$ can be expanded in a power series in r in a certain neighborhood of $r = 0$.

A. $S(\nu, E)$ is a meromorphic function of ν throughout the whole ν plane for fixed E. Indeed, ν occurs in the Schrödinger equation (1) and in the boundary condition for R_ν as $r \to 0$ in an analytic manner. Therefore R_ν is an entire function of ν for fixed E and hence the coefficients $a(\nu)$ and $b(\nu)$ are entire functions of ν. Equation (3) then implies that the only singularities of $S(\nu)$ can be poles. Furthermore $\nu = \infty$ can be an essential singularity for $S(\nu)$.

B. For fixed ν, $S(\nu, E)$ is a single-valued function of E in a plane with a cut running along the real negative axis. The point $E = 0$ is a branch point of $S(\nu, E)$. The physical reason for the appearance of the singularity at $E = 0$ is that for this value of E one can no longer distinguish the incoming wave from the outgoing wave. For the solution $R_{\nu,E}$ the point $E = 0$ is regular.

C. $S(\nu, E)$ possesses the unitarity property:

$$S(\nu, E) S^*(\nu^*, E^*) = 1. \tag{4}$$

This follows from the reality of $U(r)$. Indeed, from Eq. (1) and the boundary condition for $R_{\nu,E}$ it follows that

$$R_{\nu^*, E^*} = R^*_{\nu, E}. \tag{5}$$

With our choice of the cut, complex conjugate values of k correspond to complex conjugate values of E. Substituting the asymptotic form of $R_{\nu,E}$ for $r \to \infty$ into Eq. (5), we find

$$a(\nu^*, E^*) = b^*(\nu, E), \qquad b(\nu^*, E^*) = a^*(\nu, E).$$

This leads to Eq. (4). We emphasize the fact that for negative values of E one has to consider as complex conjugate points which are situated on opposite sides of the cut. For negative E the following equality holds:

$$S(\nu^*, E) = -e^{2i\pi\nu^*} S^*(\nu, E). \tag{6}$$

From (4) it follows that for real $E > 0$, to any pole of $S(\nu, E)$ in a point ν_k there corresponds a zero of $S(\nu, E)$ in the point ν_k^*. For $E < 0$, Eq. (6) implies that the poles are either situated on the real axis, or occur in pairs in complex conjugate points.

As E varies, a pole of $S(\nu, E)$ describes a trajectory $\nu_n(E)$ (a Regge trajectory). A Regge trajectory can neither start nor end in points corresponding to values of E inside the domain in which $S(\nu, E)$ is analytic in E. The same is also true for the zeros of $S(\nu, E)$.

The solution of Eq. (1) with the potential (2) has the form

$$R_\nu(r) = \begin{cases} J_\nu(k_1 r)/\sqrt{r}, & r < a, \\ [A_\nu H_\nu^{(1)}(kr) + B_\nu H_\nu^{(2)}(kr)]/\sqrt{r}, & r > a, \end{cases} \tag{7}$$

where $k_1^2 = 2\mu(E - U)$, J_ν and H_ν are Bessel and Hankel functions, respectively, A_ν and B_ν are constants which are determined from the continuity of R_ν and dR_ν/dr in the point $r = a$.

According to the definition (3), we find

$$S(\nu) = -\frac{x_1 J_\nu'(x_1) H_\nu^{(2)}(x) - x J_\nu(x_1) H_\nu^{(2)\prime}(x)}{x_1 J_\nu'(x_1) H_\nu^{(1)}(x) - x J_\nu(x_1) H_\nu^{(1)\prime}(x)}, \tag{8}$$

where $x = ka$, $x_1 = k_1 a$. The poles of $S(\nu)$ are determined by the equation

$$x_1 J_\nu'(x_1)/J_\nu(x_1) = x H_\nu^{(1)\prime}(x)/H_\nu^{(1)}(x), \tag{9}$$

since the denominator and numerator of the fraction in the right side of Eq. (8) cannot vanish simultaneously.

We go on to an examination of the solutions of Eq. (9).

2. THE QUASICLASSICAL ASYMPTOTIC BEHAVIOR OF BESSEL AND HANKEL FUNCTIONS FOR COMPLEX VALUES OF THE ARGUMENT AND OF THE INDEX

It is possible to solve Eq. (9) explicitly for arbitrary values of the energy, only for the case of a quasiclassical potential well. The condition of quasiclassical behavior

$$\mu U_0 a^2 \gg 1 \tag{10}$$

has the physical meaning of the existence of a large number of energy levels within the well. Since

$$x_1^2 - x^2 = 2\mu U_0 a^2, \tag{11}$$

either x or x_1 has to be large, and one may use the asymptotic expression for the corresponding function. We shall be interested in complex values the indices ν. Therefore we have to investigate in detail the asymptotic behavior of Bessel and Hankel functions for complex values of both ν and the arguments.

The functions under investigation are particular solutions of the Bessel equation

$$\frac{d^2 Z_\nu}{dx^2} + \frac{1}{x}\frac{dZ_\nu}{dx} + \left(1 - \frac{\nu^2}{x^2}\right) Z_\nu = 0. \tag{12}$$

In the quasiclassical approximation Eq. (12) has approximate solutions of the form

$$Z_\pm(x, \nu) = (x^2 - \nu^2)^{-1/4} \exp(\pm i\Phi(x, \nu)), \tag{13}$$

$$\Phi(x, \nu) = \int_\nu^x \sqrt{1 - \frac{\nu^2}{\xi^2}}\, d\xi = \sqrt{x^2 - \nu^2} - \nu \arccos\frac{\nu}{x}. \tag{14}$$

In order that one can use the solution (13) the inequality $|\Phi(\nu, x)| \gg 1$ must be satisfied.

The general solution $Z_\nu(x)$ can be represented in the form

$$Z_\nu(x) = aZ_+ + bZ_-, \tag{15}$$

where a and b do not depend on x. It makes sense to write the solution in the form (15) only if both terms in the right side of (15) are of the same order of magnitude, since the solutions $Z_\pm$ are themselves approximate, and hence the subdivision into incoming and outgoing waves is determined with an accuracy of the order of $1/|\Phi(\nu, x)|$. The coefficients a and b attain an exact significance only for $x \to \infty$ and $x \to 0$, where $|\Phi(\nu, x)| \to \infty$. Asymptotically, for $|x| \gg |\nu|$, we have

$$Z_{\pm}(x, \nu) \approx \frac{1}{\sqrt{x}} \exp\left(\pm ix \mp i\frac{\pi\nu}{2}\right) \quad (16)$$

and for $x \to 0$

$$Z_{\pm}(x, \nu) \approx \frac{e^{-i\pi/4}}{\sqrt{\nu}} \left(\frac{x}{2\nu e}\right)^{\pm\nu}. \quad (17)$$

We note, that there is no one-to-one correspondence between the signs on both sides of the equalities (16) and (17). This is due to the fact that upon going around the point $x = \nu$ the quantity $(1 - \nu^2/x^2)^{1/2}$ changes sign and therefore the definition of $Z_{\pm}(x, \nu)$ is in itself ambiguous. Nevertheless, for the true solution $Z_\nu(x, \nu)$ the point $x = \nu$ is regular, so that only the asymptotic expressions of type (15) change in this point.

