CALCULATION OF THE DOUBLET NEUTRON-DEUTERON SCATTERING LENGTH IN THE THEORY OF ZERO RANGE FORCES

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The scattering length for neutron-deuteron scattering in the $S_{1/2}$ state is calculated within the framework of the theory of zero range forces. The value obtained is 0.48×10^{-13} cm and is in satisfactory agreement with the experimental value of Hurst and Alcock.

INTRODUCTION

TER-MARTIROSYAN and Skornyakov [1] and Gribov [2] have obtained equations for the determination of the wave function of a system of three particles in the limiting case $r_0 \to 0$, where r_0 is the range of the forces. A study of these equations shows [3] that they have, in general, a non-unique solution. The general solution of these equations contains an arbitrary constant which depends on the energy and other quantum numbers of the system. The reason for this non-uniqueness of the solution of the Ter-Martirosyan and Skornyakov equations is the collapse of the particles into the center. The only exception to this is the case where the wave function of the system does not contain a part which is symmetric under the interchange of the coordinates of an arbitrary pair of particles, i.e., where the three particles can never be at the same place, so that there can be no collapse into the center. Under this condition the wave function is expressed in terms of the experimental two-particle scattering lengths. We have this situation, for example, in the problem of neutron-deuteron scattering in the state $S = 3/2$, where S is the spin of the system (quartet scattering). A quartet scattering length in good agreement with the experimental data of Hurst and Alcock [4] was calculated in [1].

If the spin of the neutron-deuteron system is $S = 1/2$ (doublet scattering), the collapse into the center occurs and the equations of [1] have a non-unique solution. It has been shown earlier [3] that the non-uniqueness of the solution of these equations can be avoided if, in addition to the triplet and singlet nucleon-nucleon scattering length, one more experimental parameter is given, for example, the binding energy of tritium. Thus the scattering amplitude for neutron-deuteron scattering in the $S = 1/2$ state is expressed in terms of three experimental parameters: the triplet and singlet nucleon-nucleon scattering lengths and the energy of the three-nucleon bound state.

In order to solve the Ter-Martirosyan and Skornyakov equations for $S = 1/2$, we must transform them in such a way as to separate explicitly the arbitrary constant contained in the general solution of these equations.

In the present paper we propose a method for the numerical solution of the equations of Ter-Martirosyan and Skornyakov for $S = 1/2$. We restrict ourselves to the case where the energy of the incident neutron is smaller than the disintegration energy of the deuteron, so that only elastic scattering is possible. The equations obtained in this paper were solved on the electronic computer of the I. V. Kurchatov Institute of Atomic Energy, under the condition that the momentum of the neutron is zero in the center of mass system (c.m.s.). S. A. Frolova, L. F. Kananikhina, and L. S. Tint participated in setting up the computational program. The calculated value of the doublet scattering length is in good agreement with the value given by Hurst and Alcock.

1. TRANSFORMATION OF THE EQUATIONS OF TER-MARTIROSYAN AND SKORNYAKOV

As was shown earlier, [3] the collapse into the center occurs only in the S state. Therefore only the S wave part of the scattering amplitude presents difficulties in the solution of the equations. According to Ter-Martirosyan and Skornyakov, [1] these equations have the form ($\hbar = M = 1$, M is the nucleon mass)

$$\frac{\gamma_E(k) - \bar{\alpha}_i}{k^2 - k_0^2} a_i(k, k_0) = \eta_i \frac{L_E(k, k_0)}{4kk_0}$$

$$+\sum_{r=1}^{2}\frac{1}{\pi}\int_0^\infty \frac{L_E(k,k')}{kk'}\overline{M}_{jr}\frac{a_r(k',k_0)k'^2\,dk'}{k'^2-k_0^2-i\delta}. \quad (1)$$

