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SHIFTS AND BROADENING OF ENERGY LEVELS OF SINGLE-ELECTRON ATOMS AND IONS IN A HIGH-TEMPERATURE PLASMA

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The shifts and widths of energy levels of single-electron atoms and ions in an equilibrium high temperature plasma can be calculated by employing the spectral representation and a Bethe-Salpeter equation for the electron-ion Green's function. The calculated quantities are expressed in terms of the mass and vertex operators obtained by the diagram technique. The level shifts are proportional to the square root of the temperature and density for ions and to the first powers of these quantities for hydrogen atoms. The width is always proportional to the density.

INTRODUCTION

A consistent quantum-statistical theory of the influence exerted by a plasma on the energy levels of single-electron atoms and ions requires that the levels be regarded as two-particle excitations of the system, having properties described by a two-particle electron-ion Green's function.[1] In the present paper this theory is developed for a high-temperature plasma.

For a low-temperature plasma the indicated procedure has been used in a recent paper by Kudrin and Tarasov,[2] who showed that the level shift calculated in [1], which is proportional to the square root of temperature and electron density, disappears for hydrogen atoms; these results agree with our findings. However, these authors did not calculate the energy shift of hydrogen atoms in the first nonvanishing approximation. In the present paper we have calculated this shift, which is proportional to temperature and electron density, in addition to the widths of atomic and ionic levels.

We shall consider an equilibrium plasma at the temperature $T = \beta^{-1}$ as a system of nonrelativistic electrons and ions undergoing instantaneous Coulomb interaction. For ions of charge $Z > 1$ we shall consider the influence of electrons and ions of charge Z on an ion of charge $Z - 1$. The interaction Hamiltonian $H_{\text{int}} = H_{ee} + H_{ei} + H_{ii}$ of this system is given in [1].

In order to describe one- and two-particle excitations we shall use temperature-dependent Green's functions expanded in Fourier series with respect to a "time coordinate" which is the reciprocal of temperature.[3] This is convenient for our problem, in which the principal effects depend on operators with zero-point frequencies.

TWO-PARTICLE EXCITATIONS

To describe single-electron atoms and ions we shall use the causal electron-ion Green's function

$$G(x, y; x', y') = S_0^{-1}\,\mathrm{Sp}\,\{\rho T\psi(x)\,\varphi(y)\,\varphi^+(y')\,\psi^+(x')\,S(\beta)\};$$

here ρ is a statistical operator, $\psi, \psi^+ (\varphi, \varphi^+)$ are the creation and annihilation operators of electrons (ions) in the temperature-dependent representation of the interaction, $S(\beta)$ is the Matsubara S matrix, and $x \equiv (\mathbf{x}, x_0)$, where x_0 is the reciprocal of temperature.

We shall write (1) in the energy representation, transforming from the particle coordinates to the relative coordinates ξ and center-of-mass coordinates η:

$$\xi = x - y, \ \eta = \nu x + \nu' y, \ \nu + \nu' = 1, \qquad (2)$$

with the primed variables introduced analogously. It is easily shown that each term in (1) contains the factor $\exp[(\epsilon_e + \epsilon_i)(\eta - \eta')]$, where $\epsilon_e + \epsilon_i$ is the electron-ion excitation energy. Therefore the spectral representation of G will in the $i\omega_k$ plane have poles $\epsilon_e + \epsilon_i$, where ω_k is the parameter of a Fourier transform in $\eta - \eta'$. It therefore becomes our problem to determine the poles of G (or the zeros of G^{-1}) in the $i\omega_k$ plane. This can be done using a Bethe-Salpeter equation for the electron-ion Green's function.[4,5]

For the given system of nonrelativistic particles the equation for the temperature-dependent function G will be easily obtained, for example, by expanding $S(\beta)$ in (1) in normal derivatives; this procedure yields

$$(G_e^{-1}G_i^{-1} + V)\,G = 1, \tag{2a}$$

where G_e and G_i are the one-particle electron and ion Green's functions, and V is the most general compact vertex operator for the electron-ion interaction. The equation for the zeros of G^{-1} is

$$\int dx'\,dy'\,[G_e^{-1}(x - x')\,G_i^{-1}(y - y') + V(x, y; x', y')]\,\psi(x', y') = 0; \tag{3}$$

here integration over x_0 and y_0 is performed from 0 to β.