The solutions $Z_\nu(x, \nu)$ are of the same order of magnitude only within a very narrow strip around the "level line" L:

$$\operatorname{Im} \Phi(x, \nu) = 0. \quad (18)$$

Let us consider first the behavior of the line L for $|x| \ll \nu$. In this region Eq. (18) goes over into the equation of the logarithmic spiral:

$$\ln|x| \operatorname{Re}\nu - \arg x \cdot \operatorname{Im}\nu = \text{const}. \quad (19)$$

One must, however, take into account the fact that the origin is an essential singularity of type $x^{\pm\nu}$ for the function $Z(x, \nu)$ and the latter becomes single-valued only on a many-sheeted Riemann surface. We shall always deal only with one sheet $|\arg x| < \pi$ (the "physical sheet"), i.e., we choose to place the cut along the negative x axis. The turns of the spiral (19) are situated on different sheets of the Riemann surface. On the physical sheet the spiral comes out from under the cut, straightens out, and goes into the point $x = \nu$.

Three branches of the level line meet in the point $x = \nu$, since in the neighborhood of this point Eq. (18) has the form

$$\operatorname{Im} (x - \nu)^{3/2}/\sqrt{\nu} = 0. \quad (20)$$

We already know that one of the three branches, L_1, goes into the origin coiling up into a spiral. The other two branches (L_2, L_3) go off into the directions $\operatorname{Re} x \to \pm\infty$, so that at infinity these two branches tend asymptotically to the straight lines $\operatorname{Im} x = \pm \frac{1}{2}\pi \operatorname{Im}\nu$. A more detailed analysis shows that for $\operatorname{Re}\nu > 0$, $\operatorname{Im}\nu > 0$ the left branch L_2 approaches the asymptote from above and the right branch L_3 approaches it from beneath. Under a change of sign the left and the right branches change places. The approximate form of the level line for the case $\operatorname{Re}\nu > 0$, $\operatorname{Im}\nu > 0$ is shown in Fig. 1. On the same figure we have displayed the level line L' which goes through the other turning point $x = -\nu$.

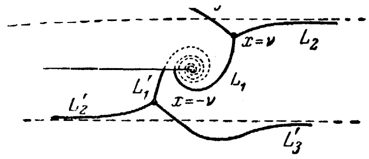

FIG. 1

For real values of ν the form of the line L changes slightly, since now two branch points $x = \pm\nu$ appear on it, in place of only one (cf. Fig. 2). In this case there exist no level lines leading from the points $x = \pm\nu$ into point $x = 0$.

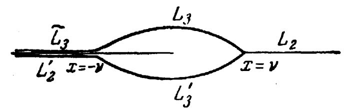

FIG. 2

As one moves along the level line the coefficients in the linear combination (15) remain the same (of course, with the above mentioned precisions). The coefficients may change (as mentioned) only upon passing the "turning points" $x = \pm\nu$, where the solutions $Z_{\pm}$ are no longer good approximations. Let us see in which way the knowledge of the coefficients on one of the branches of the level line allows one to find the coefficients in the linear combination (15) on the other branches[1]. In order to do this we shall go over from one branch to the other far away from the turning point, so that our path always lies within the domain of validity of the quasiclassical approximation.

Let the solutions have the form $a_i Z_+ + b_i Z_+$ on the lines L_i (Fig. 3), respectively. If Z_+ is large in region I, then evidently $a_1 = a_2$ since one must obtain the same function as one moves in the region I from the branch L_1 to the branch L_2. Similarly one obtains that $b_2 = b_3$.

One cannot equate the constants a_3 and a_1, since the quantity $(x^2 - \nu^2)^{1/2}$ changes sign on going around the point $x = \nu$. Let us place a cut

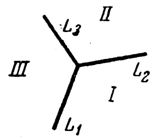

FIG. 3

[1]The idea of the following line of reasoning is due to Furry.[6]

along the line L_3. Let us denote the solutions on the upper side of the cut by $Z_\pm$, as before, and on the lower side of the cut we denote them by $\tilde{Z}_\pm$. It follows from (13) that

$$Z_\pm = -i\tilde{Z}_\mp. \tag{21}$$

The reasoning carried out above shows that $b_1 = b_3 = -ia_3$, with the coefficient b_2 undetermined.

Let us pose a more restricted problem, supposing that $a_1 = 0$, $b_1 = 1$, and that the solution is represented by an exponential which is small inside the region I. Then, as a consequence of what was said before, $a_2 = 0$ and the solution equals Z_- within the whole section I, including its boundary lines L_2 and L_3. This implies $b_2 = b_3 = 1$. For such a solution the asymptotic forms are the following:

$$Z = \begin{cases} Z_- & \text{on } L_1 \text{ and } L_2, \\ Z_- + iZ_+ & \text{on } L_3. \end{cases} \tag{22}$$

Similarly one can obtain the other solutions:

$$Z = \begin{cases} Z_+ & \text{on } L_1, \\ \tilde{Z}_+ = +iZ_- & \text{on } L_3, \\ Z_+ + iZ_- & \text{on } L_2. \end{cases} \tag{23}$$

$$Z = \begin{cases} Z_+ & \text{on } L_2 \text{ and } L_3, \\ Z_+ - iZ_- & \text{on } L_1. \end{cases} \tag{24}$$

Let us apply the preceding reasoning to the solutions of the Bessel equation. The Bessel function $J_\nu(x)$ is defined as that solution which behaves like a small exponential in the domain bounded by the lines L_1 and L_3. Indeed, $J_\nu(x)$ behaves as x^ν as $x \to 0$ on any sheet of its Riemann surface. Therefore $J_\nu(x)$ is represented by the scheme (23). The same scheme also represents the function $J_\nu(x)$ on the line L' (evidently, with the replacement of the branches L_i by L_i'). To determine the asymptotic behavior of the function $J_\nu(x)$ in the lower x half-plane one may also use the well-known relation $J_\nu(e^{-i\pi}x) = e^{-i\nu\pi}J_\nu(x)$. The function $H^{(1)}(x)$ is defined by its asymptotic behavior as $x \to \infty$:

$$H_\nu^{(1)}(x) \sim \exp\{i(x - \pi\nu/2)\}.$$

Consequently, $H_\nu^{(1)}(x)$ is represented by a small exponential in the domain bounded by the branches L_2 and L_3 and this function corresponds to the scheme (24). On the branch L' the function $H_\nu^{(1)}(x)$ is represented by the following scheme

$$H_\nu^{(1)}(x) = e^{i\nu\pi} \begin{cases} Z_+(-\nu, x) & \text{on } L_1 \text{ and } L_3', \\ Z_+(-\nu, x) + iZ_-(-\nu, x) & \text{on } L_2'. \end{cases} \tag{25}$$

Similarly one can find the asymptotic forms of the function $H_\nu^{(2)}(x)$.

Near the "turning point" $x = \nu$ all mentioned functions are expressible in terms of the Airy function $\mathrm{Ai}(\zeta)$, where

$$\tfrac{2}{3}\zeta^{3/2} = \Phi(x, \nu). \tag{26}$$

The argument of ζ is assumed to vanish on that branch of the level line on which the function $Z_\nu(x)$ is represented by two exponential functions.