Here the functions $a_1(k, k_0)$ and $a_2(k, k_0)$ are equal to the functions $a_0(k, k_0)$ and $b_0(k, k_0)$ in [1], so that the S-wave elastic scattering amplitude is $a_1(k_0, k_0)$. The quantities $\overline{\alpha}_1^{-1}$ and $\overline{\alpha}_2^{-1}$ are the triplet and singlet nucleon-nucleon scattering lengths, k_0 is the momentum of the incident neutron in the c.m.s., $\frac{3}{4}k_0^2 - \overline{\alpha}_1^2 = E$, where E is the energy of the system. The quantities η_j have the values $\eta_1 = 1$, $\eta_2 = 3$, $\gamma_E(k) = \sqrt{\frac{3}{4}k^2 - E}$, and the function $L_E(k, k')$ has the form

$$L_E(k,k') = \ln\frac{k^2+k'^2+kk'-E}{k^2+k'^2-kk'-E}$$

Finally, the matrix $\overline{M}_{jr}$ is equal to

$$\overline{M} = \begin{pmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{pmatrix}.$$

According to [3], we must compute the wave function of tritium in order to determine the arbitrary constant in the solution of the system (1). The wave function of tritium satisfies (1) without the free term. Since the system (1) without the free term has a solution for arbitrary E, it is necessary to give the experimental binding energy of tritium E_T for a determination of the wave function of that nucleus.

In the present paper we shall be interested in the case $E < 0$, when only elastic scattering of the neutron on the deuteron is possible. In this case it is easily verified by direct substitution that the functions $a_j(k, k_0)$ can be written in the form

$$a_j(k,k_0) = \overline{a}_j(k,k_0)/[1 - ik_0\overline{a}_1(k_0,k_0)],$$

where $\bar{a}_j(k, k_0)$ is a real quantity which satisfies the system of equations

$$\frac{\gamma_E(k)-\overline{\alpha}_j}{k^2-k_0^2}\overline{a}_j(k,k_0) = \eta_j\frac{L_E(k,k_0)}{4kk_0} + \sum_{r=1}^{2}\frac{1}{\pi}\int_0^\infty\frac{L_E(k,k')}{kk'}\overline{M}_{jr}\frac{\overline{a}_r(k',k_0)k'^2\,dk'}{k'^2-k_0^2}, \quad (3)$$

where the integral is taken in the sense of the principal value.

To transform the system (3) we make use of the substitution proposed by Minlos and Faddeev.[5] We introduce instead of the variable k the new variable t,

$$k = \sqrt{-E}\,(1-t^2)/t\sqrt{3} \quad (4)$$

and instead of the functions $\bar{a}_j(k, k_0)$ the functions

$$\overline{\varphi}_{1,2}(t) = \sqrt{-E}\,\frac{k}{k^2-k_0^2}$$

$$[\overline{a}_1(k,k_0)(\gamma_E(k)-\overline{\alpha}_1) \pm \overline{a}_2(k,k_0)(\gamma_E(k)-\overline{\alpha}_2)]. \quad (5)$$

The functions $\overline{\varphi}_1(t)$ and $\overline{\varphi}_2(t)$ satisfy the equations

$$\overline{\varphi}_1(t) = -\frac{\sqrt{3}\pi}{2x_0}\widetilde{q}(t,t_0) - 2\sum_{j=1}^{2}\int_0^1\widetilde{q}(t,t')M_{1j}(t')\overline{\varphi}_j(t')\,dt' - 2\int_0^1\widetilde{q}(t,t')\overline{\varphi}_1(t')\frac{dt'}{t'}, \quad (6)$$

$$\overline{\varphi}_2(t) = \frac{\sqrt{3}\pi}{4x_0}\widetilde{q}(t,t_0) + \sum_{j=1}^{2}\int_0^1\widetilde{q}(t,t')M_{2j}(t')\frac{dt'}{t'}\overline{\varphi}_j(t'), \quad (7)$$

where

$$\widetilde{q}(t,t') = q\left(\frac{t}{t'}\right) - q(tt'), \quad q(\beta) = -\frac{2}{\sqrt{3}\pi}\ln\frac{1+\beta+\beta^2}{1-\beta+\beta^2}. \quad (8)$$

The matrix $M_{jr}(t)$ is equal to

$$M(t) = \begin{pmatrix} \frac{\alpha_1}{1+t^2-2\alpha_1 t} + \frac{\alpha_2}{1+t^2-2\alpha_2 t} & \frac{\alpha_1}{1+t^2-2\alpha_1 t} - \frac{\alpha_2}{1+t^2-2\alpha_2 t} \\ \frac{\alpha_1 t}{1+t^2-2\alpha_1 t} - \frac{\alpha_2 t}{1+t^2-2\alpha_2 t} & 1 + \frac{\alpha_1 t}{1+t^2-2\alpha_1 t} + \frac{\alpha_2 t}{1+t^2-2\alpha_2 t} \end{pmatrix}, \quad (9)$$

where $\alpha_j = \overline{\alpha}_j/\sqrt{-E}$, and the quantity x_0 is equal to

$$x_0 = k_0/\sqrt{-E} = 2\sqrt{(\alpha_1^2-1)/3}. \quad (10)$$

Finally, $t_0 = \alpha_1 - \sqrt{\alpha_1^2 - 1}$.