Equation (3) will now be written with the variables (2). We note that since the operator G (and therefore G^{-1} and V) possesses translational invariance in a state of statistical equilibrium these operators can be represented as functions of ξ, ξ', and $\eta - \eta'$. The Fourier transform of (3) in spatial variables can be obtained for any ν; we select $\nu = m_e/(m_e + m_i)$, where m_e and m_i are the electron and nuclear masses.[4]

We now consider the Fourier transform of (3) in the "time" variables x_0 and y_0. The operators G_e, G_i, and G are antiperiodic functions of x_0, y_0, x_0', y_0'; therefore their Fourier expansion in these variables contains only odd harmonics with the frequencies $\omega_{2n+1} = (2n + 1)\pi\beta^{-1}$;[6] the function $\psi(x', y')$ has a similar property.[1)] In calculating the operators G_e^{-1}, G_i^{-1}, and V by means of the diagram technique along with the antiperiodic dependence of these functions we encounter vertices with δ-function dependence on $x - x'$ and $y - y'$. Also, a δ function of the time variable is contained in the Coulomb potential.

We first consider a transformation of the part of (3) containing no δ-function vertices. The integrand in (3) contains only even harmonics, i.e., periods β of x_0' and y_0'. Transforming to the time coordinates (2), we obtain

$$\omega_{2m}x_0' + \omega_{2n}y_0' = (2m + 2n)\,\pi\beta^{-1}\eta_0' + (2m\nu_0' - 2n\nu_0)\,\pi\beta^{-1}\xi_0'. \tag{3*}$$

It follows that the integrand will have the period 2β for ξ_0' only when $\nu_0 = \nu_0' = 1/2$; this is the only choice of ν_0 that makes a Fourier transformation possible for the temperature-dependent functions in the cases of all possible masses.

Transforming in (3) to the variables (2) with the indicated choices of the numbers ν and ν_0, we obtain

$$\frac{1}{4}\int d\xi'\,d\eta'\,\{G_e^{-1}[\eta - \eta' + \nu'(\xi - \xi')] \times G_i^{-1}[\eta - \eta' - \nu(\xi - \xi')] + V(\xi, \xi', \eta - \eta')\}\,\psi(\xi', \eta') = 0. \tag{3a}$$

Here and hereinafter integration with respect to the time coordinates extends over the interval $(-\beta, \beta)$. Integration over this region is possible in virtue of the indicated periodicity of the integrand; the integral is not changed thereby in the cases of terms containing time-dependent δ functions. In applying (3a) and subsequent equations to such terms we shall multiply their Fourier transforms by 2^v, where v is the number of δ-function vertices, and we shall also double the Fourier transform of the Coulomb potential, taking $u(k) = u(k) = 4\pi e^2 k^{-2}$.

The Fourier transformation of (3a) gives

$$G_e^{-1}(\nu K + p)\,G_i^{-1}(\nu' K - p)\,\psi(p, K) + \beta^{-1}(2\pi)^{-3}\int dq\,V(p, -q, K)\,\psi(q, K) = 0; \tag{4}$$

here the four-dimensional vectors have a discrete time component $p \equiv (p, \omega_p)$ and the four-dimensional integral denotes spatial integration and time summation. The quantities G_e^{-1} and G_i^{-1} have only odd harmonics, while $V(p, -q, K)$ and $\psi(q, K)$ have only even harmonics in ω_k and a complete spectrum in ω_p and ω_q. However, in (4) and subsequently summation over ω_q must be performed only over frequencies having parity opposite to that of $\omega_k/2$. The Fourier transform of V is

$$V(p, q, K) = \frac{2^v}{8}\int d\xi\,d\xi'\,d\eta\,V(\xi, \xi', \eta)\,e^{i(p\xi + q\xi' + K\eta)}, \tag{5}$$

where $p\xi \equiv p\xi + \omega_p\xi_0$; the other quantities transform similarly.