In the case of real values of ν the character of the level lines changes in such a way that there is no level line connecting the points $x = 0$ and $x = \nu$ (cf. Fig. 2). Therefore one must solve anew the problem of finding the asymptotic behavior of $J_\nu(x)$ in this case. If $\nu > 0$ then $J_\nu(x)$ is, by definition, represented by a small exponential $Z_+(\nu, x)$ inside the "eye" (Fig. 2) and therefore also on its boundaries L_3 and L_3'. On the branches L_2, L_2', $\tilde{L}_3$ the function $J_\nu(x)$ is represented by two exponentials according to the scheme:

$$J_\nu(x) = \begin{cases} Z_+ + iZ_- & \text{on } L_2, \\ Z_+ - iZ_- & \text{on } \tilde{L}_3. \end{cases} \tag{27}$$

For $\nu < 0$, one cannot apply the same reasoning, since inside the "eye" $J_\nu(x)$ is represented by a large exponential, at least for values of ν which are not close to integral values. In this case one may make use of the connection between $J_\nu(x)$ and $H_\nu(x)$:

$$J_\nu(x) = \tfrac{1}{2}(H_\nu^{(1)}(x) + H_\nu^{(2)}(x)).$$

The asymptotic forms of $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are as follows:

$$H_\nu^{(1)} = \begin{cases} Z_+ & \text{on } L_2,\ L_3 \\ Z_+ + iZ_- & \text{on } L_3' \end{cases}; \tag{28}$$

$$H_\nu^{(2)} = \begin{cases} -iZ_- & \text{on } L_2,\ L_3', \\ -i(Z_- + \tilde{Z}_+) & \text{on } L_3, \\ -(iZ_- + Z_+(1 - e^{i\nu\pi})) & \text{on } \tilde{L}_3. \end{cases}$$

One can see from these expressions that, as before, $J_\nu(x)$ is represented by the scheme (27).

For negative integer ν the function $J_\nu(x)$ becomes $J_{-\nu}(x)$. Therefore, near negative integer values of ν the asymptotic form of $J_\nu(x)$ inside the "eye" is not represented by a large exponential. In order to clarify the situation in this case one must use the formula

$$J_\nu(x) = e^{i\nu\pi}J_{-\nu}(x) - ie^{i\nu\pi}\sin\nu\pi H_\nu^{(1)}(x). \tag{29}$$

In what follows we shall be interested in the asymptotic form of $J_\nu(x)$ for values of ν lying close to an integer, when x is inside the "eye." In this case we have

$$J_\nu(x) = e^{-i\pi/4}(Z_+ - 2\sin\nu\pi Z_-)/\sqrt{2\pi}. \quad (30)$$

The term in Z_- is the principal one when $|\sin\nu\pi| \sim 1$, but in the neighborhood of integral values of ν the role of the term in Z_+ increases.

3. THE LINES NEAR WHICH THE POLES OF $S(\nu, E)$ ARE SITUATED FOR FIXED ENERGY

Let us return to Eq. (9), which determines the position of the poles of $S(\nu)$. We fix the energy E, i.e., the position of the points x and x_1. Inside the domains bounded by the level lines L and L′ the functions $J_\nu(x)$ and $H_\nu(x)$ are represented by one of the solutions of the form (16), as was shown above. If x and x_1 were simultaneously situated inside such domains, it would be impossible to satisfy Eq. (9), since this equation would take the form $(x^2 - \nu^2)^{1/2} = \pm(x_1{}^2 - \nu^2)^{1/2}$. It is necessary that either $J_\nu(x_1)$ or $H'_\nu(x)$ be represented by a superposition of Z_+ and Z_- with nonvanishing coefficients. For this it is necessary either for x_1 to lie on L_2 (or L'_2), or for x to lie on L_1 (or on L'_2). Since $x_1 > 0$ the possibility of x_1 lying on L'_2 is excluded. The solutions of Eq. (9) must be real in this case and must approximately satisfy the condition that x_1 be situated on the branch L_2.

From Fig. 2 it is obvious that a necessary condition for this is $\nu < x_1$. Our reasoning till now has referred only to the case $\nu > 0$. For $\nu < 0$ the same asymptotic forms are valid provided $|\nu| < x_1$, and the asymptotic form (30) is valid for $|\nu| > x_1$. In both cases $J_\nu(x_1)$ is represented by a superposition of Z_+ and Z_-, and consequently it is possible for Eq. (9) to have a solution. Thus one of the lines near which the poles of $S(\nu)$ are situated is the straight line $\nu < x_1$ (the line M_1 in Figs. 4 and 5).

The second line of poles M_2 is determined by the condition that x be situated on the branch L_1. The quantity x is purely imaginary for $E < 0$ (we agree to put $x = i\xi$, $\xi > 0$) and is real for $E > 0$ ($x > 0$). In the first case ($E < 0$) the poles are aligned near the imaginary axis, so that $|\operatorname{Im}\nu| > \xi$ (Fig. 4). In the second case ($E > 0$), as can be seen from Fig. 1, the line of poles is situated in the complex ν-plane and possesses two branches symmetrical about the origin. They start out from the points $\nu = \pm x$ under an angle of $\pi/3$ with respect to the real ν axis and are determined by the equation

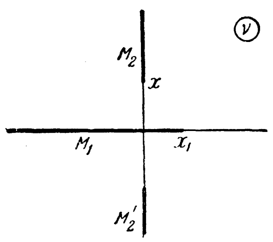

FIG. 4

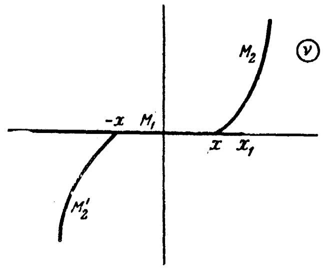

FIG. 5

$$\operatorname{Im}\Phi(\nu, \pm x) = 0 \quad (31)$$

(evidently, one must ake care that ν be situated on the branch L'_2). For large ν Eq. (31) yields asymptotically

$$\operatorname{Re}\nu = \operatorname{Im}\nu\left(\pi/2 \ln\frac{2|\nu|}{xe}\right). \quad (32)$$

The form of the curves M_2 and M'_2 is shown in Fig. 5. The form of the M-curves will be determined more precisely below.

4. THE "PHYSICAL" SERIES OF POLES

We define as "physical" the series of poles situated near the real straight line (M_1 in Figs. 4 and 5), since for certain values of the energy the poles of this series correspond to stationary states in the potential well.

We start from the case of negative values of the energy $E > -U_0$. In this case $x = i\xi$ is purely imaginary and $x_1 > 0$. Let us also suppose that $x_1 \gg 1$. Then $J_\nu(x_1)$ and $\exp\{i(\nu\pi/2 + \pi/2)\}H_\nu^{(1)}(x)$ are real for real ν. Consequently, Eq. (9) has purely real solutions. According to (27) we have on the pole line, for $|\nu| < x_1$ (note that x_1 lies on L_2):

$$J_\nu(x_1) = \frac{2(x_1^2 - \nu^2)^{-1/4}}{\sqrt{2\pi}}\cos\left[\Phi(x, \nu)_1 - \frac{\pi}{4}\right]. \quad (33)$$

The function $H_\nu^{(1)}(x)$ is represented by one exponential Z_- only. Substituting (33) in (9) we find

$$\operatorname{tg}(\Phi(x_1, \nu) - \pi/4) = \sqrt{(\xi^2 + \nu^2)/(x_1^2 - \nu^2)}. \quad (34)^*$$

The right hand side of Eq. (34) is a slowly varying function of ν and the left hand side is a quickly varying one.