The system of equations (6) and (7) has a non-unique solution as a consequence of the fact that the kernel in the last term on the right-hand side of (6) has a singularity for $t \to 0$, $t' \to 0$, $t/t' \to$ const. The kernel of (7) also has a singularity of this type, but, as will be shown in Appendix 1, this singularity does not lead to a non-uniqueness in the solution of the system (6) and (7).

In order to separate out the arbitrary constant in the solution of the system (6) and (7), we shall solve (6) for the function $\overline{\varphi}_1(t)$, regarding all terms on the right hand side of (6) except the last one as the inhomogeneity. In this way we obtain some new equation for the functions $\overline{\varphi}_j(t)$. As shown in Appendix 1, this equation has the form

$$\overline{\varphi}_1(t) = A(\alpha_1)\sin(s_0\ln t) + \frac{\pi\sqrt{3}}{4x_0}Q(t,t_0) + \sum_{j=1}^{2}\int_0^1 Q(t,t')M_{1j}(t')\overline{\varphi}_j(t')\,dt', \quad (11)$$

where $A(\alpha_1)$ is an arbitrary constant, $s_0 \approx 1$ is the positive root of the equation

$$1 - \frac{8\,\mathrm{sh}(\pi s/6)}{\sqrt{3}\,s\,\mathrm{ch}(\pi s/2)} = 0, \quad (12)^*$$

*ch = cosh, sh = sinh.

and

$$Q(t, t') = Q(t/t') - Q(tt'), \tag{13}$$

with

$$Q(\beta) = \frac{8}{\pi\sqrt{3}} \int_0^\infty \frac{\cos(s \ln \beta)\,\mathrm{sh}(\pi s/6)\,ds}{s\,\mathrm{ch}(\pi s/2) - (8/\sqrt{3})\,\mathrm{sh}(\pi s/6)}. \tag{14}$$

The function $Q(\beta)$ obviously has the property $Q(\beta) = Q(1/\beta)$. Besides, as shown in Appendix 1, the function $Q(\beta)$ is, for $\beta < 1$, given by the series

$$Q(\beta) = -\frac{\sin(s_0 \ln \beta)}{f(s_0)} + \sum_{n=1}^{\infty} \frac{1}{C_n} \beta^{s_n}, \tag{15}$$

$$f(s_0) = -\frac{1}{s_0} + \frac{\pi}{6}\,\mathrm{cth}\,\frac{\pi s_0}{6} - \frac{\pi}{2}\,\mathrm{th}\,\frac{\pi s_0}{2}, \tag{16*}$$

$$C_n = -\frac{1}{s_n} + \frac{\pi}{6}\,\mathrm{ctg}\,\frac{\pi s_n}{6} + \frac{\pi}{2}\,\mathrm{tg}\,\frac{\pi s_0}{2}, \tag{17†}$$

where the quantities s_n are the positive roots of the equation

$$s \cos\frac{\pi s}{2} - \frac{8}{\sqrt{3}} \sin\frac{\pi s}{6} = 0. \tag{18}$$

Equations (11) and (7) determine the functions $\bar{\varphi}_j(t)$. The solution of this system can evidently be written in the form

$$\bar{\varphi}_j(t) = A(\alpha_1)\,\varphi_j^{(1)}(t) + \varphi_j^{(2)}(t), \tag{19}$$

where the functions $\varphi_j^{(1)}(t)$ satisfy the equations

$$\varphi_1^{(1)}(t) = \sin(s_0 \ln t) + \sum_{j=1}^{2} \int_0^1 Q(t, t')\,M_{1j}(t')\,\varphi_j^{(1)}(t')\,dt',$$

$$\varphi_2^{(1)}(t) = \sum_{j=1}^{2} \int_0^1 \tilde{q}(t, t')\,M_{2j}(t')\,\varphi_j^{(1)}(t')\,\frac{dt'}{t'}, \tag{20}$$