ENERGY SPECTRUM OF TWO-PARTICLE EXCITATIONS

We select in the operator G^{-1} the zeroth and higher-order terms with respect to the density N_e of free electrons. For this purpose we use the Dyson equations

$$G_e^{-1} = S_e^{-1} - M_e, \quad G_i^{-1} = S_i^{-1} - M_i,$$

where S_e (S_i) is the one-particle Green's function for noninteracting electrons (ions), and M_e and M_i are the corresponding mass operators. We write the vertex operator in the form

$$V = V_0 + V_1,$$

[1)] This follows from the relation between ψ and G.

where V_0 is of zeroth order and V_1 is of higher order in N_e. Equation (4) then becomes

$$(S_e^{-1}S_i^{-1} + V_0 + W)\psi = 0,$$
$$W = -S_e^{-1}M_i - M_eS_i^{-1} + M_eM_i + V_1, \qquad (6)$$

where W approaches zero with N_e.

As already shown, the eigenvalues of this operator in the $i\omega_k$ plane give the energy spectrum of the electron-ion excitations. Since the terms of the order N_e in (6) are small, we shall solve this equation by the perturbation method.[1] A method of integrating the Bethe-Salpeter equation has been developed in[4,5] for quantum electrodynamics. This method can be extended to quantum statistics and to temperature-dependent functions in the following manner.

We first consider the zeroth approximation of (6). In the momentum representation we have

$$S_e^{-1}(\nu K + p) = i(\omega_k/2 + \omega_p) + \mu_e - T_e(\mathbf{p}),$$
$$T_e(\mathbf{p}) = (\nu\mathbf{K} + \mathbf{p})^2/2m_e;$$
$$S_i^{-1}(\nu' K - p) = i(\omega_k/2 - \omega_p) + \mu_i - T_i(\mathbf{p}),$$
$$T_i(\mathbf{p}) = (\nu'\mathbf{K} - \mathbf{p})^2/2m_i.$$

The vertex operator V_0 corresponds to the Coulomb interaction of an electron and nucleus:

$$V_0(x, y; x', y')$$
$$= -Zu(\mathbf{x} - \mathbf{y})\,\delta(x_0 - y_0)\,\delta(x - x')\,\delta(y - y');$$

in the momentum representation we have

$$V_0(p, -q, K) = -Zu(\mathbf{p} - \mathbf{q}). \qquad (7)$$

Thus in the zeroth approximation (6) becomes

$$S_e^{-1}(\nu K + p)\,S_i^{-1}(\nu' K - p)\,\psi(p, K)$$
$$- \beta^{-1}(2\pi)^{-3} Z\int dqu(\mathbf{p} - \mathbf{q})\,\psi(q, K) = 0. \qquad (8)$$

Here only ψ is integrated over q_0.

Introducing the function

$$\varphi(\mathbf{q}) \equiv \beta^{-1}\int dq_0\psi(q, K),$$

we write (8) as

$$\psi(p, K)$$
$$= S_e(\nu K + p)\,S_i(\nu' K - p)\,Z(2\pi)^{-3}\int d\mathbf{q}u(\mathbf{p} - \mathbf{q})\,\varphi(\mathbf{q}).$$

Applying the operation[2)] $\beta^{-1}\int dp_0$ to this equation, we obtain

$$\beta^{-1}\int dp_0S_e(\nu K + p)\,S_i(\nu' K - p)$$
$$= -F^{-1}(\mathbf{p}, K)\,[1 - f_e(T_e(\mathbf{p})) - f_i(T_i(\mathbf{p}))],$$
$$F(\mathbf{p}, K) = i\omega_k + \mu - T(\mathbf{p}),$$
$$\mu = \mu_e + \mu_i,\ T(\mathbf{p}) = \mathbf{K}^2/2M + \mathbf{p}^2/2m, \qquad (9)$$

where M is the ion mass, m is the reduced mass, and f is the Fermi function; terms of f are of the order N and must be considered only in first approximation.

Thus in zeroth approximation we obtain

$$[i\omega + \mu - T(\mathbf{p})]\,\varphi(\mathbf{p}) = -Z(2\pi)^{-3}\int d\mathbf{q}u(\mathbf{p} - \mathbf{q})\,\varphi(\mathbf{q}),$$

i.e., the Schrödinger equation for a single-electron ion[7] having the momentum **K**, total energy $i\omega_k + \mu = K_0 = \mathbf{K}^2/2M + E$, and the wave function φ. For $\psi(p, K)$ we obtain

$$\psi(p, K) = -F(\mathbf{p}, K)\,S_e(\nu K + p)\,S_i(\nu' K - p)\,\varphi(\mathbf{p}). \qquad (10)$$