*tg = tan, arctg = $\tan^{-1}$, ctg = cot.

Equation (33) yields the series of poles ν_n:

$$\Phi(x_1, \nu_n) = n\pi + \pi/4 + \gamma(n), \qquad (35)$$

where

$$\gamma(n) = \operatorname{arctg}\sqrt{(\xi^2 + \nu^2)/(x_1^2 - \nu^2)}. \qquad (36)$$

It is not difficult to find the distance $\delta\nu_n$ between neighboring poles, which is a slowly varying function of ν:

$$\delta\nu_n = -\pi/\arccos\frac{\nu_n}{x_1}. \qquad (37)$$

It is easy to count the number of poles between $\nu = -x_1$ and $\nu = x_1$, the result being approximately equal to x_1, and consequently the average distance between the poles equals two. The distance between poles increases from left to right. For $\nu_n = -x_1$ we have $\delta\nu = -1$.

Equations (33)—(37) lose their validity for $|\nu - x_1| \lesssim x_1^{1/3}$. In this case one must express the functions $J_\nu(x)$ in terms of the Airy functions $\mathrm{Ai}(\zeta)$, where ζ is defined by Eq. (26). The first poles of the physical series are determined by the equation

$$\nu_n = x_1 - (x_1/2)^{1/3}\zeta_n, \qquad (38)$$

where ζ_n $(n \sim 1)$ are the zeros of the Airy functions.

Equations (33)—(37) lose their validity also for $\nu < x_1$ $(|\nu + x_1| \gg x_1^{1/3}$. In this case one has to use the asymptotic form (30). The solutions of Eq. (9) are close to integral values $\nu = -m$. Taking this into account one obtains an equation for the determination of $\sigma = \pi(\nu + m)$:

$$(e^{i\Phi_m} + 2\sigma e^{-i\Phi_m})/(e^{i\Phi_m} - 2\sigma e^{-i\Phi_m}) = \sqrt{(\xi^2 + m^2)/(m^2 - x_1^2)} = y_m, \qquad (39)$$

whence

$$\sigma = {}^1/_2 e^{2i\Phi_m}(y_m - 1)/(y_m + 1), \quad \Phi_m = \Phi(-m, x_1). \qquad (40)$$

The inequality $\exp(i\Phi(-m, x_1)) \ll 1$ implies that $\sigma \ll 1$ and that the poles are indeed close to negative integral points. For $m \gg x_1$ the quantities σ decrease with increasing m according to the formula

$$\sigma = \frac{m^2}{\eta U_0^2 a^2}\frac{1}{(m!)^2}\left(\frac{x_1}{2m}\right)^{2m}. \qquad (41)$$

The values of the energy for which a pole $\nu_n(E)$ has a half-integral positive value correspond to stationary states. In this case the condition (35) coincides with the Bohr quantization rule for a rectangular spherical potential well.

Let us now consider the case $x_1 \ll 1$ $(\xi \gg 1)$. The corresponding energy values are close to the bottom of the well $E + U_0 \ll U_0$ so that there can be no stationary state with such an energy. In this case one has to use a power expansion of $J_\nu(x_1)$. Obviously, the poles of the physical series are situated near the negative integral values of ν and Eq. (41) remains valid without any restrictions on m.

In the case $x_1 \sim 1$ and $\nu \sim 1$, (9) takes the form

$$x_1 J_\nu'(x_1)/J_\nu(x_1) \approx -\xi. \qquad (42)$$

The roots of this equation are situated to the left of the point $\nu = x_1$, at a distance ~ 1 from each other.

The complete picture of the motion of the physical series of poles is as follows: for values of E close to U_0 the poles are "attached" to the negative integral points. As the energy increases the poles that are situated to the left of $-x_1$ remain almost stationary, whereas the poles situated to the right of $-x_1$ move in such a manner, that the first pole comes after x_1 at a distance $x_1^{1/3}$ and the average distance between the poles in the interval $(-x_1, x_1)$ equals 2. This picture is in agreement with the principle of non-disappearance of poles.

Let us go on to the case $E > 0$. We shall consider first small values of the energy $x \ll 1$. The asymptotic forms of the functions $J_\nu(x_1)$ remain unchanged, but the $H_\nu^{(1)}(x)$ become almost purely imaginary. It is nevertheless very important to determine the small real part of $H_\nu^{(1)}(x)$, since it is this part that determines the imaginary part of the poles. We shall use for this purpose the well-known relation

$$H_\nu^{(1)}(x) = i\,\frac{e^{-i\nu\pi}J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi},$$

which for small x yields

$$H_\nu^{(1)}(x) \approx -\frac{i}{\pi}\left(\frac{x}{2}\right)^{-\nu}\Gamma(\nu) + \left(\frac{x}{2}\right)^{\nu}\Big/\Gamma(\nu+1). \qquad (43)$$

For $|\nu| < x_1$ the poles are determined by the equation (cf. (34))

$$\operatorname{tg}(\Phi(x,\nu) - \pi/4) = (|\nu| - i2\pi(\Gamma(|\nu|)^{-2}x^{2|\nu|})/\sqrt{x^2 - \nu^2}. \qquad (44)$$

It follows from here that

$$\operatorname{Im}\nu = \frac{2\pi(x/2)^{2|\nu|}\sqrt{x_1^2 - \nu^2}}{(\Gamma(|\nu|))^2 x_1^2 \arccos(\nu/x_1)} > 0. \qquad (45)$$

Equation (45) is valid in the region $\nu \ln x \gg 1$. For small $\nu \lesssim 1/\ln x$ one must use a different relation:

$$\operatorname{Im}\nu \approx 4\nu^2/x_1\,[(x/2)^{\nu} - (x/2)^{-\nu}]^2, \qquad (46)$$

which asymptotically coincides with (45) as $1/\ln x \ll \nu \ll x_1$.

In the region $\nu < x_1$, where the poles are close to integral points, we obtain

$$\operatorname{Im} \nu = \frac{8i\Phi_m (x/2)^{2m}}{(\Gamma(m))^2 \sqrt{m^2 - x_1^2 (1-y_m)^2}}, \qquad (47)$$

where Φ_m and y_m are defined in Eqs. (39) and (40).

As x_1 increases, the poles for which $-x_1 < \operatorname{Re}\nu < x_1$ move along behind x_1 in such a manner that $\operatorname{Re}(\nu_n - x_1)$ remains almost constant. When the pole ν_n passes near the value $\nu_0 \sim m + 1/2$, there appears a resonance in this state ($l = m$). For a resonance to appear it is essential that $\operatorname{Im} \nu_n << 1$. It can be seen from Eqs. (45) and (46) that a resonance is possible if $x/\nu_0 \ll 1$. For $x = 0$ the poles are situated on the real axis at distances ~ 1 from the half-integral positive points. They will reach the vicinity of these points for a variation $\delta x_1 \sim 1$, i.e., for $\delta x \sim x_1/x$; for the appearance of a resonance it is necessary that $\delta x \ll \nu$, i.e., $x_1/\nu \ll x \ll \nu$. This condition can be satisfied for large values of $x_1^2 - x^2$ for values of $\nu_0 \gg (x_1)^{1/2}$, i.e., for states with large angular momenta.