and the functions $\varphi_j^{(2)}(t)$ are solutions of the system

$$\varphi_1^{(2)}(t) = \frac{\sqrt{3}\pi}{4x_0} Q(t, t_0) + \sum_{j=1}^{2} \int_0^1 Q(t, t')\,M_{1j}(t')\,\varphi_j^{(2)}(t')\,dt',$$

$$\varphi_2^{(2)}(t) = \frac{\sqrt{3}\pi}{4x_0} \tilde{q}(t, t_0) + \sum_{j=1}^{2} \int_0^1 \tilde{q}(t, t')\,M_{2j}(t')\,\varphi_j^{(2)}(t')\,\frac{dt'}{t'}. \tag{21}$$

It follows from (20) and (21) that $\varphi_j^{(2)}(t) \to 0$ for $t \to 0$, whereas the functions $\varphi_j^{(1)}(t)$ are equal to

$$\varphi_1^{(1)}(t) = \sin(s_0 \ln t) + \nu(\alpha_1)\cos(s_0 \ln t), \tag{22}$$

$$\varphi_1^{(2)}(t) = \mu(\alpha_1)\cos(s_0 \ln t). \tag{23}$$

The constants $\nu(\alpha_1)$ and $\mu(\alpha_1)$ in (22) and (23) are expressed in terms of certain integrals over $\varphi_j^{(1)}(t)$ and $\varphi_j^{(2)}(t)$, which we shall not write down in view of their complicated form. To verify (22) and (23) one must substitute formulas (13) and (15) for $Q(t, t')$ in (20) and (21) and use the fact that the integrals over the region $t' < t$ go to zero if $t \to 0$.

Since the functions $\varphi_1^{(1)}(t)$ and $\varphi_1^{(2)}(t)$ have no definite limit for $t \to 0$, the systems of equations (20) and (21) are not very convenient for a numerical solution. However, as shown in Appendix 2, the systems (20) and (21) can be transformed into three systems of equations for the three pairs of functions $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$, which have the property that $\varphi_j(t) \to 0$, $\chi_j(t) \to 0$, and $\Psi_j(t) \to 0$ for $t \to 0$. The systems of equations for the determination of the functions $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$ will not be given here in view of their complicated structure. The formulas connecting the functions $\varphi_j^{(1)}(t)$ and $\varphi_j^{(2)}(t)$ with the functions $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$ are given in Appendix 2.

To solve numerically the equations of Ter-martirosyan and Skornyakov, (1), we must therefore transform them to the equivalent system of equations (6) and (7) by the change of variables (4). The general solution of the equations (6) and (7) is given in terms of the solution of the equations (20) and (21) according to (19), and contains a single arbitrary constant, $A(\alpha_1)$. The equations (20) and (21) or the equivalent equations for the functions $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$ have a unique solution and can be solved numerically.

2. DETERMINATION OF THE CONSTANT $A(\alpha_1)$

Let us now turn to the determination of the constant $A(\alpha_1)$. It follows from [3] that the constant $A(\alpha_1)$ can be determined from the requirement that the following relation be satisfied for $k \to \infty$:

$$a_1(k, k_0) + a_2(k, k_0) = C(E)\,[a_1(k) + a_2(k)], \tag{24}$$

where $a_j(k)$ are the functions $a_j(k, k_0)$ for tritium and $C(E)$ is a constant independent of k. The functions $a_j(k)$ are determined by solving the system (1) with the free term set equal to zero and $E = E_T$. The functions $\varphi_j^{(2)}(t)$ are therefore zero for tritium.

The functions $\varphi_j^{(1)}(t)$ for tritium, which we shall denote by $\tilde{\varphi}_j(t)$, are solutions of the system (20) with

$$\alpha_1 = \alpha_1^0 = \bar{\alpha}_1/\sqrt{-E_T}, \qquad \alpha_2 = \alpha_2^0 = \bar{\alpha}_2/\sqrt{-E_T}.$$

Expressing the quantities $a_j(k, k_0)$ and $a_j(k)$ in terms of the functions $\varphi_j^{(1)}(t)$, $\varphi_j^{(2)}(t)$, and $\tilde{\varphi}_j(t)$ and using (4) for the relation between k and t, we obtain for $k \to \infty$

*cth = coth, th = tanh.