Equation (6) containing the eigenvalue is nonlinear and requires a special form of perturbation theory developed in [4] for time-dependent Green's functions.[3)] For temperature-dependent Green's functions the same method leads to the following formula for a correction to the energy level E of a state φ under a perturbation W:

$$\Delta E = \delta E - i\Gamma = \beta^{-1}\int dp\tilde{\psi}(p, K)\,W\psi(p, K)\big|_{i\omega_k + \mu = K_0 + i\delta}$$
$$+ Z\int d\mathbf{p}\,d\mathbf{q}\varphi^*(\mathbf{p})u(\mathbf{p} - \mathbf{q})\,[f_e(T_e(\mathbf{q})) + f_i(T_i(\mathbf{q}))]\,\varphi(\mathbf{q}), \qquad (11)$$

where $\tilde{\psi}$ is obtained from (10) through the replacement of φ by φ^*. Here δE is the shift and Γ is the width of the given level.

It follows from perturbation theory that in calculating matrix elements we must consider ω_k as belonging to a discrete spectrum, since (9) is fulfilled only in this case, and that we must replace $i\omega_k + \mu$ by $K_0 + i\delta$ only in the final result. The infinitesimal imaginary term added to K_0 will be w written explicitly only in calculating the widths.

MASS AND VERTEX OPERATORS

Mass and vertex operators in the diagram technique can easily be constructed for a high-temperature plasma. First- and second-order diagrams for an electron mass operator are given in [1], where it is shown that the main contribution to

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[2)] Summation here and subsequently can be performed using the Poisson formula, for example.

[3)] In [2] this theory was developed for temperature-dependent Green's functions; however, the resulting formula is not exact, a term ~N being absent.

level shifts comes from second-order divergent diagrams. The divergences are removed by considering the Debye screening, i.e., by a diagram summation leading to replacement of the Coulomb potential u(k) by the effective potential

$$\tilde{u}(k) = u(\mathbf{k})/[1 - u(\mathbf{k})P(k)], \tag{12}$$

where P(k) is the polarization operator.[1]

The foregoing also applies to the ion-mass and vertex operators. We shall therefore consider diatrams where the Coulomb potential is replaced by the effective potential. Whenever the Coulomb potential makes a negligibly small contribution we shall use the potential

$$\bar{u}(k) \equiv \tilde{u}(k) - u(\mathbf{k}) = u^2(\mathbf{k}) \cdot P(k)/[1 - u(\mathbf{k})P(k)]. \tag{13}$$

In first approximation the electron polarization operator can be represented by

$$P_e(\mathbf{k}, k_0) = -(N_e\beta/\lambda_e k)\,[e^{-z^2}\,\mathrm{Erf}\,i(z) + e^{-z^{*2}}\,\mathrm{Erf}\,i(z^*)],$$

$$z = \lambda_e k/2 + i\beta k_0/2\lambda_e k, \quad \lambda_e^2 = \beta/2m_e; \tag{14}$$

here Erfi is the error function of imaginary argument.[8] Approximate expressions for P_e with small k are given in [1]. The ion polarization operator P_i is obtained from (14) through the replacement of N_e by N_i and of λ_e by $\lambda_i = (\beta/2m_i)^{1/2}$.

For Z > 1 we consider only the contributions of electrons and nuclei to the polarization operator. It follows from the theory of ionization equilibrium [9] that at high temperatures the density of (Z − n)-charged ions is of the order $N_e(N_e\lambda_e^3)^n$; therefore their contribution to the polarization operator can be neglected.

The principal contribution to the shift for ions comes from three diagrams, each containing one effective-potential line (the heavy dashed line in Fig. 1). The corresponding operators are

$$M_{e1}(p) = -\beta^{-1}(2\pi)^{-3}\int dq S_e(p-q)\,\bar{u}(q), \tag{15}$$

$$V_3(p, -q, K) = -Z\bar{u}(p-q). \tag{16}$$

The ion mass operator for diagram 2 is obtained when (15) is multiplied by Z^2 and S_e is replaced by S_i.

