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OSCILLATIONS OF A LOW-PRESSURE INHOMOGENEOUS PLASMA

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The methods of geometric optics are extended to electrodynamics with spatial dispersion, in which case the field equations are integral equations. This method is applied to the problem of stability of a magnetically confined plasma. The dispersion relations for longitudinal oscillations are obtained. Analysis of the dispersion relation for the limiting cases of longwave and shortwave oscillations yields the necessary and sufficient conditions for plasma instability. It is shown, in particular, that if the electron-to-ion temperature ratio is independent of coordinates a weakly inhomogeneous low-pressure plasma confined by a magnetic field is almost always unstable against shortwave oscillations.

INTRODUCTION

In recent years a large number of papers devoted to the theory of stability of spatially inhomogeneous plasma contained by a strong magnetic field have appeared in the literature. The conclusions drawn in this work are of great interest. However, in many papers [1-10] the results are obtained by a method that is incorrect and appear to be rather strange. The unusual feature is the fact that the spectrum of characteristic oscillations of the inhomogeneous plasma turns out to be a function of position.

Several authors have shown [11-13] that the solution of the stability problem for a weakly inhomogeneous plasma can be obtained by methods of geometric optics. This approach not only avoids the strange spatial dependence of the oscillation spectrum of the plasma but sheds light on the significance of certain results obtained earlier [1-10] and leads to many new results.

The methods of geometric optics are based on the theory of asymptotic solutions of differential equations. In [11-13] this method has been used in cases in which the investigation of plasma stability could be carried out within the framework of differential equations of second order. However, in a great number of interesting cases the field equations that describe the small oscillations of the inhomogeneous plasma are integral equations. Hence, the problem arises of formulating a method of geometric optics for cases in which the field equations are integral equations.

It should be emphasized that integral field equations are a characteristic feature of electrodynamics with spatial dispersion. For example, in a weakly inhomogeneous medium the relation between the electric field E and the induction D (cf. [14]) can be written in the form

$$D_i'(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \int d\mathbf{r}' \hat{\varepsilon}_{ij}(t - t', \mathbf{r} - \mathbf{r}', \mathbf{r}) E_j(\mathbf{r}', t'). \quad (0.1)$$

By weak inhomogeneity we mean that the dependence on $\mathbf{r}$ is much weaker than on $\mathbf{r} - \mathbf{r}'$.

In geometric optics the field is written in the form

$$\mathbf{E} = \mathbf{E}_0 e^{i\psi(\mathbf{r})}, \quad (0.2)$$

where $\psi(\mathbf{r})$ is the eikonal while the amplitude $\mathbf{E}_0$ is a slowly varying function of the coordinates and the time. We shall be interested below in monochromatic waves, in which case $\psi = -\omega t + \psi_1$. The gradient of the function $\psi_1(\mathbf{r})$ determines the wave vector of the field oscillation, $\nabla\psi_1(\mathbf{r}) = \mathbf{k}(\mathbf{r})$. Then in view of the rapid dependence of $\hat{\epsilon}_{ij}$ on $\mathbf{r} - \mathbf{r}'$, we can write as an approximation

$$D_i'(\mathbf{r}, t) \approx e^{-i\omega t + i\psi_1(\mathbf{r})} \varepsilon_{ij}(\omega, \nabla\psi_1, \mathbf{r}) E_{0j}, \quad (0.3)$$

where [14]

$$\varepsilon_{ij}(\omega, \nabla\psi_1, \mathbf{r}) = \int_0^{\infty} dt \int d\mathbf{r}' e^{i\omega t - i\nabla\psi_1 \mathbf{r}'} \hat{\varepsilon}_{ij}(t, \mathbf{r}', \mathbf{r}). \quad (0.4)$$

The expression in (0.3) is approximate and corresponds to the zeroth approximation of geometric optics, in which we neglect spatial derivatives of both E_{0j} and ϵ_{ij}.