Resonances in a state with $\nu_0 \sim 1$ are possible if there exist poles close to small half-integral ν for $E = 0$.

The form of the dependence of $\operatorname{Im}(\nu)$ on ν with a maximum at $\nu = 0$, and the monotonic decrease as $|\nu|$ increases up to x_1, have a simple physical explanation. The effective potential $U_{eff} = U + \nu^2/2\mu r^2$ possesses a potential barrier. The penetration of particles through this barrier for $E > 0$ leads to the appearance of $\operatorname{Im} \nu$. It is clear that as ν^2 increases, the height and width of the barrier increase, which in turn explains the decrease of $\operatorname{Im} \nu$.

By means of simple physical considerations it is not difficult to obtain the dependence of $\operatorname{Im} \nu$ on ν and E, at least up to slowly varying factors. We recall first that the average number of oscillations which the particle performs inside the well before leaving it is equal to the inverse of the probability P of barrier penetration for one passage. However, according to the well-known formula

$$P = \exp\left\{-2\int_a^{\nu/k} \sqrt{\nu^2/r^2 - k^2}\, dr\right\} = \exp\{2i\Phi(x, \nu)\}. \qquad (48)$$

(Eq. (48) is true up to a factor in front of the exponential.) On the other hand, the average number of oscillations equals $(2\pi \operatorname{Im} n)^{-1}$, where n is the number of the level ($n = I/\hbar$, where I is the adiabatic invariant).

Hence we find

$$\operatorname{Im} \nu = \operatorname{Im} n \frac{\partial \nu}{\partial n} = \frac{1}{2\pi P}\frac{\partial \nu}{\partial n}. \qquad (49)$$

Let us determine $\partial\nu/\partial n$ from the Bohr quantization condition (35):

$$\frac{\partial \nu}{\partial n} = \frac{\partial \Phi}{\partial n} \Big/ \frac{\partial \Phi}{\partial \nu} = -\frac{\pi}{\arccos \nu}. \qquad (50)$$

Eqs. (48), (49), (50) again yield (45) up to the factors $(x_1^2 - \nu^2)^{1/2}/x_3^2$. The missing factors can be obtained by making Eq. (48) more precise. (One has then to take into account that the quasiclassical approximation is not applicable near $r = a$, where the potential suffers a discontinuity.)

Let us go over to the case $x, x_1 \gg 1$. Here there are two possibilities.

A. $x_1 - x \gg x_1^{1/3}$, which is equivalent to $E \ll U_0 (\mu U_0 a^2)^{1/2}$. In this case there are a multitude of poles between x and x_1. We start with the poles for which $\operatorname{Re} \nu < x$. For these poles Eq. (9) takes on the form

$$\operatorname{tg}(\Phi(\nu, x_1) - \pi/4) = -i\sqrt{(x^2-\nu^2)/(x_1^2-\nu^2)} = -iy_\nu. \qquad (51)$$

The Bohr condition has the previous form [cf. Eq. (35)], but $\gamma(n)$ is now a complex quantity, so that the right hand side of the equation is complex. We find from (51)

$$\operatorname{Im} \nu = \frac{1}{2\arccos(\nu/x_1)} \ln\frac{1+y_\nu}{1-y_\nu}. \qquad (52)$$

According to Eq. (52) the maximum of $\operatorname{Im}(\nu)$ is realized for a positive value $\nu_0(x) = x^2/x_1$* and equals $\nu/(x^2 - \nu^2)^{1/2}$. Thus, the maximum is displaced towards large ν as the energy increases. For small energies ($E \ll U_0$, $x \ll x_1$)

$$\operatorname{Im} \nu \approx 2\sqrt{x^2 - \nu^2}/\pi x_1 \leqslant 1. \qquad (53)$$

For $E \sim U_0$, $x \sim x_1$ Eq. (52) has a maximum for $\nu_0 \sim x$, the value of which is of the order of one. Finally, for $E \gg U_0$ and $x - \nu \gg x_1 - x$ ($1 - y_\nu \ll 1$), we obtain from (52)

$$\operatorname{Im} \nu \approx \frac{1}{2\arccos(\nu/x_1)} \ln\frac{x^2-\nu^2}{x_1(x_1-x)} \sim \ln\frac{E}{U_0}. \qquad (54)$$

Equation (54) is valid far from the maximum of $\operatorname{Im} \nu$ and to the left of it. The maximum of $\operatorname{Im}(\nu)$ is situated at a value ν_0 such that $x - \nu_0 \sim x_1 - x$, in the case $E \gg U_0$. At the maximum $\operatorname{Im} \nu \sim (x_1/(x_1 - x))^{1/2}$. Then $\operatorname{Im} \nu$ falls off to values $1/x_1^{1/3}(x_1 - x)$ for $x - \nu \sim x^{1/3} \ll x_1 - x$.

The poles which are situated between x and x_1 correspond physically to quasi-stationary states (evidently, for values of ν close to half-integral values), since the energy of these states lies below the potential barrier. Therefore, for such poles $\operatorname{Im} \nu$ is exponentially small. In order to find $\operatorname{Im} \nu$

we represent $H_\nu^{(1)}(x)$, according to (43), in the form of the sum

$$H_\nu^{(1)}(x) = (\nu^2 - x^2)^{-1/4} [-i\pi^{-1} e^{-i\Phi(\nu, x)} + e^{i\Phi(\nu, x)}]. \quad (55)$$

From here we obtain a formula analogous to (45):

$$\mathrm{Im}\,\nu = 2\pi e^{-2i\Phi(\nu, x)} \sqrt{2(\nu - x)/x_1} \qquad (\nu - x \gg x_1^{1/3}). \quad (56)$$

The orders of magnitude of Im ν for poles situated to the right and left of x, for $|\nu - x| \sim x_1^{1/3}$, coincide if $x_1 - x \sim x_1^{1/3}$.

B. $x_1 - x \ll x_1^{1/3}$. In this case there are no poles between x and x_1. For Im ν we obtain Eq. Eq. (52). The nearest pole is situated at a distance of the order of $x_1^{1/3}$ from x_1. For poles for which $x - \nu \ll x$, we expand $\Phi(\nu, x)$ in powers of $(x_1 - \nu)/x_1$. We obtain then from (51)

$$\nu = x_1 - \frac{3^{2/3}}{2} x_1^{1/3}\left(n\pi + \frac{i}{3}\ln\frac{8(x_1 - \nu)}{x_1 - x}\right). \quad (57)$$

Equation (57) is valid also for n = 1, since it yields $x - \nu_1 \gg x^{1/3}$.

5. THE "UNPHYSICAL" SERIES OF POLES

We call "unphysical" the series of poles situated near the lines going to $\pm\infty$ (M_2 and M_2' in figs. 4, 5) on which the function $H_\nu^{(1)}(x)$ is represented by a superposition of Z_+ and Z_-. The poles of this series do not correspond to any bound states.