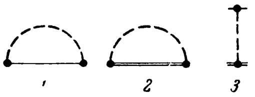

FIG. 1

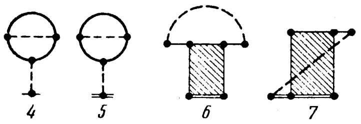

FIG. 2

We must also consider the four diagrams that are diagonal in $x_0 - y_0$, each containing two effective potential lines (Fig. 2), and the four diagrams with a "ladder" vertex (the cross-hatched rectangle in Fig. 2). Corresponding to diagram 4 we have the mass operator

$$M_{e4} = \beta^{-2}(2\pi)^{-6}\tilde{u}(0)\int dp\,dq S_e^2(p)\,\bar{u}(p-q)\,S_e(q)$$

$$\cong \frac{1}{2(1+Z)}\frac{e^2}{d}, \tag{17}$$

and to diagram 5 the same expression multiplied by −Z; these diagrams thus cancel each other exactly for the hydrogen atom. Two additional diagrams for M_e and M_i, which are not shown in Fig. 2, are obtained from 4 and 5 through the substitution of an ion loop for an electron loop. The corresponding mass operators are obtained through the multiplication of (17) by $-Z^2$ and Z^3; these are also canceled for the hydrogen atom.

We also give the vertex operator describing multiple electron-ion interaction. This operator is obtained by summing ladder diagrams:

$$V_l(\mathbf{p}, -\mathbf{q}, K)$$

$$= (2\pi)^3\,[T(\mathbf{p}) - i\omega_k - \mu]\sum_n \frac{T(\mathbf{q}) - K_{0n}}{i\omega_k + \mu - K_{0n}}\varphi_n(\mathbf{p})\,\varphi_n^*(\mathbf{q}); \tag{18}$$

here the summation is performed over the entire ion energy spectrum. This vertex has been considered by Galitskiĭ,[10] and its effect on the ion has been taken into account in [2]. One must also consider the compact diagrams 6 and 7 (Fig. 2) containing the ladder potential and one effective-potential line, and also two analogous ion diagrams.4) Diagram 6 corresponds to the operator

$$V_6(p, -q, K)$$

$$= \beta^{-1}(2\pi)^{-3}\int dr\tilde{u}(r)\,S_e(\nu K + p - r)\,V_l(p - \nu' r, -q$$

$$+ \nu' r, K - r)\,S_e(\nu K + q - r); \tag{19}$$

the additional operators are constructed analogously.

For the hydrogen atom the main contribution to the shift comes from eight nondiagonal diagrams containing two effective-potential lines. Figure 3 does not show three ion diagrams analogous to the electron diagrams 8, 9, and 10, The diagrams in

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4) Noncompact diagrams are taken into account by the Bethe-Salpeter equation.

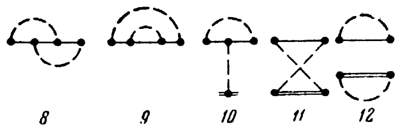

FIG. 3

Fig. 3 correspond to the operators

$$M_{e8}(p) = \beta^{-2}(2\pi)^{-6}\int dq\,dr\,S_e(p-q)\,\bar{u}(q)$$

$$\times S_e(p-q+r)\,\bar{u}(r)\,S_e(p+r); \tag{20}$$

$$M_{e9}(p) = \beta^{-2}(2\pi)^{-6}\int dq\,dr\,S_e(p-q)\,\bar{u}(q)$$

$$\times S_e(p-q+r)\,\bar{u}(r)\,S_e(p-q); \tag{21}$$

$$V_{10}(p,-q,K) = Z\beta^{-1}(2\pi)^{-3}\,\bar{u}(p-q)$$

$$\times\int dr\,S_e(\nu K+p-r)\,\bar{u}(r)\,S_e(\nu K+q-r), \tag{22}$$

$$V_{11}(p,-q,K) = -Z^2\beta^{-1}(2\pi)^{-3}$$

$$\times\int dr\,S_e(\nu K+p-r)\,\bar{u}(r)$$

$$\times S_i(\nu' K-q-r)\,\bar{u}(r-p+q). \tag{23}$$

Ion mass operators for diagrams analogous to 8 and 9 are obtained through multiplication of (20) and (21) by Z^4 and the replacement of S_e by S_i; the ion vertex analogous to 10 is obtained through multiplication of (22) by Z^4 and the replacement of S_e by S_i. The operator for diagram 12, corresponding to the product M_eM_i in (6), equals the product of (15) by the analogous expression for ions.

