CONCENTRATION OF A LIGHT WAVE IN A WEAKLY INHOMOGENEOUS DIELECTRIC

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We consider the transmission of a plane wave through a smoothly stratified inhomogeneity. At angles of incidence close to that corresponding to total reflection from the layer the energy density inside the inhomogeneity can become many times greater than the mean energy density in the wave. The possible connection between this effect and the granulated structure of radiation from laser crystals is discussed.

THE propagation of waves in inhomogeneous media has been treated by a number of workers.[1-3] It is the purpose of the present work to call attention to the fact that an appreciable field concentration is possible even when the inhomogeneity is weak ($\Delta n/n \ll 1$). The field concentration effect is usually accompanied by the generation of heat which, in turn, causes a change in the optical inhomogeneity. This effect may be responsible for the field formation in crystals used in lasers. We shall treat the problem of field concentration in a wave transmitted through a weak optical inhomogeneity in the simplest possible terms.

Assume that in an infinite nonabsorbing nonmagnetized isotropic medium there is a smooth plane stratified inhomogeneity on which, at an angle θ, is incident a plane polarized monochromatic wave. We compute the ratio of the energy density inside the stratification to the energy far from the stratification as a function of θ. The electric field $\mathbf{E}_\theta$ is assumed to be perpendicular to the plane of incidence. Taking the y axis parallel to $\mathbf{E}_\theta$ and the z axis perpendicular to the stratification, we write the following equations for the complex field amplitudes:

$$\partial^2 E_\theta/\partial x^2 + \partial^2 E_\theta/\partial z^2 + k^2(z) E_\theta = 0, \qquad k(z) = \omega c^{-1}\sqrt{\varepsilon(z)}.$$

The quantity $E_\theta(x, z)$ is then given by

$$E_\theta(x, z) = X_\theta(x) Z_\theta(z), \qquad X_\theta(x) = X \exp(ikx\sin\theta),$$

$$d^2 Z_\theta/dz^2 + [k^2(z) - k^2\sin^2\theta] Z_\theta = 0, \qquad k = \omega c^{-1}\sqrt{\varepsilon},$$

where ϵ and θ are the values of the dielectric constant of the medium and the angle of incidence at far distances from the stratification.

We shall assume that $\epsilon(z)$ is given by the Epstein function for a symmetric stratification [1,2] (Fig. 1a):

$$\varepsilon(z) = \varepsilon[1 - 4Me^{mz}/(1 + e^{mz})^2].$$

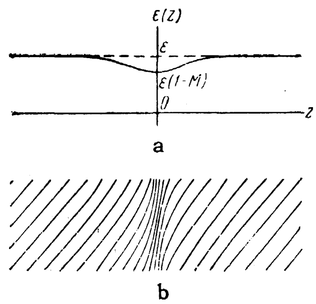

FIG. 1. The variation of $\epsilon(z)$ in a symmetric Epstein layer (a) and refraction in an Epstein layer (b).

We now compute the quantity

$$Q(\theta, z) = \varepsilon(z)\,|E_\theta(z)|^2/\varepsilon\,|E_\theta(-\infty)|^2$$

which is the ratio of energy density inside the stratification to that in the incident wave far from the stratification.[1)] When

$$Z_\theta = F_\theta(\xi)\,\xi^{1/2\, iS\cos\theta}(1-\xi)^\alpha, \qquad \xi = -e^{mz}$$

the function $F_\theta(\xi)$ satisfies the hypergeometric equation

$$\frac{d^2F_\theta}{d\xi^2} - \frac{(2\alpha + iS\cos\theta + 1)\xi - (1 + iS\cos\theta)}{\xi(1-\xi)}\frac{dF_\theta}{d\xi}$$

$$- \frac{\alpha(\alpha + iS\cos\theta)}{\xi(1-\xi)}F_\theta = 0,$$

$$\alpha = {}^1/_2\,(1 + \sqrt{1 - 4S^2M}), \quad S = 2k/m.$$

The parameter S and the stratification thickness are related by the expression $S \cong 4l/\lambda$ where λ is the wavelength far from the inhomogeneity.

[1)]The ratio of the energy density inside the inhomogeneity to the mean value far from the inhomogeneity is affected by the reflected wave as well as the incident wave. It differs from Q by the factor

$$R(\theta) = \{1 + \exp[2\pi S(\cos\theta - \sqrt{M})]\}/\{2 + \exp[2\pi S(\cos\theta - \sqrt{M})]\},$$

which ranges between ½ and 1.