We consider first the case $x_1^2 > 0$, $x^2 < 0$, $-x^2 \gg 1$. In the upper ν-half-plane the poles are situated above the point x at a distance $\sim x^{1/3}$. In the discrete spectrum ($x^2 < 0$) the poles in the lower half-plane are situated in points which are complex conjugates of the poles in the upper half-plane of ν. The functions $J_\nu(x)$ and $H_\nu^{(1)}(x)$ have the forms

$$J_\nu(x) \sim Z_+, \quad H_\nu^{(1)}(-x) \sim Z_+ - Z_-$$
$$= (x^2 - \nu^2)^{-1/4} \sin(\Phi(\nu, x) - \pi/4). \quad (58)$$

From (57), (58), and (9) we obtain

$$\mathrm{tg}(\Phi(\nu, x) - \pi/4) = i\sqrt{(x_1^2 - \nu^2)/(x^2 - \nu^2)} = iy_\nu. \quad (59)$$

From this we derive the "Bohr quantization rule" for the unphysical series:

$$\Phi(\nu, x) = n\pi + \pi/4 + \gamma(n). \quad (60)$$

For the solutions (60), Im $\nu \gg$ Re ν. In analogy to (52) we find

$$\mathrm{Re}\,\nu = \ln\frac{1 - y_\nu}{1 + y_\nu} \Big/ 2\ln\left|\frac{\nu}{x} + \sqrt{\frac{\nu^2}{x^2} - 1}\right|. \quad (61)$$

Asymptotically, for large $|\nu| \gg x_1^2 - x^2$, we obtain

$$\mathrm{Im}\,\nu \approx \frac{n\pi}{\ln|\nu/x|} = \frac{n\pi}{\ln n},$$
$$\mathrm{Re}\,\nu = -1 + \ln\left|\frac{x_1^2 - x^2}{x^2}\right| \Big/ 2\ln|\nu|. \quad (62)$$

However, if $|x|^{4/3} \ll |x^2 - \nu^2| \ll x_1^2 - x^2$, then $y_\nu \gg 1$ and

$$\mathrm{Re}\,\nu \approx -\sqrt{\frac{x^2 - \nu^2}{x_1^2 - \nu^2}} \Big/ 2\ln\left|\frac{\nu}{x} + \sqrt{\frac{\nu^2}{x^2} - 1}\right|. \quad (63)$$

For $\nu/x - 1 \ll 1$ Eq. (63) yields

$$-\mathrm{Re}\,\nu \approx |x|/\sqrt{x_1^2 - x^2}.$$

The poles are situated to the right of the straight line Im $\nu = -1$ and approach it asymptotically as $n \to \infty$, coming closer together logarithmically[2)].

Let us consider now the behavior with respect to the x-variable of the poles of the unphysical series in the neighborhood of the essential singularity x = 0 of $S(\nu, x)$.

We use Eq. (60). For $x \to 0$ we have $\Phi(\nu, x) = -i\nu\ln(ex/2\nu)$ and thus $\nu_n = in\pi/\ln(ex/2\nu)$. Consequently, for fixed n the poles approach logarithmically (in energy) the point $\nu = 0$ (coalescence of poles). Our calculation cannot be considered rigorous, since we have used Eq. (60) which is valid only for large ν. In order to carry out a rigorous calculation we represent the Hankel function $H_\nu^{(1)}(x)$ for small x in the form

$$H_\nu^{(1)}(x) = \frac{i}{\sin\nu\pi}\left[\frac{e^{-i\nu\pi}}{\Gamma(1+\nu)}\left(\frac{x}{2}\right)^\nu - \frac{1}{\Gamma(1-\nu)}\left(\frac{x}{2}\right)^{-\nu}\right] \quad (64)$$

[we have retained the first terms of the series for $J_\nu(x)$ and $J_{-\nu}(x)$]. As regards $J_\nu(x_1)$, this function will be, as before, represented by a single exponential Z_+.

From (9) we find

$$\mathrm{ctg}\,\varphi = \sqrt{(x_1^2 - \nu^2)/\nu^2} = -iy, \quad (65)$$

$$\varphi = -\frac{i}{2}\ln\left[\left(\frac{x}{2}\right)^{2\nu} e^{-i\nu\pi}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}\right]. \quad (66)$$

It is convenient to introduce the quantity A:

$$A = \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} e^{-i\pi(\nu - 1/2) - 2\nu}\nu^{2\nu}. \quad (67)$$

For $|\nu| \gg 1$ we have $\ln A \ll 1$. From (65) and (66) we obtain

$$\mathrm{Im}\,\nu = \frac{n\pi}{\ln|\nu/x|}, \quad (68)$$

2)When E < U the poles are situated to the left of Im $\nu = -1$, since $x_1^2 < 0$. For such energies (lower than the bottom of the well) the physical series of poles splits into two (one series with even numbers, the other with odd numbers), and one of the series goes off into the upper half-plane to the left of the imaginary axis, as E increases, and the other goes into the lower half-plane.

the same as has been obtained in the cruder estimate. Further we find

$$\mathrm{Re} = \left[2\ \mathrm{Im}\,\nu\left(\arg x - \frac{\pi}{2}\right) + \ln\frac{y+1}{y-1} - \ln A\right] \Big/ 2\ \ln\left|\frac{x}{2\nu}\right|. \qquad (69)$$

For $|\nu/x_1| \gg 1$ we have $y \approx 1 - x_1^2/2\nu^2$ and (69) yields

$$\mathrm{Re}\,\nu \approx \left[n\pi\left(\ln\left|\frac{\nu}{x}\right|\right)^{-1}\left(\arg x - \frac{\pi}{2}\right) + \ln\left|\frac{2\nu}{x_1}\right|\right] \Big/ \ln\left|\frac{x}{\nu}\right|. \qquad (70)$$

Thus, for $|\nu| \gg x_1$

$$\mathrm{Re}\,\nu \approx -n\pi\ \ln^{-2}\left|\frac{x}{\nu}\right|\left(\arg x - \frac{\pi}{2}\right) - 1. \qquad (71)$$

When x goes from imaginary "positive" values to real positive values, $\arg(x - \pi/2)$ changes from zero to $-\pi/2$. For $\arg x = 0$ and sufficiently large n, $\mathrm{Re}\,\nu \approx n\pi^2/2\ln^2|n| \gg 1$ for $|n| \gg 1$.

Finally, for $x^2 \gg 1$ we get $x_1^2 \gg 1$ the functions $J_\nu(x)$ and $H_\nu^{(1)}(x)$ are given by Eqs. (57) and (58), and the equation for the location of the poles coincides with (59). For $|\nu| \gg x^2, x_1^2$ we obtain the result (62) $\mathrm{Im}\,\nu \approx n\pi/\ln|n/x|$.

The expression for $\mathrm{Re}\,\nu$ changes, since now $\arg x = 0$:

$$\mathrm{Re}\,\nu \simeq \mathrm{Im}\,\nu/\ln|n/x|. \qquad (72)$$

As already stated, the poles closest to the point x are situated at a distance $x^{1/3}$ from that point. In this region

$$H_\nu^{(1)}(x) \approx Ai\,[\Phi(\nu, x)] \approx Ai\left[\frac{2}{3}\,\frac{2}{x}(\nu - x)^{3/2}\right]. \qquad (73)$$

If $\lambda_1 \sim 1$ is the first zero of the Ai function, then

$$\nu - x \sim x^{1/3}\lambda_1^{2/3}. \qquad (74)$$

These estimates of the order of magnitude are valid for $x \sim 1$.