Substituting (0.3) in Maxwell's equations we obtain in the same zeroth approximation

$$|(\nabla\psi_1)^2\delta_{ij} - (\nabla\psi_1)_i(\nabla\psi_1)_j - \omega^2 c^{-2}\varepsilon_{ij}(\omega, \nabla\psi_1, \mathbf{r})| = 0, \quad (0.5)$$

which represents a generalization of the usual eikonal equation to the case of a medium with spatial dispersion. The difference lies in the dependence of the dielectric tensor on $\mathbf{k} = \nabla\psi_1$ which, obviously leads to extensive complications in the analysis and solution of the generalized eikonal equation.

The higher approximations in the method of geometric optics for a medium with spatial dispersion can be obtained by making more precise the right side of Eq. (0.3) and by taking account of the appropriate derivatives in the field equations. We shall not go into the development of the general theory here. However, in the solution of the problem given below (stability of magnetic containment of a low pressure plasma) we will actually give geometric-optics solutions with accuracy up to the first approximation, and will also consider some of the central problems in geometric optics—the question of turning points, the Stokes phenomena, and the spectrum of characteristic frequencies of the field equations.[1)]

In the first section of the present work we give a brief derivation of the equation for the longitudinal oscillations of a weakly inhomogeneous collisionless plasma contained by a strong magnetic field. We limit ourselves to a one-dimensional geometry so that all equilibrium quantities characterizing the plasma depend only on the single coordinate x. The force lines of the external magnetic field are assumed to be straight and parallel to the z axis. In the second section we solve the equations of the longitudinal field oscillation in a plasma with accuracy to first order in geometric optics. In the last and third section we investigate the oscillation spectrum and obtain stability criteria for an inhomogeneous plasma in certain particular cases.

I. INTEGRAL EQUATION FOR LONGITUDINAL OSCILLATIONS OF AN INHOMOGENEOUS PLASMA

Before we obtain the equation for small oscillations of the field of an inhomogeneous plasma, we must first determine the particle distribution functions (electrons and ions) in the equilibrium state. Assuming that there is no electric field in the equilibrium state, we have

$$v_\perp \cos\varphi\, \partial f_0/\partial x - \Omega\partial f_0/\partial\varphi = 0. \tag{1.1}$$

Here, f_0 is the equilibrium distribution function while $\Omega = eB/mc$ is the particle Larmor frequency.

In the case of interest, a low pressure plasma, the magnetic pressure $B^2/8\pi$ is much greater than the kinetic pressure p and the inhomogeneity in the magnetic field can be neglected compared with the density and temperature variations in the plasma. If the plasma inhomogeneity is small enough, specifically, if the particle Larmor radius is small compared with the characteristic scale size of the inhomogeneity L,

$$v_T/\Omega L \ll 1, \tag{1.2}$$

where $v_T = \sqrt{T/m}$ is the particle thermal velocity, then the solution of (1.1) can be written in the form

$$f_0(x, \varphi, v_\perp, v_z) = \left(1 + \frac{v_\perp \sin\varphi}{\Omega}\frac{\partial}{\partial x} - \frac{1}{4}\frac{v_\perp^2 \cos 2\varphi}{\Omega^2}\frac{\partial^2}{\partial x^2} + \ldots\right) F(v_\perp, v_z, x). \tag{1.3}$$

Because the problem is homogeneous in time and the coordinates y and z, in plasma states that are close to equilibrium, the particle distribution function can be written in the form $f(x)\exp\{-i\omega t + ik_y y + ik_z z\}$. In this case, if we consider only longitudinal plasma oscillations ($\nabla\times\mathbf{E} = 0$, $\mathbf{E} = -\nabla\Phi$), the function f(x) is given by the equation[2)]

$$-i(\omega - k_z v_z - k_y v_\perp \sin\varphi) f + v_\perp \cos\varphi\, \partial f/\partial x - \Omega\, \partial f/\partial\varphi = (e/m)(\nabla\Phi)\, \partial f_0/d\mathbf{v}. \tag{1.4}$$