†ctg = cot, tg = tan.

$$a_1(k, k_0) + a_2(k, k_0)$$
$$= A(\alpha_1)[\sin(s_0 z) + \nu(\alpha_1)\cos(s_0 z)] + \mu(\alpha_1)\cos(s_0 z), \quad (25)$$

$$a_1(k) + a_2(k) = \sin(s_0 z/\lambda) + \nu(\alpha_1^0)\cos(s_0 z/\lambda), \quad (26)$$

where

$$z = \ln(\sqrt{-E/3}/k), \qquad \lambda = \alpha_1^0/\alpha_1. \quad (27)$$

It follows from (25) and (26) that (22) holds only if

$$A(\alpha_1) = \frac{\mu(\alpha_1)[\cos(s_0 \ln\lambda) + \nu(\alpha_1)\sin(s_0\ln\lambda)]}{\nu(\alpha_1^0)[\cos(s_0\ln\lambda) - \nu(\alpha_1)\sin(s_0\ln\lambda)] - \rho}, \quad (28)$$

$$\rho = \sin(s_0\ln\lambda) + \nu(\alpha_1)\cos(s_0\ln\lambda). \quad (29)$$

3. DETERMINATION OF THE DOUBLET NEUTRON-DEUTERON SCATTERING LENGTH

To calculate the doublet neutron-deuteron scattering length we must solve the system (1) for $E = -\overline{\alpha}_1^2$, i.e., for $k_0^2 = 0$. For this purpose it is in turn necessary to solve the system (20) and (21) for $\alpha_1 = 1$, i.e., for $x_0 = 0$. Moreover, the determination of the constant $A(\alpha_1)$ requires the knowledge of the wave function of tritium, i.e., the solution of the system (20) and (21) for $\alpha_1 = \alpha_1^0$, $\alpha_2 = \alpha_2^0$.

As explained in Sec. 1, it is convenient from a practical point of view not to solve the system (20) and (21) but the equivalent system of equations for the functions $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$. The equations for these functions were solved numerically on an electronic computer.

According to the Gauss quadrature formula[6]

$$\int_0^1 f(t)\,dt = \sum_{k=1}^{n} A_k f(t_k), \quad (30)$$

where t_k are the points and A_k the coefficients of the quadrature formula. The systems of integral equations for $\varphi_j(t)$, $\chi_j(t)$, and $\Psi_j(t)$ have been replaced by algebraic equations. The Gauss quadrature formulas were very convenient for the solution of this problem, since they give good accuracy and the distribution of the points t_k permits one to take the singularities of the integral equations into account correctly. The computation of the functions $Q(\beta)$ for $\beta > 0.75$ was carried out with the help of specially derived quadrature formulas of the Gauss type which are convenient for calculations of principal value integrals with high accuracy. For $\beta < 0.75$, $Q(\beta)$ was calculated with the help of the representation (15), taking into account the first 32 terms of the series.

The systems of algebraic equations were solved by the method of Gauss with selection of a main element. The problem was solved for three values of n: n = 15, n = 16, and n = 22. A comparison of the results for different n showed that the quadrature formulas employed in the calculation gave the function $a_1(k, k_0)$ accurately to two significant places.

For the triplet and singlet nucleon-nucleon scattering lengths and the binding energy of tritium we took the following values:

$$\overline{\alpha}_1^{-1} = 4{,}32\cdot 10^{-13}\,\text{cm}$$

$$\overline{\alpha}_2^{-1} = -25\cdot 10^{-13}\,\text{cm} \qquad E_T = -8.32\,\text{MeV}.$$

As a result of the calculations the doublet neutron-deuteron scattering length $-a_1(0,0)$ came out to be

$$-a_1(0,0) = 0{,}11\cdot\overline{\alpha}_1^{-1} = 0.48\cdot 10^{-13}\,\text{cm}.$$

The experimental value for this quantity, obtained by Hurst and Alcock,[4] is

$$-a_1(0,0) = (0.16 \pm 0.07)\,\overline{\alpha}_1^{-1} = (0.7 \pm 0.3)\cdot 10^{-13}\,\text{cm}.$$

Thus the theory of zero range forces describes in a completely satisfactory manner the scattering of a neutron on a deuteron at low energies.