SHIFTS AND WIDTHS OF ENERGY LEVELS

The foregoing equations enable the calculation of energy level shifts and widths. The energy correction corresponding to the electron mass operator of diagram 1 is obtained from (11), (15), and (6); after summing over ω_p we have

$$\Delta E_1 = \beta^{-1}(2\pi)^{-3}\int d\mathbf{p}\,d\mathbf{q}\,|\varphi(\mathbf{p})|^2$$

$$\times\sum_{\omega_q}\frac{\bar{u}(q)}{T_e(\mathbf{p}-\mathbf{q})+T_i(\mathbf{p})-i\omega_k-\mu+i\omega_q}.$$

The principal contribution to ΔE_1 comes from the term with $\omega_q = 0$ and the region of small q.[1]5) Therefore the shift and width are

$$\delta E_1 = \beta^{-1}(2\pi)^{-3}\int\frac{d\mathbf{p}\,|\varphi(\mathbf{p})|^2}{T(\mathbf{p})-K_0}\int d\mathbf{q}\,\bar{u}(\mathbf{q},0)$$

$$= -\beta^{-1}\frac{e^2}{d}\int\frac{d\mathbf{p}\,|\varphi(\mathbf{p})|^2}{\mathbf{p}^2/2m-E},$$

$$\Gamma_1 = -\beta^{-1}\pi(2\pi)^{-3}\int d\mathbf{p}\,d\mathbf{q}\,|\varphi(\mathbf{p})|^2\,\bar{u}(\mathbf{q},0)\,\delta(T_e(\mathbf{p}-\mathbf{q})$$

$$+T_i(\mathbf{p})-K_0). \tag{24}$$

The shift for diagram 2 is obtained from (24) through multiplication by Z^2.

The energy correction corresponding to the vertex 3 is also determined by the zeroth harmonic of the effective potential; after summation over ω_p we have

$$\Delta E_3 = -Z\beta^{-1}(2\pi)^{-3}\int d\mathbf{p}\,d\mathbf{q}$$

$$\times\frac{\varphi^*(\mathbf{p})\,\varphi(\mathbf{p}-\mathbf{q})\,\bar{u}(\mathbf{q},0)\,[T(\mathbf{p})+T(\mathbf{p}-\mathbf{q})-2K_0]}{[T_e(\mathbf{p})+T_i(\mathbf{p}-\mathbf{q})-K_0-i\delta]\,[T_e(\mathbf{p}-\mathbf{q})+T_i(\mathbf{p})-K_0-i\delta]}. \tag{25}$$

The resulting shift is obtained from (24) through multiplication by $-2Z$; therefore the combined shift from diagrams 1, 2, and 3 is

$$\delta E^i = -(Z-1)^2\,T\frac{e^2}{d}\int d\mathbf{p}\,\frac{|\varphi(\mathbf{p})|^2}{\mathbf{p}^2/2m-E}; \tag{26}$$

this increases with temperature and density as $(TN_e)^{1/2}$ and is due to the Coulomb interaction between an ion and the plasma.

For the hydrogen atom this interaction is absent and (21) vanishes; the same result was obtained in [2]. By substituting in (26) the hydrogen functions in momentum representation,[7] we obtain

$$\delta E^i_{nl} = -\left(\frac{Z-1}{Z}\right)^2 T\,\frac{a}{d}\,\frac{8n^3\Gamma(n+l+1)\,\Gamma(l+{}^7\!/_2)\,\Gamma(n-l-3)}{\Gamma(n+l+4)\,\Gamma(l+{}^3\!/_2)\,\Gamma(n-l)\,\Gamma(-2)}$$

$$\times\,{}_4F_3\left[\begin{matrix}-n+l+1, & n+l+1, & l+{}^7\!/_2, & 3; & 1\\ -n+l+4, & n+l+4, & l+{}^3\!/_2 & & \end{matrix}\right], \tag{27}$$

where ${}_4F_3$ is a generalized hypergeometric function,[8]6) and a and d are the Bohr and Debye radii. Specifically,

$$\delta E^i_{n,\,n-1} = -T\,\frac{a}{d}\left(\frac{Z-1}{Z}\right)^2\frac{n^2(n+{}^3\!/_2)}{n+1},$$

$$\delta E^i_{n0} = -T\,\frac{a}{d}\left(\frac{Z-1}{Z}\right)^2\frac{3}{2}\,n^2,\qquad n = 2, 3, \ldots \tag{28}$$

For the combined width of the energy level from diagrams 1, 2, and 3 we obtain

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5) Terms with nonzero frequencies here and subsequently give only small corrections to the shifts and widths.