The solution of this equation that satisfies the periodicity condition in φ is of the form

$$f(x) = -\frac{ie}{2\pi m\Omega}\int dx_1 \Phi(x_1) \int dk_x e^{ik_x(x-x_1)} \int_\infty^\varphi d\varphi' \mathbf{k}\, \frac{\partial f(x', \varphi')}{\partial\mathbf{v}} \times \exp\left\{\frac{i}{\Omega}\int_\varphi^{\varphi'} (\omega - \mathbf{kv})\, d\varphi''\right\}, \tag{1.5}$$

where $\mathbf{k} = \{k_x, k_y, k_z\}$ while $x' = x + [v_\perp(\sin\varphi - \sin\varphi')]/\Omega$. Using Eq. (1.5) we find the charge density induced in the plasma

$$\rho(x) = \sum e\int f d\mathbf{v} = -\frac{1}{4\pi}\int dx_1\, G(x - x_1, x)\, \Phi(x_1), \tag{1.6}$$

where

1) In many concrete problems the "integral" aspects of the field equations are not important. In these cases, as shown in [15], the integral terms can be neglected in the first approximation and their contribution considered by means of perturbation theory, using the asymptotic characteristic functions of the differential field equations. The results obtained by this direct calculation[15] can obviously be also obtained by the method given here.

2) Plasma oscillations with phase velocities appreciably smaller than the Alfvén velocity can be regarded as longitudinal. In these cases the magnetic field of the wave can be neglected and the complete system of Maxwell's equations reduces to the Poisson equation for the potential Φ(x).

$$G(x - x_1, x) = -2\pi \int dk_x\, e^{ik_x(x-x_1)} D(x, \mathbf{k}), \quad (1.7)$$

$$D(x, \mathbf{k}) = -\sum \frac{ie^2}{m\Omega} \int d\mathbf{v} \int_{\infty}^{\varphi} d\varphi'$$
$$\times \exp\left\{\frac{i}{\Omega}\int_{\varphi}^{\varphi'} (\omega - \mathbf{kv})\, d\varphi''\right\} \mathbf{k} \frac{\partial f(x', \varphi')}{\partial \mathbf{v}}. \quad (1.8)$$

The summation is taken over all charged particles (electrons and ions).

For a weakly inhomogeneous plasma in which $F(v_\perp, v_z, x)$ is a Maxwellian distribution function with inhomogeneous temperature $T(x)$ and density $N(x)$, keeping the first terms in the expansion (1.3), we have from (1.8)

$$D(x, \mathbf{k}) = D_1(x, k_x^2, k_y, k_z) + ik_x D_2(x, k_x^2, k_y, k_z); \quad (1.9)$$

$$D_1(x, \mathbf{k})$$
$$= -\sum e^2 \left\{\frac{N}{T} - \sum_{n=-\infty}^{\infty}\left(1 + \frac{nk_y}{k_\perp^2}\frac{\partial}{\partial x}\right)\left(\frac{\omega}{T} - \frac{k_y}{m\Omega}\frac{\partial}{\partial x}\right)\right.$$
$$\left. \times \frac{N}{\omega - n\Omega} J_+\left(\frac{\omega - n\Omega}{k_z v_T}\right) A_n\left(\frac{k_\perp^2 v_T^2}{\Omega^2}\right)\right\},$$
$$D_2(x, \mathbf{k}) = -\sum e^2 \sum_{n=-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{\omega}{T} - \frac{k_y}{m\Omega}\frac{\partial}{\partial x}\right)\frac{N}{\omega - n\Omega}\frac{v_T^2}{\Omega^2}$$
$$\times J_+\left(\frac{\omega - n\Omega}{k_z v_T}\right) A_n'\left(\frac{k_\perp^2 v_T^2}{\Omega}\right). \quad (1.10)$$

Here, $k_\perp^2 = k_x^2 + k_y^2$, $A_n(x) = e^{-x} I_n(x)$, $I_n(x)$ is the Bessel function of imaginary argument, $J_+(z) = ze^{-z^2/2} \int_{i\infty}^{z} d\tau\, e^{\tau^2/2}$, and the primes denote differentiation with respect to the argument.

We shall be interested in the time development of initial field perturbations or perturbations in the particle distribution functions. In the case of longitudinal plasma oscillations this problem reduces to the problem of finding the eigenvalues of the equation

$$\Delta\Phi(x) = -4\pi\rho(x) \quad (1.11)$$

with specified boundary conditions on the function $\Phi(x)$.[3)]

II. SOLUTION OF THE INTEGRAL WAVE EQUATION

In the general case Eq. (1.11) is an integro-differential equation for which an exact solution would be difficult. Following the general approach given in the introduction we find the asymptotic solution of (1.11) to first order in geometric optics. The essential point here is the weak dependence of the kernel $G(x - x_1, x)$ on the argument x, which makes it possible to reduce the integral equation (1.11) to some nonlinear first-order differential equation containing a small parameter. The expansion of the solution of this equation in the small parameter then allows us to formulate an approximate solution for the integral equation (1.11).