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APPENDIX I

We shall solve (6) for $\overline{\varphi}_1(t)$, regarding all terms on the right-hand side of (6) except the last one as the inhomogeneity. The solution is written in the form

$$\overline{\varphi}_1(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty V(s)\sin(s\ln t)\,ds. \quad (A.1)$$

We substitute (A.1) in (6), multiply both sides of (6) by $\sqrt{2/\pi}\, t^{-1}\sin(s\ln t)$ and integrate over t from zero to one. By virtue of the equality

$$\frac{2}{\pi}\int_0^1 \sin(s\ln t)\sin(s'\ln t)\frac{dt}{t} = \delta(s - s'), \quad (A.2)$$

we obtain an algebraic equation for the determination of $V(s)$:

$$V(s)\left[1 - \frac{8}{\sqrt{3}}\frac{\operatorname{sh}(\pi s/6)}{s\operatorname{ch}(\pi s/2)}\right] = \Phi(s), \quad (A.3)$$

$$\Phi(s) = \sqrt{\frac{2}{\pi}}\int_0^1 \sin(s\ln t) f(t)\frac{dt}{t}, \quad (A.4)$$

where $f(t)$ is the sum of all terms on the right hand side of (6) except the last one. It follows from (A.3) that

$$V(s) = \mathrm{P}\frac{\Phi(s)}{1 - 8\,\mathrm{sh}(\pi s/6)/\sqrt{3}\, s\,\mathrm{ch}(\pi s/2)} + \sqrt{\frac{\pi}{2}}A(\alpha_1)\,\delta(s - s_0), \tag{A.5}$$

where $A(\alpha_1)$ is an arbitrary constant, $s_0 > 0$ is the root of Eq. (12), and the symbol P indicates that integrals over s containing this term are to be understood in the sense of the principal value.

In order to obtain (11), we must substitute (A.5) in (A.1) and use the explicit expression for $\Phi(s)$. The first term on the right-hand side of (11) comes from the term $\delta(s - s_0)$ in (A.5) and represents the solution of (6) with $f(t) \equiv 0$.

Let us explain now why the systems (20) and (21) have a unique solution. Doubts about the uniqueness of the solution of these systems might arise because of the singular nature of $\tilde{q}(t,t')$ for $t \to 0$, $t' \to 0$, $t/t' \to$ const. The equations in (20) and (21) containing $\tilde{q}(t,t')$ have the following structure

$$\eta(t) = f_1(t) + \int_0^1 \tilde{q}(t,t')\,\eta(t')\frac{dt'}{t'}, \tag{A.6}$$

where the function $f_1(T)$ is the sum of all terms on the right-hand side of the second equation (20) or (21) except the term written explicitly in (A.6), and $\eta(t)$ is either $\varphi_2^{(1)}(t)$ or $\varphi_2^{(2)}(t)$. The solution of (A.6) can be sought by the above-mentioned method in the form

$$\eta(t) = \sqrt{\frac{2}{\pi}}\int_0^\infty V_1(s)\sin(s\ln t)\,ds. \tag{A.7}$$

For $V_1(s)$ we obtain the equation

$$V_1(s)\left[1 + \frac{4}{\sqrt{3}}\frac{\mathrm{sh}(\pi s/6)}{s\,\mathrm{ch}(\pi s/2)}\right] = \Phi_1(s),$$

$$\Phi_1(s) = \sqrt{\frac{2}{\pi}}\int_0^1 f_1(t)\sin(s\ln t)\frac{dt}{t}. \tag{A.8}$$

Since the equation

$$1 + 4\,\mathrm{sh}(\pi s/6)/\sqrt{3}s\,\mathrm{ch}(\pi s/2) = 0$$

has no roots in the region $s > 0$, $V_1(s)$ is uniquely determined by (A.8). Substituting the value obtained for $V_1(s)$ in (A.7), we obtain an equation with a kernel of the Fredholm type which has a unique solution.