6) For the sake of brevity we have omitted the transition to the limit, as a result of which Γ(−2) cancels one of the factors in the numerator.

$$\Gamma = -\pi\beta^{-1}(2\pi)^{-3}\int d\mathbf{p}\,d\mathbf{q}\,\bar{u}(\mathbf{q},0)\{[Z^2|\varphi(\mathbf{p})|^2 - Z\varphi^*(\mathbf{p})\varphi(\mathbf{p}-\mathbf{q})]\,\delta\,(T_e(\mathbf{p}) + T_i(\mathbf{p}-\mathbf{q}) - K_0) + [|\varphi(\mathbf{p})|^2 - Z\varphi^*(\mathbf{p})\,\varphi(\mathbf{p}-\mathbf{q})]\,\delta\,[T_e(\mathbf{p}-\mathbf{q}) + T_i(\mathbf{p}) - K_0)\}. \qquad (29)$$

This expression differs from zero only for $K^2/2M > |E|$.

At temperatures $T \ll M|E|/m_e$ an appreciable contribution to Γ comes only from the first term. For an ion we have approximately

$$\Gamma^i = Z(Z^2-1)\frac{2\pi e^4 N_e}{m_i}\int d\mathbf{p}\,\frac{|\varphi(\mathbf{p})|^2\,|v'\mathbf{K}-\mathbf{p}|}{(\mathbf{p}^2/2m - E)^2} \sim \frac{e^2}{a}\left(\frac{m_e}{m_i}\frac{T}{|E|}\right)^{1/2} N_e a^3. \qquad (30)$$

Here integration is performed over the region $T_e(\mathbf{p}) \leq K_0$. Thus we have a width $\sim T^{1/2}N$ for an ion.

At temperatures $T \gg M|E|/m$ both terms in (29) give identical contributions and $Z(Z-1)$ in (30) is replaced by $(Z-1)(Z^2-1)$. For a hydrogen atom we have the width

$$\Gamma^a = -\beta^{-1}\pi(2\pi)^{-3}\int d\mathbf{p}\varphi^*(\mathbf{p})\,\nabla_{\mathbf{p}}\varphi(\mathbf{p})\int d\mathbf{q}\bar{u}(\mathbf{q},0)\,\delta\,(T_e(\mathbf{p}) + T_i(\mathbf{p}-\mathbf{q}) - K_0); \qquad (31)$$

this expression contains the interaction energy of an atomic dipole moment with the field $\bar{u}(q, 0)$. At temperatures $T > M|E|/m_e$ we add to the δ function in (31) a second δ function given in (29).

We now consider the diagrams in Fig. 2. It follows from (11) and (6) that the shift corresponding to mass operators that are diagonal in $x_0 - y_0$ (which are independent of frequency in the momentum representation) is given by

$$\Delta E = \int d\mathbf{p}\,|\varphi(\mathbf{p})|^2 M_e(v\mathbf{K} + \mathbf{p}) \qquad (32)$$

in the case of the electron mass operator, and similarly in the case of the ion mass operator, and has no imaginary part.

The shift of the mass operator 4, which is still independent of momentum, is M_{e4} [see (17)]; multiplying (17) by $-Z$, $-Z^2$, and Z^3, we obtain the shifts associated with diagram 5 and the analogous ion diagrams. The combined shift for the four diagonal diagrams with two effective-potential lines is

$$\delta E^i = \tfrac{1}{2}(Z-1)^2 e^2/d; \qquad (33)$$

this is proportional to $N_e^{1/2}$, decreases with increasing temperature as $T^{-1/2}$, and vanishes for hydrogen. At high temperatures it is considerably smaller than the shift (27).

The shift for the four diagrams containing a ladder potential is, in virtue of (11) and (18), given by

$$\delta E^i = -\tfrac{1}{4}(Z-1)^2\frac{e^2}{d}\frac{T}{\mathbf{k}^2/2M}; \qquad (34)$$

this is of the same order as (33) and vanishes for hydrogen atoms.