According to (1.7) the dependence of the kernel of the integral equation (1.11) on the variable x is determined completely by the function $D(x, \mathbf{k})$, which is a slowly varying function of the coordinate x; specifically, it changes significantly over distances of order L (the characteristic scale size of the inhomogeneity). At the same time the kernel $G(x - x_1, x)$ varies much more rapidly as a function of the difference $x - x_1$.

In a uniform plasma the function $D(x, \mathbf{k})$ is independent of coordinates; in this case the kernel $G(x - x_1, x)$ is a function only of the difference $x - x_1$. This situation allows us to find solutions of (1.11) for a uniform plasma in the form $e^{ik_x x}$ where k_x is obviously independent of x. In the case of a weakly inhomogeneous plasma, following the method of geometric optics we seek solutions of (1.11) in the form

$$\Phi(x) = C \exp\left\{i \int^{x} k_x(x')\, dx'\right\}, \quad (2.1)$$

where the function $k_x(x)$ is assumed to be a slowly varying function of x. Substituting the solution (2.1) in (1.11), expanding the function $\int_x^{x_1} k_x(x')\, dx'$ in a Taylor series about the point x and keeping only the first two terms in the expansion,

$$\Phi(x_1) \approx \Phi(x)\, e^{i(x_1 - x)k_x(x)}\left[1 + \tfrac{1}{2} i (x_1 - x)^2 k_x'(x)\right],$$

under the assumption that $k_x'(x)(x_1 - x)^2 \ll 1$ we obtain the following differential equation[4)] for determining the function $k_x(x)$:

$$k_x^2 + k_y^2 + k_z^2 - ik_x' = 4\pi\left(1 - \tfrac{1}{2} ik_x' \partial^2/\partial k_x^2\right) D(x, \mathbf{k}). \quad (2.2)$$

Thus, we have obtained a nonlinear first-order equation for $k_x(x)$. A simplifying feature in the solution of this equation in the geometric optics approximation is the presence of a small param-

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3)We note that our derivation of Eq. (1.11) is more general and simpler than the corresponding derivation given by Mikhaĭlovskiĭ.[16]

4)From the derivation of (2.2) it is evident that the requirement $(x_1 - x)^2 k_x'(x) \ll 1$ reduces to the condition

$$k_x' \partial^2 D(x, k)/\partial k_x^2 \ll D(x, \mathbf{k}).$$

eter of order k'_x/k_x^2. In the zeroth approximation in this parameter $k_x(x)$ is given by the algebraic equation

$$k_x^2 + k_y^2 + k_z^2 - 4\pi D\,(x,\,\mathbf{k}) = 0. \qquad (2.3)$$

Everywhere below by k_x we mean the solution of this equation.

The next approximation gives the correction $\delta k_x(x)$:

$$\frac{\delta k_x}{k_x} = \frac{i}{2k_x}\left[\ln k_x\left(1 - 4\pi\,\frac{\partial}{\partial k_x^2}\,D\,(x,\,\mathbf{k})\right)\right]' \ll 1. \quad (2.4)$$

The requirement that this correction be small is essentially the requirement for the validity of the geometric optics approximation for determining the asymptotic solution of the integral equation (1.11).

In this approximation the solutions of (2.2) become

$$\Phi\,(x) = C\left\{k_x\left(1 - 4\pi\,\frac{\partial}{\partial k_x^2}\,D\,(x,\mathbf{k})\right)\right\}^{-1/2}\exp\left\{i\int^x k_x(x')\,dx'\right\}, \qquad (2.5)$$

where the functions $k_x(x)$ are determined from (2.3). It is evident that the condition in (2.4) is violated close to the turning point, where $k_x(x) = 0$, and the point where $1 - 4\pi\,\partial D(x,\mathbf{k})/\partial k_x^2 = 0$. In the vicinity of these points (2.5) is not valid. We assume below that the function $1 - 4\pi\,\partial D(x,\mathbf{k})/\partial k_x^2$ does not vanish; solutions of the integral equation close to the turning point are discussed in the next section.