Finally, let us obtain formula (15) for $Q(\beta)$. Let us rewrite (14) in the form

$$Q(\beta) = \frac{4}{\pi\sqrt{3}}\int_{-\infty}^{\infty}\frac{\beta^{-is}\,\mathrm{sh}(\pi s/6)\,ds}{s\,\mathrm{ch}(\pi s/2) - (8/\sqrt{3})\,\mathrm{sh}(\pi s/6)}$$

$$= \frac{4}{\pi\sqrt{3}}\int_L \frac{\beta^{-is}\,\mathrm{sh}(\pi s/6)\,ds}{s\,\mathrm{ch}(\pi s/2) - (8/\sqrt{3})\,\mathrm{sh}(\pi s/6)} - \frac{\sin(s_0\ln\beta)}{f(s_0)}, \tag{A.9}$$

where the integral over the contour L goes along the real axis from $-\infty$ to $+\infty$, going around the poles at $s = \pm s_0$ on the upper side, and

$$f(s_0) = \frac{d}{ds}\frac{8}{\sqrt{3}}\frac{\mathrm{sh}(\pi s/6)}{s\,\mathrm{ch}(\pi s/2)}\bigg|_{s=s_0}. \tag{A.10}$$

The integral over the contour L is easily calculated by the method of residues. It can be shown that in the upper half plane of s the denominator of the integrand in (A.9) has only pure imaginary roots $s = is_n$. This leads to formulas (15), (17), and (18).

APPENDIX I

Let us consider, for example, the system (20). We write $Q(t,t')$ in the form

$$Q(t,t') = \overline{Q}(t,t') + \frac{2}{f(s_0)}\frac{(1-t^2)\cos(s_0\ln t)}{1 + t^2 + 2\alpha_1 t}\sin(s_0\ln t'). \tag{A.11}$$

It follows from (13) and (15) that $\overline{Q}(t,t') \to 0$ for $t \to 0$, $t' \neq 0$. We rewrite the first equation of (20) in the form

$$\varphi_1^{(1)}(t) = \sin(s_0\ln t) + \frac{(1-t^2)\,\nu(\alpha_1)}{1+t^2+2\alpha_1 t}\cos(s_0\ln t)$$
$$+ \sum_{j=1}^{2}\int_0^1 \overline{Q}(t,t')\,M_{1j}(t')\,\varphi_j^{(1)}(t')\,dt', \tag{A.12}$$

$$\nu(\alpha_1) = \frac{2}{f(s_0)}\sum_{j=0}^{2}\int_0^1 \sin(s_0\ln t)\,M_{1j}(t)\,\varphi_j^{(1)}(t)\,dt. \tag{A.13}$$

The functions $\varphi_j^{(1)}(t)$ are written in the form

$$\varphi_1^{(1)}(t) = \sin(s_0\ln t) + \frac{(1-t^2)\,\nu(\alpha_1)}{1+t^2+2\alpha_1 t}\cos(s_0\ln t)$$
$$+ \frac{1-t^2}{1+t^2+2\alpha_1 t}\left[\varphi_1(t) + \nu(\alpha_1)\,\chi_1(t)\right], \tag{A.14}$$

$$\varphi_2^{(1)}(t) = \frac{1-t^2}{1+t^2+2\alpha_1 t}\left[\varphi_2(t) + \nu(\alpha_1)\,\chi_2(t)\right]. \tag{A.15}$$

It then follows from (22) that the functions $\varphi_j(t)$ and $\chi_j(t)$ go to zero for $t \to 0$. In order to obtain equations for the functions $\varphi_j(t)$ and $\chi_j(t)$ we must substitute (A.14) and (A.15) in (A.12) and in the second equation of (20) and compare the expressions for equal powers of $\nu(\alpha_1)$.

In order to determine the parameter $\nu(\alpha_1)$ we must substitute expressions (A.14) and (A.15) for $\varphi_j^{(1)}(t)$ in (A.13) and solve the resulting equation for $\nu(\alpha_1)$. The system (21) can be transformed in a similar way. The functions $\varphi_j^{(2)}(t)$ are expressed in terms of the functions $\chi_j(t)$ and $\Psi_j(t)$ according to

$$\varphi_1^{(2)}(t) = \frac{1-t^2}{1+t^2+2\alpha_1 t}[\mu(\alpha_1)\cos(s_0 \ln t)$$

$$+\mu(\alpha_1)\chi_1(t) + \Psi_1(t)], \qquad (A.16)$$

$$\varphi_2^{(2)}(t) = \frac{1-t^2}{1+t_2+2\alpha_1 t}[\mu(\alpha_1)\chi_2(t) + \Psi_2(t)]. \qquad (A.17)$$

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