For $x \gg 1$ there exist poles such that $x^{1/3} \ll |x - \nu_n| \ll x$. For these poles we obtain from (59), expanding $\Phi(\nu, x)$ in powers of $(\nu - x)/x$,

$$\Phi(\nu, x) \approx \frac{2^{3/2}}{3}\,e^{-i\pi/2}x^{-1/2}(\nu - x)^{3/2}. \qquad (75)$$

We obtain from (59)

$$\Phi(\nu, x) = -n\pi + \frac{\pi}{4} - \frac{i}{2}\ln\frac{1-y}{1+y}, \qquad (76)$$

whence

$$\nu = x + e^{i\pi/3}\frac{3^{2/3}}{2}x^{1/3}\left(n\pi + \frac{i}{2}\ln\frac{1-y}{1+y}\right)^{2/3}. \qquad (77)$$

For not too large values of the energy, when $|x_1 - x| \gg |x - \nu| \gg x^{1/3}$, $y \gg 1$

$$\nu = x + {}^1/_2\,e^{i\pi/3}3^{2/3}(xn^2\pi^2)^{1/3}, \qquad n \gg 1. \qquad (78)$$

However, if $|x_1 - x| \ll x^{1/3}$, then $y \approx 1 + (x_1 - x)/2(x - \nu)$ and

$$\nu = x + e^{i\pi/3}\frac{3^{2/3}}{2}x^{1/3}\left(n\pi + \frac{i}{2}\ln\left|\frac{x_1 - x}{4(x - \nu)}\right|\right)^{2/3}. \qquad (79)$$

Equation (79) is valid also for $n \sim 1$.

Note that Eq. (79) coincides formally with Eq. (57) for the physical series. This means that at large energies $E \gg U_0(2\mu U_0 a^2)^{1/2}$ these series go over continuously into each other. The angle of inclination of the pole line with respect to the real ν-axis equals $\pi/6$ for $|n\pi| \ll \ln|(x - \nu)/(x_1 - x)| \sim \ln x$. For $x \gg n\pi \gg \ln|(x - \nu)/(x_1 - x)|$ the inclination angle increases to $\pi/3$. Finally, for very large n, ($|\nu/x| \gg 1$) we obtain from (62) and (72)

$$\mathrm{Re}\,\nu/\mathrm{Im}\,\nu \approx |1/\ln n|.$$

In the third quadrant of the ν-plane the poles of the unphysical series are approximately symmetric with respect to the point $\nu = 0$ to the poles in the first quadrant.

We emphasize the fact that for $E > 0$ there does not exist an extreme right pole for the poles of the unphysical series. According to Eq. (62) and (72) $\mathrm{Re}\,\nu \approx n\pi/\ln^2 n$ increases without bound as n increases.

6. THE MOTION OF THE POLES

Until now we have investigated the locations of the poles for fixed energy. It is interesting to follow the motion of a pole with fixed number n as the energy varies. We start with the poles of the physical series. The first pole of the physical series is determined by solving the equation

$$\frac{A'i(\Phi(\nu, x_1))}{Ai(\Phi(\nu, x_1))} = \sqrt{\frac{x_1}{x_1 - \nu}}\,x\,\frac{H_\nu'(x)}{H_\nu(x)}. \qquad (80)$$

For $E < 0$ the root ν_1 of Eq. (80) moves behind x_1 at a distance 1, for $x_1 \lesssim 1$, and at a distance $\sim x_1^{1/3}$ for $x_1 \gg 1$, from the point $\nu_1 = -1$ to the point $\nu_1 \approx (2\mu U_0 a^2)^{1/2} = \alpha$, along the real ν axis ($\nu_1 \sim \alpha$ corresponds to $E = 0$). Further, ν_1 wanders off into the complex plane when $E > 0$, so that $\mathrm{Re}\,\nu_1$, as before follows at a distance $x_1^{1/3}$ behind the point x_1, and $\mathrm{Im}\,\nu_1 \sim (x/\nu_1)^{2\nu_1}$. For $x_1 \sim \alpha^{3/2}/(\ln\alpha)^{1/2}$, $\mathrm{Im}\,\nu_1$ increases to a value of the order of one, and for $x_1 \sim \alpha^{3/2}$ and $E \sim U_0\alpha$, we obtain $\mathrm{Im}\,\nu_1 \sim \mathrm{Re}(\nu_1 - x) \sim \alpha^{1/2} \sim x_1^{1/3}$. As the energy increases further $\mathrm{Re}\,\nu_1$ keeps on following behind $x_1(x)$ at a distance $\sim x_1^{1/3}$ and $\mathrm{Im}\,\nu_1$ increases as const. $x_1^{1/3}\ln(x_1/\alpha^{3/2})$ [cf. Eq. (57)].

Let us consider further the pole with number $n = \alpha$ (an "average" pole of the physical series). This pole will move from the point $\nu = -n$ for $E = U_0$ along the real axis up to a point which is approximately determined by the equation

$$\Phi(\nu, \alpha) = n\pi + \pi/4 \tag{81}$$

at $E = 0$. We assume that the solution $\nu_n(0)$ of Eq. (81) is positive and in general that $\nu_n(0) \sim \alpha$. Then, for $E > 0$ the poles ν_n will slowly move into the complex plane, so that $\mathrm{Im}\,\nu_n \sim \exp(-2i\Phi(\nu, x))$ and $\mathrm{Re}\,\nu$ increases as the energy increases, following x_1 according to the law $\Phi(\nu, x_1) = n\pi$, i.e. increases in order of magnitude at the same rate as x_1. The imaginary part $\mathrm{Im}\,\nu$ reaches a value ~ 1 at an energy determined by the condition $\Phi(\nu, x_1) \sim n\pi$ (under our assumptions this is $E \sim U_0$, $\nu_n \sim \alpha$) and further increases at first logarithmically and then, starting from $x \sim \alpha^{3/2}$ ($E \sim U_0\alpha$) according to the power law $x^{1/3}$ (cf. Eq. (57)).

Let us discuss now the behavior of a "remote" pole with a number $n \gg \alpha$. Such a pole stays very close to the point $\nu = -n$ up to energies $E \sim n/2\mu a^2$, for which x, $x_1 \sim n$. For large values of the energy the pole starts moving behind the point $x_1(x)$ with a slowly varying distance to the latter. $\mathrm{Im}\,\nu$ is determined by Eq. (52) and has the order of magnitude $\mathrm{Im}\,\nu \sim \ln[(x_1^2 - x^2)/(x_1^2 - \nu^2)]^{1/2}$. As the energy increases x_1 and ν vary in such a way that $|x_1 - \nu_n| \approx 2\pi n$, up to energies for which $|x_1 - \nu_n| \sim x_1^{1/3}$. The axis $\mathrm{Im}\,\nu$ is intercepted by the pole ν_n at an energy $E_n \sim n^2\pi^2/\mu a^2$ in the point

$$\mathrm{Im}\,\nu_n \sim \ln|U_0/E|. \tag{82}$$

For $x_1 \sim (2\pi n)^3$ the motion of the pole is determined by Eq. (57). Here $\mathrm{Im}\,\nu \sim \mathrm{Re}\,(x_1 - \nu) \sim x_1^{1/3}$. The inflection points of the trajectories of the poles of the physical series are located in a region in which $\mathrm{Im}\,\nu$ becomes of the order of one. As the number n increases, the inflection points are displaced to the left. For the poles which are situated far in the negative half-plane, the inflection points are situated near those integral points on the negative semiaxis to which these poles become attached at low energies. The trajectories intersect in a region where Eq. (57) is applicable (the points of intersection correspond to different values of the energy for different trajectories).