Finally, we point out one feature that simplifies the analysis of (2.3) in a number of cases. It follows from Eq. (1.10) that $D_2(x,\mathbf{k}) \sim k_x^{-2}\,\partial D_1(x,\mathbf{k})/\partial x \sim D_1(x,\mathbf{k})/Lk_x^2$. On the other hand, from the applicability of the geometric optics approximation $k'_x \sim k_x/L \ll k_x^2$. Hence, in the approximation used here $D(x,\mathbf{k}) \approx D_1(x,\mathbf{k})$. Taking account of this relation we note that within the framwork of geometric optics (2.3) has symmetric roots $\pm k_x(x)$ that play the role of wave numbers in the direction of the plasma inhomogeneity.

III. LOW-FREQUENCY OSCILLATION SPECTRUM OF AN INHOMOGENEOUS PLASMA

We now apply the solutions of (1.11) obtained above to the investigation of the spectrum of low-frequency longitudinal oscillations of an inhomogeneous plasma. We note that actual spectra of longitudinal plasma oscillations have already been studied in [11] and [12]. In particular, in these papers consideration was given to cases in which the integral equation (1.11) could be reduced to a second-order differential equation. This procedure is possible for long wavelength plasma oscillations, in which case the root of (2.3) satisfies the condition[5)] $k_x^2(x) \ll \max\,\{k_y^2,\,\Omega^2/v_T^2\}$. In view of the results obtained earlier in [11] and [12], we treat here two other cases of longitudinal oscillations of an inhomogeneous plasma. We limit ourselves to low-frequency oscillations defined by $k_z v_{Ti} \ll \omega \ll k_z v_{Te} \ll \Omega_i$.

For simplicity we shall assume that the wavelength of the oscillations is appreciably greater than the electron Larmor radius. Assuming that in the frequency range being considered it is sufficient to consider only the $n = 0$ terms in (1.10), we have from (2.3)[6)]

$$k_\perp^2 + k_z^2 + 4\pi e^2\left\{\frac{N}{T_e} + \frac{N}{T_i} - \left(\frac{1}{T_i} - \frac{k_y}{M\omega\Omega_i}\,\frac{\partial}{\partial x}\right)NA_0\left(\frac{k_\perp^2 v_{Ti}^2}{\Omega_i^2}\right)\right.$$
$$\left.+\, i\sqrt{\frac{\pi}{2}}\left(\frac{1}{T_e} + \frac{k_y}{M\omega\Omega_i}\,\frac{\partial}{\partial x}\right)N\,\frac{\omega}{|k_z|\,v_{Te}}\right\} = 0, \qquad (3.1)$$

where $k_\perp^2 = k_x^2 + k_y^2$. We now investigate Eq. (3.1) in the limiting cases of longwave and shortwave plasma oscillations.

In the long wavelength region, where $k_\perp^2 v_{Ti}^2 \ll \Omega_i^2$, we have from (3.1)

$$k_x^2\,(x) = -\,k_y^2 - \left\{1 - 4\pi e^2\left(\frac{1}{T_i} - \frac{k_y}{M\omega\Omega_i}\,\frac{\partial}{\partial x}\right)\frac{Nv_{Ti}^2}{\Omega_i^2}\right\}^{-1}$$
$$\times\left\{k_z^2 + 4\pi e^2\left(\frac{1}{T_e} + \frac{k_y}{M\omega\Omega_i}\,\frac{\partial}{\partial x}\right)N\left(1 + i\sqrt{\frac{\pi}{2}}\,\frac{\omega}{|k_z|\,v_{Te}}\right)\right\}. \qquad (3.2)$$

In the other limiting case, where $k_\perp^2 v_{Ti}^2 \gg \Omega_i^2$ but with the wavelength greater than the Debye radius, we have from (3.1)

$$\frac{N}{T_e} + \frac{N}{T_i} + \frac{k_y}{\sqrt{2\pi}\,k_\perp M\omega}\,\frac{\partial}{\partial x}\,\frac{N}{v_{Ti}}$$
$$+\, i\sqrt{\frac{\pi}{2}}\,\frac{k_y}{|k_z|\,M\Omega_i}\,\frac{\partial}{\partial x}\,\frac{N}{v_{Te}} = 0. \qquad (3.3)$$