The value of $Q(\theta, z)$ at the center of the stratification is

$$Q(\theta, 0) = \frac{1-M}{4\pi}\left|\frac{\Gamma(^1/_4 + ^1/_4\sqrt{1-4A^2} - ^1/_2 iA\tau)\,\Gamma(^1/_4 - ^1/_4\sqrt{1-4A^2} - ^1/_2 iA\tau)}{\Gamma(iA\tau)}\right|^2,$$

where

$$A = S\sqrt{M},\ \tau = M^{-1/2}\cos\theta.$$

We consider briefly the case $M > 0$ for which the optical density in the inhomogeneous stratification is smaller than that of the uniform medium, assuming that the density in the stratification does not differ greatly from that of the medium but that the relative thickness of the stratification is large: $M \ll 1$, $A \gg 1$. In this case, for angles that are not too close to the angle corresponding to total reflection $\theta_0 = \cos^{-1}\sqrt{M}$ we have $Q(\theta, 0) \cong \cos\theta/\sqrt{|\cos^2\theta - M|}$. This formula can be easily obtained from geometric optics: it describes the concentration of energy at the center of the inhomogeneity due to the refraction and concentration of rays in the medium of lower density (Fig. 1b).

At angles close to θ_0, we find $Q(\theta, 0) \cong \sqrt{A} \times \exp[-2S^2(\theta - \theta_0)^2]$, that is to say, when $|\theta - \theta_0| < 1/2S$ the energy density at the center of the inhomogeneity is $\sqrt{A}$ times greater than the mean value (Fig. 2).

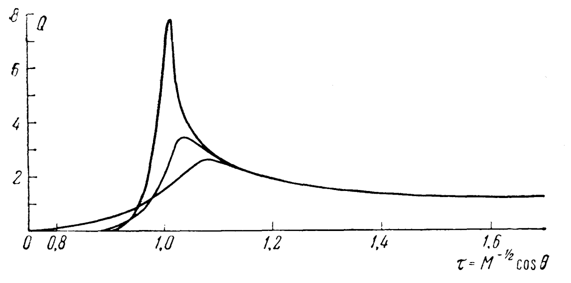

FIG. 2. Field concentration as a function of angle of incidence at the stratification; A = 100; 10; 5 (in order of decreasing maxima).

If $M < 0$ a region of guided propagation is produced in the dielectric. We shall not consider this case here. It is our opinion that as far as lasers are concerned the region of guided propagation is not of great interest since it is isolated from the remaining medium and is thus not convenient for generation.

The estimate $Q(\theta_0, 0) \cong \sqrt{A}$ is obtained under several simplifying assumptions, one of which is that $\epsilon(z)$ is real at the frequency being considered. Absorption in a solid dielectric means, first of all, that there will be enhanced heat generation at points of local field concentration; the local heating can have a marked effect on the inhomogeneity. For example, if field concentration in a dielectric that is initially uniform causes a heating of several degrees, an inhomogeneity will be formed (in the heated channel) for which the value $|M| \cong \epsilon^{-1}\Delta T|d\epsilon'/dT|$ is 10^{-3}—10^{-4}; if $d\epsilon'/dT < 0$, $l \cong 5 \times 10^{-2}$ cm, and $\lambda = 8 \times 10^{-5}$ cm we have $Q(\theta_0, 0) \cong 5$—10. Depending on the sign of the thermal effect the field concentration of the light wave can be strengthened or weakened.

These considerations also apply to a laser crystal. In order to obtain a negative balance between absorption and amplification one usually chooses a material in which the absorption is a minimum; at the generation frequency; the primary source of heating here is the loss due to the transformation of the acquired energy in the generated radiation. For example, to obtain photons at the generation frequency one must use radiationless transitions in a ruby laser. A simple calculation shows that an increase in the intensity of induced transitions means that there is an increase in the probability for spontaneous radiationless transitions; hence, field concentration of the generated radiation leads to local heating. The sign of $d\epsilon'/dT$ determines the thermal stability of the uniformity of the field associated with the light wave; since generation occurs in a region of anomalous dispersion the sign of $d\epsilon'/dT$ can depend on frequency.

We consider briefly the field distribution in a laser crystal. Usually the mirrors at the ends are perpendicular to Ox—the direction of the optic axis. One or more discrete modes, characterized by the wave vectors **k** participate in the oscillation. As a consequence of fluctuations the wave vectors for each mode are smeared to some extent both in magnitude and direction. In a cylindrical crystal the highest Q will be that of the axial modes whose wave vectors make small angles with the Ox axis. The inhomogeneity of the field in the axial modes can only have a stratified structure since any inhomogeneity in the Ox direction is rapidly equalized as a result of induced radiation effects. We use the notation (y_1, z_1), (y_2, z_2), ... to denote segments parallel to Ox along which the optical path between mirrors is a minimum. These segments surround regions of lower optical density in which the field is concentrated. Thus, at small intensities individual saturation "wavelets" (y_1, z_1), (y_2, z_2), ... are generated; the greater bulk of the crystal remains essentially in the linear amplification mode. Each mode corresponds to an interference pattern determined by the geometry of the crystal and the distribution of $\epsilon(y, z)$. The radiation of all the small portions of one mode appearing as maxima in this pattern is coherent.

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