When $E < 0$ the poles of the unphysical series move down towards the point $\nu = 0$ as E increases. For $x^2 < 0$, $|x| \ll 1$, Eqs. (68) and (69) imply that near $x = 0$ and $\nu = 0$

$$\mathrm{Re}\,\nu/\mathrm{Im}\,\nu = |\nu/x_1|^2. \tag{83}$$

Near $x = 0$ and $\nu = 0$ and for $x^2 > 0$ we obtain

$$\mathrm{Im}\,\nu/\mathrm{Re}\,\nu \approx \ln|\nu/x| \gg 1. \tag{84}$$

When x^2 goes from negative to positive values, the poles of the unphysical series situated in the upper half-plane pass through the point $\nu = 0$ from the second into the first quadrant of the ν-plane. The poles in the lower and upper half-planes are always symmetrical about the axis $\mathrm{Re}\,\nu = 0$ when $E < 0$ ($x^2 < 0$). For $E > 0$ these poles pass through the point $\nu = 0$ again into the third quadrant of the ν plane and move approximately symmetrically with respect to the point $\nu = 0$ to the poles in the first quadrant [these poles correspond to $n < 0$ in Eqs. (68)–(71)] [3].

For $x^2 \gg 1$ and poles with a large number n, Eqs. (62) and (72) yield

$$\mathrm{Im}\nu \approx \frac{n\pi}{\ln n}\left(1 + \frac{\ln x}{\ln n}\right), \tag{85}$$

$$\mathrm{Re}\nu \approx \frac{\ln \nu}{\ln n}\left(1 + \frac{\ln x}{\ln n}\right). \tag{86}$$

We recall that we assume here that $\mathrm{Re}\,\nu \gg x$. For energies for which $n/x \ll 1$, Eq. (77) is valid so that the motion of the poles of the unphysical series is analogous to the motion of the first poles of the physical series (for the same energies). Equation (77) goes almost smoothly over into Eq. (57) [the one is obtained from the other by changing the sign of n].

7. RELATION BETWEEN THE NUMBER OF LEVELS AND THE NUMBER OF RESONANCES

The number of levels inside a quasiclassical spherically symmetric potential well can be found with asymptotic accuracy. Indeed, since the number of levels is large, one can connect the number N of levels with the positions of the poles $\nu_n(E)$ at $E = 0$ in the following manner

$$N = \sum_{n=0}^{n_{max}} \nu_n(0), \tag{87}$$

where n_{max} corresponds to the pole lying closest to zero. Changing the sum in (87) into an integral and integrating by parts we obtain:

$$N = \int_0^{n_{max}} \nu_n dn = \int_0^{\alpha} n\, d\nu. \tag{88}$$

The dependence of n on ν is determined by the quantization condition

$$\Phi(\nu_n, \alpha) = \int_{\nu}^{\alpha} \sqrt{1 - \nu_n^2/\xi^2}\, d\xi = n\pi,$$
$$\alpha = x_1|_{E=0} = \sqrt{2\mu U_0 a^2}. \tag{89}$$

[3]In order to put poles for $x^2 < 0$ and for $x^2 > 0$ in correspondence with each other one can go over from imaginary x to real positive x along a circular arc $|x| = \epsilon \ll 1$; Eqs. (64)–(71) obviously remain valid.

Substituting n from (89) into (88) and carrying out the integration, we get

$$N = \alpha^2/8. \quad (90)$$

It is easy to find the total number of resonances (i.e., of quasistationary states in the problem under consideration).

Indeed, a quasistationary state corresponds to a value of ν close to a half-integer, for $E > 0$ and $\mathrm{Im}\,\nu \ll 1$. As has been shown in Sec. 4, the inequality $\nu < x$ is a necessary and sufficient condition for the smallness of $\mathrm{Im}\,\nu$. Thus, a pole ν_n corresponds to resonances up to values of the energy for which $\nu_n = x = x_n$. Consequently, the number of resonances M is determined, with the same accuracy as N, by the formula

$$N = \sum_{n=0}^{n_{max}} (x_n - \nu_n(0)) = \sum_{n=0}^{n_{max}} x_n - N. \quad (91)$$

The computation of $X = \sum_{n=0}^{n_{max}} x_n$ is carried out in the same way as the computation of N, taking into account the relations

$$\Phi(x_{1n}, x_n) = n\pi, \quad x_{1n}^2 - x_n^2 = \alpha^2. \quad (92)$$

The result of the integration yields $X = \alpha^2/4$ and

$$M = N. \quad (93)$$

It is interesting to investigate the relationship between N and M in the case of a sufficiently general quasiclassical potential U(r). In this case the number of levels is given as before by Eqs. (87) and (88), but the Bohr quantization rule, which determines the relation between ν_n and n, has the usual form

$$\int_{r_1(\nu)}^{r_2(\nu)} p_r dr = n\pi, \quad p_r = \sqrt{2\mu(E - U(r) - \nu^2/2\mu r^2)}, \quad (94)$$

where $r_1(\nu)$ and $r_2(\nu)$ are the turning points.

Let us put $E = 0$ in (94) and substitute n from (94) into (88). The integration will respect to ν can be carried out explicitly and yields

$$N = -\frac{1}{4}\int_0^\infty U(r)\, r dr. \quad (95)$$

Unfortunately, one does not succeed in deriving such an elegant and closed expression for the number of resonances M. A given pole ν_n corresponds to resonances up to an energy E_n, which equals the maximum value of the potential barrier $U(r) + \nu^2/2\mu r^2$. Let the corresponding value of r be r_{2n}. It is easy to obtain the system of equations for the quantities

$$\nu_n(E_n) \equiv x_n, \quad E_n, \quad r_{2n} = r_2(\nu_n), \quad r_{1n} = r_1(\nu_n),$$

$$E_n = U(r_{2n}) + \frac{1}{2} U'(r_{2n})\, r_{2n}, \quad U'(r_{2n}) + \frac{x^2}{\mu r_{2n}^3} = 0 \quad (96)$$

[one may add (94) to these equations]. Then M is given by the double integral

$$M = \frac{1}{\pi}\int_0^\infty dr \int_0^x dx \sqrt{2\mu(E(x) - U(r) - x^2/2\mu r^2)} - N, \quad (97)$$

where $E(x) \equiv E_n$ is determined from (94) and (96) by eliminating r_n. From (95) and (97) it is clear that M and N are of the same order of magnitude ($\sim\mu U_0 a^2$, where a is the characteristic length of the potential). This relation does not depend on the details of the potential and in order for it to be applicable it is only necessary that the number of stationary states be large.

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