Hence,

$$k_x^2\,(x) = -\,k_y^2 + \frac{1}{2\pi}\,\frac{k_y^2 v_{Ti}^4\,[(N/v_{Ti})']^2}{N^2\omega^2\,(1 + T_i/T_e)^2}$$
$$\times\left\{1 - i\sqrt{2\pi}\,\frac{k_y v_{Ti}^2\,(N/v_{Ti})'}{|k_z|\,\Omega_i\,N\,(1 + T_i/T_e)}\right\}. \qquad (3.4)$$

The relations in (3.2) and (3.4) can now be considered aside from their relation to the original

5)It is evident that this condition is always satisfied for a plasma at zero temperature.

6)Here and everywhere below the ions are assumed to be singly charged, i.e., $|e_i| = |e_e|$. As a result $N_e = N_i = N(x)$.

integro-differential equation (1.11). Specifically, (3.2) and (3.4), which give the zero-order geometric optics solution of the integro-differential equation (1.11), are at the same time the zero-order geometric optics solution of the second-order differential equation

$$\Phi''(x) + k_x^2(x)\,\Phi(x) = 0. \qquad (3.5)$$

In this sense (1.11) and (3.5) are "equivalent."

The following point should be emphasized. The relation in (3.2), which corresponds to longwave oscillations, arises when the integro-differential equation (1.11) reduces to a second-order differential equation. In this sense, the writing of (3.5) is trivial. On the other hand, (3.4) is obtained as a result of an approximate solution of an essentially integral equation.[7)] In this sense, the formation of an equivalent second-order equation (3.5) is, in our opinion, of great importance.

We are now in a position to use many of the results of the theory of asymptotic solutions of second-order differential equations, carrying them over by means of the approximate equivalent equation (3.5) to the case of interest here, namely the theory of asymptotic solutions of integral equations. We recall briefly here some of the important points in the theory of asymptotic solutions of (3.5) that are pertinent here. This is necessary here for where the theory is well known to the physicist for real $k_x^2(x)$, for example from quantum mechanics, it is not well known for complex $k_x^2(x)$. We shall not go into details here but refer the reader to the specialized literature; [18-22] we shall only recall the situation as regards the Stokes phenomenon and the eigenvalue spectra.

The asymptotic solution of Eq. (3.5) is of the form

$$\Phi(x) = C_{(+)}\exp\left\{i\int^{x} k_x(x')\,dx'\right\} + C_{(-)}\exp\left\{-\,i\int^{x} k_x(x')\,dx'\right\}. \qquad (3.6)$$

In the general case the complex plane of x can be divided into regions in which the coefficients $C_{(\pm)}$ have definite values. Under these conditions the transition from one such region to another leads to a discontinuous change in the coefficients (Stokes phenomenon).

The simplest case is the one in which the real x axis does not go outside of one of the Stokes regions over the entire range of values of physical interest. In this case the asymptotic eigenvalue spectrum is given by

$$\int k_x(x)\,dx = n\pi, \qquad (3.7)$$

where n is an integer appreciably greater than unity while the integration is carried out over the entire physical range of values of x. If we do not use special dissipative or real boundary conditions the calculation of corrections to the spectrum in (3.7) in the worst case leads to the appearance of a real term of order unity on the right side of (7)[8)].

If, however, the physical region of values of the coordinate x encompasses several Stokes regions, than both the equation which determines the spectrum of characteristic frequencies and the constants $C_{(\pm)}$ in (3.6) can be obtained by an analysis of the solutions of (3.5) in the vicinity of the turning points, given by

$$k_x^2(x) = 0. \qquad (3.8)$$

In the vicinity of the turning points the solution of (3.5) is given in terms of Bessel functions. The asymptotic behavior and the Stokes phenomena for these functions are well known. By requiring the coincidence of this asymptotic solution with (3.6) we determine the constants $C_{(\pm)}$. Since the turning point lies on the Stokes line which divides the Stokes regions in this way we find the constants in (3.6) for neighboring Stokes regions.