Let us now consider the diagrams in Fig. 3. All eight diagrams give real energy corrections differing only in their numerical coefficients; here the diagrams 8, 9, and 10 and the analogous ion diagrams correspond to the coefficients 1, 1, $-4Z$, Z^4, Z^4, and $-4Z^3$; the diagrams $6Z^2$ and $-6Z^2$ correspond to the diagrams 11 and 12. Thus the combined energy level shift for the diagrams in Fig. 3 is

$$\delta E = -2(Z^4 - 2Z^3 - 2Z + 1)\,T^2\frac{e^4}{d^2}\int\frac{d\mathbf{p}\,|\varphi(\mathbf{p})|^2}{(\mathbf{p}^2/2m - E)^3}. \qquad (35)$$

For ions this shift is always considerably smaller than (27) and can be neglected.

However, (35) does not disappear for $Z = 1$ and determines the shift for hydrogen atoms.[7)] The integral in (35) is also expressed in terms of generalized hypergeometric functions:[8]

$$\delta E^a_{nl} = TN_e a^3\, 2^{10}\,\pi\,\frac{n^7\Gamma(n+l+1)\,\Gamma(l+{}^{11}\!/_2)\,\Gamma(n-l-5)}{\Gamma(n+l+6)\,\Gamma(l+{}^3\!/_2)\,\Gamma(n-l)\,\Gamma(-4)} \times {}_4F_3\left[\begin{matrix}-n+l+1,\ n+l+1,\ l+{}^{11}\!/_2,\ 5;1\\ -n+l+6,\ n+l+6,\ l+{}^3\!/_2\end{matrix}\right]. \qquad (36)$$

Specifically,

$$\delta E^a_{n,n-1} = TN_e a^3 2^5\pi n^6\,(n + {}^7\!/_2)\,(n + {}^5\!/_2)/(n+1)\,(n+2);$$
$$\delta E^a_{20} = TN_e a^3 2^6 138\pi,\ \delta E^a_{n0} = TN_e a^3 n^6 140\pi,\ n = 3, 4, \ldots \qquad (37)$$

The shifts of hydrogen levels increase with temperature and density as TN. This dependence on density is associated with the short-range forces between atoms and plasma charges.

The foregoing shifts must be compared with the level shift determined by the second term in (11), for which we have approximately

$$\delta E_{nl} \simeq e^{\beta\mu_e}\int d\mathbf{p}\,d\mathbf{q}\,\varphi^*(\mathbf{p})\,u\,(\mathbf{p}-\mathbf{q})\,\varphi(\mathbf{q}) = 16\pi^{3/2}\frac{|E_2|}{n^2}\left(\frac{|E_1|}{T}\right)^{3/2} N_e a^3. \qquad (38)$$

This shift decreases with increasing temperature and at high temperatures is considerably smaller than (36). Thus the shifts and broadening of both atomic and ionic energy levels are determined by the diagrams containing the effective potential.

[7)] The nonzero frequencies in the diagrams of Fig. 1 give for hydrogen atoms the shift $\sim N_e T^{-1}$. At high temperatures its ratio to (36) is of the order $(e^2/aT)^2$.

The foregoing calculated shifts of levels lead to shifts in the frequencies of emitted photons; these shifts are negative for ions and are positive for atoms. For atoms this result does not disagree with experimental results;[11] no data are available for ions.

The conditions for applying the foregoing results are determined first, by the possibility of approximating the operators M_e, M_i, and V by the given diagrams; secondly, by the validity of the approximations performed in calculating the matrix elements; and thirdly, by the validity of the perturbation theory as applied to (6). The first and third conditions are fulfilled in any case for a high-temperature plasma, when the ionization energy is small compared with the thermal energy ($Z^2e^2/2aT \ll 1$). Thus these conditions provide a lower limit of the temperature region. The third condition leads to the requirement $\delta E_n \ll E_n$; this condition is violated for large TN_e because of the rapid increase of δE_n with n. In this way upper limits of temperature and density are obtained: the upper limit of $N_e a^3 \cdot aT/e^2$ is of the order $Z^8/9(Z-1)^2n^8$ for ions and $10^{-3}n^{-8}$ for hydrogen atoms.

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