Without going into greater detail on this question (it is much simpler to consider each case individually[9)]) we give here only the final result for the most interesting case, in which the imaginary part of the function $k_x^2(x)$ is small. The dispersion equation for the spectrum of characteristic frequencies is of the form in (3.7) except that n is replaced by $n + 1/2$ while the integration is taken between the turning points. For simplicity we assume that the transparency region $\mathrm{Re}\, k_x^2(x) > 0$.

Having in mind the fact that the imaginary part of $k_x^2(x)$ is small, separating real and imaginary parts, and neglecting the small contribution due to integration along the imaginary axis, we can write the two following relations, which give the frequency ω and growth rate γ of the oscillations:

[7)]This integral equation is actually equivalent to

$$\Phi(x) = f(x)\int dx_1 \Phi(x_1)\, K_0(|k_y(x - x_1)|),$$

where $K_0(x)$ is the Macdonald function.

[8)]It should be noted that the difference between (2.5) and the first-order solution (3.5) does not change the dispersion relation (3.7). In the vicinity of the turning point of the function $k_x^2(\omega, x)$ where one joins the asymptotic solutions of (3.5) it can be assumed that $1 - 4\pi\partial D(\omega, \mathbf{k})/\partial k_x^2 = \mathrm{const}$ while the expression in (2.5) coincides with the solution of (3.5).

[9)]Cf. [17], where a corresponding analysis is given for particular assumptions with regard to the form of the function $k_x^2(x)$.

$$\int dx\ \operatorname{Re} k_x(\omega, x) = n\pi, \qquad (3.9)$$

$$\int dx \left\{ \operatorname{Im}\ k_x(\omega, x) + \gamma \frac{\partial}{\partial\omega} \operatorname{Re} k_x(\omega, x) \right\} = 0, \qquad (3.10)$$

where the integration is taken over the region of transparency between the projections on the real axis of the complex turning points (3.8).[10)]

After these general considerations we return to the question of stability of an inhomogeneous plasma with respect to the low-frequency oscillations indicated above. The oscillation spectrum has been studied in [11] for dense plasma in which $V_A^2 = B^2/4\pi NM \ll c^2$. In the case of a low-density plasma, where $V_A^2 \gg c^2$, using (3.2), (3.9), and (3.10) we obtain the following dispersion equations for the frequency and growth rates of the oscillations in an inhomogeneous plasma:

$$\int dx \sqrt{Q(\omega x)}$$

$$= \int dx \left[-k_y^2 - k_z^2 - \frac{4\pi e^2 N}{T_e}\left(1 + \frac{T_e}{T_i}\frac{k_y V_d}{\omega}\right)\right]^{1/2} = n\pi,$$

$$\gamma = \sqrt{\frac{\pi}{2}}\frac{\omega^2}{|k_z|}\left\{\int dx \frac{N}{T_i}\frac{k_y V_d}{\omega\sqrt{Q(\omega x)}}\right\}^{-1}$$

$$\times \int dx \frac{N}{T_i}\frac{k_y V_d}{\omega v_{Te}} Q^{-1/2}(\omega, x)\left[1 + \frac{T_i}{T_e}\frac{\omega}{k_y V_d} - \frac{1}{2}\frac{\partial \ln T_e}{\partial \ln N}\right], \qquad (3.11)$$

where $V_d = (v_{Ti}^2/\Omega_i)\,\partial \ln N/\partial x$.

From the first equation in (3.11) it follows that

$$N k_y V_d / T_i\,\omega \leqslant -(k_y^2 + k_z^2 + r_D^2)/4\pi e^2, \qquad (3.12)$$

where $r_D = (T_e/4\pi e^2 N)^{1/2}$ is the electron Debye radius. By virtue of this inequality we obtain the instability condition for an inhomogeneous plasma (positive growth rate)

$$\int \frac{k_y N V_d}{\omega T_i v_{Te}}\left[1 + \frac{T_i}{T_e}\frac{\omega}{k_y V_d} - \frac{1}{2}\frac{\partial \ln T_e}{\partial \ln N}\right]\frac{dx}{\sqrt{Q(\omega, x)}} < 0. \qquad (3.13)$$

Using (3.12) and (3.13) we obtain the local instability condition

$$(k_y^2 + k_z^2)\, r_D^2/[1 + (k_y^2 + k_z^2)\, r_D^2] - \tfrac{1}{2}\partial \ln T_e/\partial \ln N > 0, \qquad (3.14)$$

satisfaction of which over the entire range of transparency of the plasma is patently sufficient. Whence it follows, in particular, that oscillations characterized by the wavelengths greater than the electron Debye radius are excited if $\partial \ln T_e/\partial \ln N \le 0$ in the entire transparency region. Oscillations with wavelengths smaller than the electron Debye radius are excited when $\partial \ln T_e/\partial \ln N < 2$. If, $\partial \ln T_e/\partial \ln N \ge 2$ over the entire transparency region of the plasma the longwave oscillations considered here cannot be excited and the inhomogeneous plasma is stable with respect to such oscillations.

For the shortwave oscillations of an inhomogeneous plasma the dispersion equations are, in accordance with Eqs. (3.4), (3.9), and (3.10),

$$\int dx \sqrt{Q(\omega, x)} = \int dx \left\{ k_y^2 \left[\frac{1}{2\pi}\frac{v_{Ti}^4\,[(N/v_{Ti})']^2}{N^2\omega^2(1 + T_i/T_e)^2} - 1\right]\right\}^{1/2} = n\pi,$$

$$\gamma = -\sqrt{\frac{\pi}{2}}\frac{k_y^2}{|k_z|\Omega_i}\left\{\int \frac{k_y^2 + Q(\omega,x)}{\sqrt{Q(\omega, x)}}dx\right\}^{-1}$$

$$\times \int dx \frac{k_y^2 + Q(\omega,x)}{\sqrt{Q(\omega,x)}}\frac{v_{Ti}^3}{v_{Te}N(1 + T_i/T_e)}\frac{\omega}{k_y}\left(\frac{N}{v_{Ti}}\right)'$$

$$\times \left[1 + \frac{1}{2}\frac{\partial \ln (T_i/T_e)}{\partial \ln (N/T_i^{1/2})}\right]. \qquad (3.15)$$

From these equations we obtain the sufficiency condition for instability of an inhomogeneous plasma:

$$\int dx \frac{k_y^2 + Q(\omega,x)}{\sqrt{Q(\omega,x)}}\frac{v_{Ti}^3}{v_{Te}N(1 + T_i/T_e)}\frac{\omega}{k_y}$$

$$\times \left(\frac{N}{v_{Ti}}\right)'\left(1 + \frac{1}{2}\frac{\partial \ln (T_i/T_e)}{\partial \ln (N/T_i^{1/2})}\right) < 0. \qquad (3.16)$$

In addition to this condition we have a local instability condition, which is of more restrictive meaning, but which exhibits a more concrete form. Specifically, noting that the quantity $(N/v_{Ti})'\omega/k_y$ is negative over the entire plasma transparency region [as follows from (3.3)] we have from (3.16)

$$1 + \tfrac{1}{2}\partial \ln (T_i/T_e)/\partial \ln (N/T_i^{1/2}) > 0. \qquad (3.17)$$

This local condition is necessary but not sufficient. On the other hand, if this condition is satisfied for the entire plasma transparency region the plasma is unstable. For the inverse condition, shortwave oscillations are not excited in an inhomogeneous plasma. The instability condition in (3.16) encompasses a very wide class of inhomogeneous plasma configurations. In particular, it shows that an inhomogeneous plasma in which T_i/T_e = const is unstable. We note that the local condition for instability, corresponding to (3.14) and (3.17), has been obtained earlier by Rudakov and Mikhaĭlovskiĭ for the case of an isothermal plasma[9] (cf. also [1-4] and [8]).

In conclusion we note that the instabilities of an inhomogeneous plasma contained by a strong magnetic field investigated in the present section, in

10)These relations were used in an analysis of the spectrum and stability in [11]. We note that the proof of (3.8) and (3.9) can be obtained by another method than that given above. Perturbation theory can be used for this proof.[15]

contrast for example with those investigated in reference 12, represent a kinetic instability, since they are associated with residue terms in the kernel of the integral equation (1.11).

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